Chapter 7
Theory of Elasticity

7.1 Introduction

The classical theory of elasticity is primary a theory for isotropic, linearly elastic materials subjected to small deformations. All governing equations in this theory are linear partial differential equations, which means that the *principle of superposition* may be applied: The sum of individual solutions to the set of equations is also a solution to the equations. The classical theory of elasticity has a *theorem of uniqueness of solution* and a *theorem of existence of solution*. The theorem of uniqueness insures that if a solution of the pertinent equations and the proper boundary conditions for a particular problem is found, then this solution is the only solution to the problem. The theorem is presented and proven in Sect. 7.6.3. The theorem of existence of a solution is fairly difficult and complicated to prove and perhaps not so important, as from a practical view point we understand that a physical solution must exist. Although this chapter is primarily devoted to the classical theory of elasticity, Sect. 7.10 includes some fundamental aspects of the general theory of elasticity.

Temperature changes in a material result in strains and normally also in stresses, so-called *thermal stresses*. Section 7.5 presents the basis for determining the thermal stresses in elastic material.

Two-dimensional theory of elasticity in Sect. 7.3 presents analytical solutions to many relatively simple but important problems. Examples are thick-walled circular cylinders subjected to internal and external pressure and a plate with a hole. Analytical solutions are, apart from being of importance as solutions to practical problems, also serving as test examples for numerical solution procedures like finite difference methods and finite element methods.

An important technical application of the theory of elasticity is the theory of torsion of rods. The elementary torsion theory applies only to circular cylindrical bars. The *Saint-Venant theory of torsion* for cylindrical rods of arbitrary cross-section is presented in Sect. 7.4.

The theory of stress waves in elastic materials is treated in Sect. 7.7. The introductory part of the theory of elastic waves is mathematically relatively simple, and some of the most important aspects of elastic wave propagation are revealed, using simple one-dimensional considerations. The general theory of elastic waves is fairly complex and will only be given in an introductory exposition.

Anisotropic linearly elastic materials are presented in Sect. 7.8. Materials having different kinds of symmetry are discussed. This basis is then applied in the theory of fiber-reinforced composite materials in Sect. 7.9.

An introduction to non-linearly elastic materials and elastic materials that are subjected to large deformations, is presented in Sect. 7.10. We shall return to these topics in Sect. 11.7.

Energy methods have great practical importance, both for analytical solution and in relation to the finite element method. The basic concepts of elastic energy are presented in Sect. 7.2 and 7.10. However, the energy methods are not included in this book.

7.2 The Hookean Solid

We shall now assume *small deformations and small displacements* such that the strains may be given by the *strain tensor for small deformations* **E**, which in a Cartesian coordinate system Ox has the components:

$$E_{ik} = (u_{i,k} + u_{k,i}) \tag{7.2.1}$$

where u_i are the components of the displacement vector **u**, and $u_{i,j}$ are the displacement gradients. E_{ii} (not summed) are longitudinal strains in the directions of the coordinate axes, and E_{ij} $(i \neq j)$ are half of the shear strains for the directions $\mathbf{e}_i$ and $\mathbf{e}_j$.

A material is called *elastic*, also called *Cauchy-elastic*, if the stresses in a particle $\mathbf{r} = x_i \mathbf{e}_\mathrm{i}$ are functions only of the strains in the particle.

$$T_{ik} = T_{ik}(E, x) \Leftrightarrow \quad \mathbf{T} = \mathbf{T}[\mathbf{E}, \mathbf{r}] \tag{7.2.2}$$

These equations are the basic *constitutive equations* for *Cauchy-elastic materials*. In Sect. 7.6 another definition of elasticity is introduced, *hyperelasticity*, also called *Green-elasticity*, which reflects that a material may store deformation work as *elastic energy*, also called *strain energy*, a concept to be defined already in Sect. 7.2.2.

If the elastic properties are the same in every particles in a material, the material is *elastically homogeneous*. If the elastic properties are the same in all directions through one and the same particle, the material is *elastically isotropic*. Metals, rocks, and concrete are in general considered to be both homogeneous and isotropic materials. The crystals in polycrystalline materials are assumed to be small and their orientations so random that the crystalline structure may be neglected. Each individual crystal is normally anisotropic. By milling or other forms of

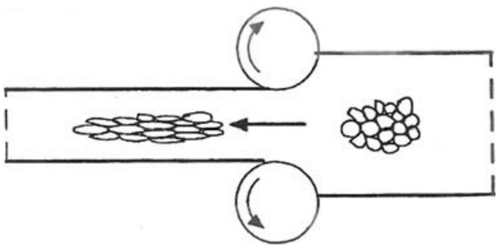


Fig. 7.2.1 Anisotropy due to milling

macro-mechanical forming of polycrystalline metals, an originally isotropic material may be anisotropic, as indicated in Fig. 7.2.1. Inhomogeneities in concrete due to the presence of large particles of gravel may for practical reasons be overlooked when the concrete is treated as a continuum. Materials with fiber structure and well defined fiber directions have anisotropic elastic response. Wood and fiber reinforced plastic are typical examples. The elastic properties of wood are very different in the directions of the fibers and in the cross-fiber direction. Anisotropic elastic materials are discussed in Sect. 7.8 and 7.9.

Isotropic elasticity implies that the principal directions of stress and strain coincide: *The stress tensor and the strain tensor are coaxial*. This may be demonstrated by the following arguments. Figure 7.2.2 shows a material element with orthogonal edges in the undeformed configuration.The element is subjected to a state of stress with principal directions parallel to the undeformed edges of the element. The deformed element is also shown. For the sake of illustration the deformation of the element is exaggerated considerably. Due to the symmetry of the configuration of stress and the isotropy of the elastic properties the diagonal planes marked p_1 and p_2 are equally deformed. This means that the element retains the right angles between its edges through the deformation. Thus the principal directions of strains coincide with the principal directions of stress.

Homogeneous elasticity implies that the stress tensor is independent of the particle coordinates. Therefore the constitutive equation of a homogeneous Cauchy-elastic material should be of the form:

$$T_{ik} = T_{ik}(E) \Leftrightarrow \quad \mathbf{T} = \mathbf{T}[\mathbf{E}] \tag{7.2.3}$$

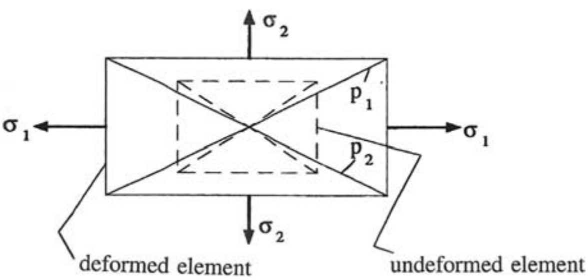


Fig. 7.2.2 Coaxial stresses and strains

If the relation (7.2.2) is linear in **E**, the material is said to be *linearly elastic*. The six coordinate stresses T_{ij} with respect to a coordinate system Ox are now linear functions of the six coordinate strains E_{ij}. For a generally anisotropic material these linear relations contain $6 \times 6 = 36$ coefficients or material parameters, which are called *elasticities* or *stiffnesses*. For a homogeneously elastic material the stiffnesses are constant material parameters. We shall now prove that for an isotropic, linearly elastic material the number of independent stiffnesses is two. Different types of anisotropy are presented and discussed in Sects. 7.8 and 7.9.

In tension or compression tests of isotropic materials a test specimen is subjected to uniaxial stress σ and experiences the strains ε in direction of the stress and ε_t in any transverse direction, i.e. normal to the stress. For a linearly elastic material the following relations may be stated:

$$\varepsilon = \frac{\sigma}{\eta}\,, \quad \varepsilon_t = -\nu\,\varepsilon = \nu\,\frac{\sigma}{\eta} \tag{7.2.4}$$

where η is the *modulus of elasticity* and ν is the *Poisson's ratio*. Values for η and ν for some characteristic materials are given in Table 7.2.1. The symbol η for the modulus of elasticity rather then the more common symbol E is used to prevent confusion between the modulus of elasticity and the strain matrix E. Throughout the book the symbol η for the modulus of elasticity will be used in constitutive equations were the bold face tensor notation or the index notation are used. However, when the xyz – notation is used and in cylindrical coordinates and spherical coordinates the more common symbol E will be used for the modulus of elasticity.

A linearly elastic material in a state of uniaxial stress: $\sigma_1 \neq 0, \sigma_2 = \sigma_3 = 0$, obtains the strains:

$$\varepsilon_1 = \frac{\sigma_1}{\eta}\,, \quad \varepsilon_2 = \varepsilon_3 = \varepsilon_t = -\nu\,\frac{\sigma_1}{\eta}$$

In a general state of triaxial stress, with principal stresses σ_1, σ_2, and σ_3, the principal strains are:

Table 7.2.1 Density ρ, modulus of elasticity $\eta \equiv E$, shear modulus $\mu \equiv G$, poisson's ratio ν, and thermal expansion coefficient α for some characteristic materials

	ρ [10^3 kg/m^3]	η, E [GPa]	μ, G [GPa]	ν	α [$10^{-6}\,°\mathrm{C}^{-1}$]
Steel	7.83	210	80	0.3	12
Aluminium	2.68	70	26	0.25	23
Concrete	2.35	20–40		0.15	10
Copper	8.86	118	41	0.33	17
Glass	2.5	80	24	0.23	3–9
Wood	0.5	4–11	fiber dir.	anisotropic	3–8
Cork				0	
Rubber	1.5		0.007	0.49	
Bronze	8.30	97	39		
Brass	8.30	97	39		19
Magnesium	1.77	40	16		25
Cast iron	7.75	103	41	0.25	11

$$\varepsilon_1 = \frac{\sigma_1}{\eta} - \frac{\nu}{\eta}(\sigma_2 + \sigma_3) = \frac{1+\nu}{\eta}\sigma_1 - \frac{\nu}{\eta}(\sigma_1 + \sigma_2 + \sigma_3) \quad \text{etc. for } \varepsilon_2 \text{ and } \varepsilon_3$$

The result follows from the fact that the relations between stresses and strains are linear such that the principal of superposition applies. The result may be rewritten to:

$$\varepsilon_i = \frac{1+\nu}{\eta}\sigma_i - \frac{\nu}{\eta}\,\mathrm{tr}T\,, \quad \mathrm{tr}T = \sigma_1 + \sigma_2 + \sigma_3 = \mathrm{tr}\;\mathbf{T}$$

The expression may also be presented in the matrix representation:

$$\varepsilon_i \delta_{ik} = \frac{1+\nu}{\eta}\sigma_i\,\delta_{ik} - \frac{\nu}{\eta}(\mathrm{tr}T)\,\delta_{ik} \tag{7.2.5}$$

This is furthermore the matrix representation of a tensor equation between the tensors **E** and **T** in a coordinate system with base vectors parallel to the principal directions of stress. The matrix equation (7.2.5) now represents the tensor equation:

$$\mathbf{E} = \frac{1+\nu}{\eta}\mathbf{T} - \frac{\nu}{\eta}(\mathrm{tr}\;\mathbf{T})\,\mathbf{1} \tag{7.2.6}$$

In any Cartesian coordinate system Ox (7.2.6) has the representation:

$$E_{ik} = \frac{1+\nu}{\eta}T_{ik} - \frac{\nu}{\eta}T_{jj}\,\delta_{ik} \tag{7.2.7}$$

From the (7.2.6) and (7.2.7) the inverse relations between the stress tensor and the strain tensor is obtained:

$$\mathbf{T} = \frac{\eta}{1+\nu}\left[\mathbf{E} + \frac{\nu}{1-2\nu}(\mathrm{tr}\;\mathbf{E})\,\mathbf{1}\right] \quad \Leftrightarrow \quad T_{ik} = \frac{\eta}{1+\nu}\left[E_{ik} + \frac{\nu}{1-2\nu}E_{jj}\,\delta_{ik}\right] \tag{7.2.8}$$

The inversion procedure is given as Problem 7.1. The relations (7.2.6, 7.2.7, 7.2.8) represent the *generalized Hooke's law* and are the constitutive equations of the *Hookean material* or *Hookean solid*, which are names for a *isotropic, linearly elastic material*. Note that normal stresses T_{ii} only result in longitudinal strains E_{ii}, and visa versa, and that shear stresses T_{ij} only result in shear strains $\gamma_{ik} = 2E_{ik}$, and visa versa. This property is in general not the case for anisotropic materials, see Sect. 7.8.

For the relations between the coordinate shear stresses and coordinate shear strains the (7.2.8) give:

$$T_{ik} = 2\mu\,E_{ik} = \mu\,\gamma_{ik}\,, \quad i \neq k \tag{7.2.9}$$

The material parameter μ ($\equiv G$ in common notation) is called the *shear modulus* and is given by:

$$\mu = \frac{\eta}{2(1+\nu)} \quad \Leftrightarrow \quad G = \frac{E}{2(1+\nu)} \tag{7.2.10}$$

Table 7.2.1 presents the elasticities $\eta\,(\equiv E), \mu\,(\equiv G)$, and ν for some characteristic materials. The values vary some with the quality of the materials listed and with temperature. The values given in the table are for ordinary, or room, temperature of 20°C. We shall find that the relationship for the shear modulus in formula (7.2.10) is not exactly satisfied. This is due to the fact that the values given in Table 7.2.1 are standard values found from different sources.

The relationship between the elastic volumetric strain ε_v and the stresses is found by computing the trace of the strain matrix from (7.2.7).

$$\varepsilon_v = E_{ii} = \frac{1+\nu}{\eta} T_{ii} - \frac{\nu}{\eta} T_{jj}\,\delta_{ii} = \frac{1-2\nu}{\eta} T_{ii} \tag{7.2.11}$$

The *mean normal stress* σ^o and the *bulk modulus* or the *compression modulus* of elasticity κ are introduced:

$$\sigma^o = \frac{1}{3} T_{ii} = \frac{1}{2} \operatorname{tr} \mathbf{T} \quad \text{the mean normal stress} \tag{7.2.12}$$

$$\kappa = \frac{\eta}{3\,(1-2\nu)} \quad \text{the bulk modulus} \tag{7.2.13}$$

Then the result (7.2.11) may now be presented as:

$$\varepsilon_v = \frac{1}{\kappa}\,\sigma^o \tag{7.2.14}$$

A more appropriate name for κ than the compression modulus had perhaps been the *expansion modulus*. For an isotropic state of stress the mean normal stress is equal to the normal stress, i.e. $\mathbf{T} = \sigma^o \mathbf{1}$.

Fluids are considered as linearly elastic materials when sound waves are analyzed. The only elasticity relevant for fluids is the bulk modulus κ. For water $\kappa = 2.1$GPa, for mercury $\kappa = 27$GPa, and for alcohol $\kappa = 0.91$GPa.

It follows from (7.2.13) that a Poisson ratio $\nu > 0.5$ would have given $\kappa < 0$, which according to (7.2.14) would lead to the physically unacceptable result that the material increases its volume when subjected to isotropic pressure. Furthermore we may expect to find that $\nu \geq 0$ because a Poisson ratio $\nu < 0$ would, according to $(7.2.4)_2$, give an expansion in the transverse direction when the material is subjected to uniaxial stress. Thus we may expect that:

$$0 \leq \nu \leq 0.5 \tag{7.2.15}$$

The upper limit for the Poisson ratio, $\nu = 0.5$, which according to (7.2.13) gives $\kappa = \infty$, characterizes an *incompressible material*. Among the real materials rubber, having $\nu = 0.49$, is considered to be (nearly) incompressible, while the other extreme, $\nu = 0$, is represented by cork, which is an advantageous property when corking bottles. As a curiosity and a historical note it may be mentioned that a theory developed by Poisson and based on an atomic model of materials, led to a universal value of ν equal to 0.25. From Table 7.2.1 it is seen that this “universal” value is not universal, although close to the values found in experiments for some important materials.

A very simple form for Hooke's law (7.2.8) is obtained when we decompose the stress tensor $\mathbf{T}$ and the strain tensor $\mathbf{E}$ into isotrops and deviators:

$$\mathbf{T} = \mathbf{T}^o + \mathbf{T}', \quad \mathbf{E} = \mathbf{E}^o + \mathbf{E}' \tag{7.2.16}$$

$$\mathbf{T}^o = \frac{1}{3}\,(\text{tr } \mathbf{T})\,\mathbf{1} = \sigma^o\,\mathbf{1}, \quad \mathbf{E}^o = \frac{1}{3}\,(\text{tr } \mathbf{E})\,\mathbf{1} = \frac{1}{3}\,\varepsilon_v\mathbf{1} \tag{7.2.17}$$

From (7.2.8) or (7.2.6) we find that:

$$\mathbf{T}^o = 3\kappa\mathbf{E}^o, \; \mathbf{T}' = 2\mu\,\mathbf{E}' \tag{7.2.18}$$

The development of these results is given as Problem 7.2. Alternative forms for Hooke's law are:

$$\mathbf{T} = 2\mu\,\mathbf{E} + \left(\kappa - \frac{2}{3}\mu\right)(\text{tr } \mathbf{E})\,\mathbf{1} \tag{7.2.19}$$

$$\mathbf{E} = \frac{1}{2\mu}\mathbf{T} - \frac{3\kappa - 2\mu}{18\mu\,\kappa}\,(\text{tr } \mathbf{T})\,\mathbf{1} \tag{7.2.20}$$

The parameters:

$$\mu \text{ and } \lambda \equiv \kappa - \frac{2}{3}\mu \tag{7.2.21}$$

are called the *Lamé constants*. The parameter λ does not have any independent physical interpretation.

For an *incompressible material*: $\varepsilon_v \equiv 0$, the mean stress: $\sigma^o = (1/3)\text{tr}\mathbf{T}$, cannot be determined from Hooke's law. For these materials it is customary to replace (7.2.8) or (7.2.19) by:

$$\mathbf{T} = -p\,\mathbf{1} + 2\mu\,\mathbf{E} \tag{7.2.22}$$

$p = p(\mathbf{r}, t)$ is an unknown pressure, which is an unknown tension if p is negative. The pressure p can only be determined from the equations of motion and the corresponding boundary conditions.

7.2.1 An Alternative Development of the Generalized Hooke's Law

The constitutive equations for a Hookean solid, or what we have called the generalized Hooke's law for general states of stress and strain, represented by (7.2.6, 7.2.7, 7.2.8), may also be found on the basis of mathematical results in Sect. 4.6.3 on isotropic tensor functions. Since the relationship between the stress tensor $\mathbf{T}$ and the strain tensor $\mathbf{E}$ is linear, we may write:

$$\mathbf{T} = \mathbf{S} : \mathbf{E} \quad \Leftrightarrow \quad T_{ij} = S_{ijkl}\,E_{kl} \tag{7.2.23}$$

$$\mathbf{E} = \mathbf{K} : \mathbf{T} \quad \Leftrightarrow \quad E_{ij} = K_{ijkl}\,T_{kl} \tag{7.2.24}$$

The fourth order tensor $\mathbf{S}$ is called the *elasticity tensor* or the *stiffness tensor*. The fourth order tensor $\mathbf{K}$ is called the *compliance tensor* or the *flexibility tensor*. Since we assume that the material model defined by the constitutive equations (7.2.23) and (7.2.24) is isotropic, each of the tensors $\mathbf{S}$ and $\mathbf{K}$ must be represented by the same matrix in all coordinate systems Ox. This implies that $\mathbf{S}$ and $\mathbf{K}$ are isotropic fourth order tensors. With reference to (4.6.30) we see that $\mathbf{S} = \mathbf{I}_4^s$, and with the symmetric 4.order isotropic tensor $\mathbf{I}_4^s$ from (4.6.31) we may write:

$$\mathbf{S} = 2\mu\,\mathbf{1}_4^s + \left(\kappa - \frac{2}{3}\mu\right)\mathbf{1}\otimes\mathbf{1} \quad \Leftrightarrow \quad S_{ijkl} = \mu\left(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}\right) + \left(\kappa - \frac{2}{3}\mu\right)\delta_{ij}\,\delta_{kl} \tag{7.2.25}$$

$$\mathbf{K} = \frac{1}{2\mu}\,\mathbf{1}_4^s - \frac{3\kappa - 2\mu}{18\mu\,\kappa}\mathbf{1}\otimes\mathbf{1} \quad \Leftrightarrow \quad K_{ijkl} = \frac{1}{4\mu}\left(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}\right) - \frac{3\kappa - 2\mu}{18\mu\,\kappa}\,\delta_{ij}\,\delta_{kl} \tag{7.2.26}$$

The material parameters have been chosen such that the result coincide with what is found above in (7.2.19) and (7.2.20). Compare (7.2.25) with (4.6.34).

7.2.2 Strain Energy

In Chap. 6 the concept of stress power was defined. The stress power per unit volume is given by the expression $\omega = \mathbf{T} : \mathbf{D}$, where $\mathbf{D}$ is the rate of deformation tensor. When we assume small deformations, we may set $\mathbf{D} = \dot{\mathbf{E}}$, and thus:

$$\omega = \mathbf{T} : \dot{\mathbf{E}} \tag{7.2.27}$$

Since the stresses are linear functions of the strains through Hooke's law (7.2.8), the work done on the material per unit volume when the state of stress is increased from zero stress to the state given by $\mathbf{T}$, is equal to:

$$W = \frac{1}{2}\mathbf{T} : \mathbf{E} \tag{7.2.28}$$

This work is recoverable in the sense that the material may perform an equal amount of work on the environment when the stresses are relieved, and W may thus be considered to be stored in the material in the form of *elastic energy per unit volume* or *strain energy per unit volume*:

$$\phi = \frac{1}{2}\mathbf{T} : \mathbf{E} = \mu\,\mathbf{E} : \mathbf{E} + \frac{1}{2}\left(\kappa - \frac{2}{3}\mu\right)(\text{tr }\mathbf{E})^2 \tag{7.2.29}$$

By linear decompositions of the stress tensor $\mathbf{T}$ and the strain tensor $\mathbf{E}$ according to (7.2.16) it may be shown, see Problem 7.7, that the strain energy consists of a *volumetric strain energy* ϕ^o and a *deviatoric strain energy* ϕ':

$$\phi = \phi^o + \phi'$$
$$\phi^o = \frac{1}{2}\mathbf{T}^o : \mathbf{E}^o = \frac{\kappa}{2}\varepsilon_v{}^2 = \frac{1}{2\kappa}(\sigma^o)^2 = \frac{1}{18\kappa}(\text{tr } \mathbf{T})^2$$
$$\phi' = \frac{1}{2}\mathbf{T}' : \mathbf{E}' = \mu \mathbf{E}' : \mathbf{E}' = \frac{1}{4\mu}\mathbf{T}' : \mathbf{T}' \tag{7.2.30}$$

For uniaxial stress:

$$\phi = \frac{1}{2}\sigma\,\varepsilon = \frac{1}{2}\eta\,\varepsilon^2 = \frac{1}{2\eta}\sigma^2 \tag{7.2.31}$$

7.3 Two-Dimensional Theory of Elasticity

The general equations of the theory of elasticity can only be solved by elementary analytical methods in a few special and simple cases. In many problems of practical interest we may however introduce simplifications with respect to the state of stress or the state of displacements, such that a useful solution may be found by relatively simple means.

7.3.1 Plane Stress

Thin plates or slabs that are loaded parallel to the middle plane, see Fig. 7.3.1, by body forces **b** and contact forces **t** on the boundary surface A, such that:

$$b_\alpha = b_\alpha(x_1, x_2, t)\ ,\quad b_3 = 0 \quad \text{in the volume } V \tag{7.3.1}$$
$$t_\alpha = t_\alpha(x_1, x_2, t)\ ,\quad t_3 = 0 \quad \text{on the surface } A \tag{7.3.2}$$

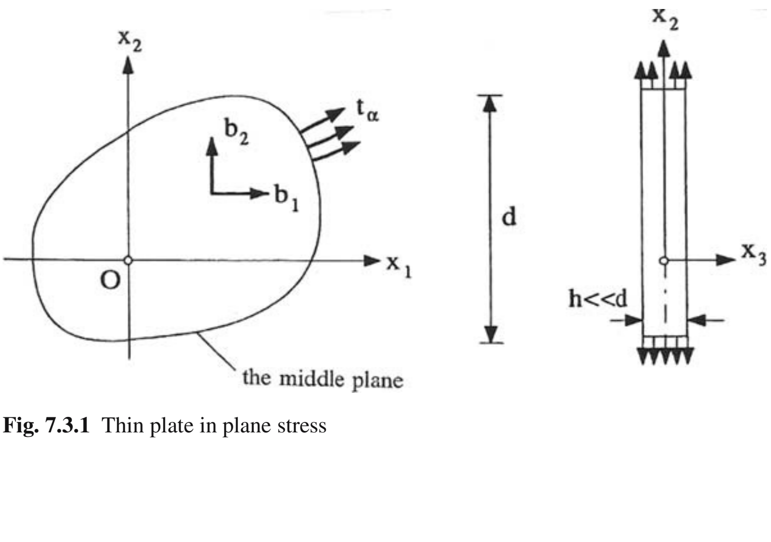


Fig. 7.3.1 Thin plate in plane stress

obtain approximately a state of plane stress:

$$T_{i3} = 0\,,\quad T_{\alpha\beta} = T_{\alpha\beta}\,(x_1, x_2, t) \tag{7.3.3}$$

A stricter analysis of the problem presented in Fig. 7.3.1 and with the loading conditions given by (7.3.1) and (7.3.2), will show that the conditions (7.3.3) for plane stress are not completely satisfied. However, the approximation (7.3.3) is acceptable for a thin plate if the stresses $T_{\alpha\beta}$ and the displacements u_α are considered to represent mean values over the thickness h of the plate. The thickness h is assumed to be much smaller than a characteristic diameter d of the plate, see Fig. 7.3.1.

The fundamental equations for a thin plate in plane stress are:

1) The Cauchy equations of motion:

$$\text{div}\ \mathbf{T} + \rho\,\mathbf{b} = \rho\,\ddot{\mathbf{u}} \quad \Leftrightarrow \quad T_{\alpha\beta},_\beta + \rho\, b_\alpha = \rho\, \ddot{u}_\alpha \tag{7.3.4}$$

The acceleration $\mathbf{a}$ has been substituted by the second material derivative of the displacement vector $\mathbf{u}$, which in turn will be represented by the second partial derivative of $\mathbf{u}$ with respect to time:

$$\ddot{\mathbf{u}} = \frac{\partial^2 \mathbf{u}}{\partial t^2} \tag{7.3.5}$$

2) *Hooke's law for plane stress*, see Problem 7.3:

$$T_{\alpha\beta} = 2\mu \left[E_{\alpha\beta} + \frac{\nu}{1-\nu} E_{\rho\rho}\, \delta_{\alpha\beta} \right],\ 2\mu = \frac{\eta}{1+\nu} \tag{7.3.6}$$

$$\sigma_x = \frac{2G}{1-\nu}\left[\varepsilon_x + \nu\,\varepsilon_y\right]\,,\ \ \sigma_y = \frac{2G}{1-\nu}\left[\varepsilon_y + \nu\,\varepsilon_x\right]\,,\ \ \tau_{xy} = G\,\gamma_{xy}\,,\ \ 2G = \frac{E}{1+\nu} \tag{7.3.7}$$

$$E_{\alpha\beta} = \frac{1}{2\mu}\left[T_{\alpha\beta} - \frac{\nu}{1+\nu} T_{\rho\rho}\,\delta_{\alpha\beta} \right],\ E_{33} = -\frac{\nu}{\eta} T_{\rho\rho} \tag{7.3.8}$$

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\,\sigma_y\right],\ \varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\,\sigma_x\right],\ \gamma_{xy} = \frac{1}{G}\tau_{xy}$$

$$\varepsilon_z = -\frac{\nu}{E}\left[\sigma_x + \sigma_y\right],\ E = \frac{G}{2\,(1+\nu)} \tag{7.3.9}$$

Note that for a convenient notation two different symbols have been used for the modulus of elasticity: $\eta \equiv E$ and for the shear modulus: $\mu = G$.

3) Strain-displacement relations:

$$E_{\alpha\beta} = \frac{1}{2}\left(u_\alpha,_\beta + u_\beta,_\alpha\right). \tag{7.3.10}$$

Equations (7.3.4), (7.3.6), and (7.3.10) represent all together 8 equations for the 8 unknown functions $T_{\alpha\beta}$, $E_{\alpha\beta}$, and u_α.

It is assumed that the contact forces, or stresses, are given on a part A_σ of the surface A of the plate, while the displacements are given over the remaining part: $A_u = A - A_\sigma$, such that:

$$t_\alpha = T_{\alpha\beta} n_\beta = t_\alpha^* \quad \text{on } A_\sigma \tag{7.3.11}$$

$$u_\alpha = u_\alpha^* \quad \text{on } A_u \tag{7.3.12}$$

t_α^* and u_α^* are prescribed functions of position. Equations (7.3.11) and (7.3.12) are thus boundary conditions in the problem governed by the (7.3.4), (7.3.6), and (7.3.10).

In an analytical solution to a problem in the theory of elasticity it is customary to choose either displacements or stresses as the primary unknown functions. In the present section we consider the first alternative. In Sect. 7.3.3 the stresses are chosen as the primary unknown functions.

When the displacements are selected as the primary unknowns, the fundamental equations are transformed as follows. The relations (7.3.10) are substituted into Hooke's law (7.3.6), and the result is:

$$T_{\alpha\beta} = \mu \left[u_{\alpha,\beta} + u_{\beta,\alpha} + \frac{2\nu}{1-\nu} u_{\rho,\rho}\, \delta_{\alpha\beta} \right] \tag{7.3.13}$$

These expressions for the stresses are then substituted into the Cauchy equations of motion (7.3.4) to give:

$$u_{\alpha,\beta\beta} + \frac{1+\nu}{1-\nu} u_{\beta,\beta\alpha} + \frac{1}{\mu} \rho \left(b_\alpha - \ddot{u}_\alpha \right) = 0 \tag{7.3.14}$$

These equations of motion are called the *Navier equations for plane stress*, named after Claude L. M. H. Navier [1785–1836]. The two displacement components u_α are to be found from the two Navier equations. The stresses may then be determined from the expressions (7.3.13), and the strains are determined from the expressions (7.3.10). Finally the boundary conditions (7.3.11) and (7.3.12) complete the solution to the problem.

Below we shall consider two problems for which the state of displacements is axisymmetrical. The result of this assumption is that the state of deformation is irrotational, or in other words the result is a state of pure strain:

$$\tilde{R}_{\alpha\beta} = 0 \Leftrightarrow u_{\alpha,\beta} = u_{\beta,\alpha} \quad \Rightarrow \quad u_{\alpha,\beta\beta} = u_{\beta,\alpha\beta} = u_{\beta,\beta\alpha}$$

The two Navier equations (7.3.14) may now be transformed into:

$$\varepsilon_{A,\alpha} + \frac{1-\nu}{2\mu} \rho \left(b_\alpha - \ddot{u}_\alpha \right) = 0 \tag{7.3.15}$$

ε_A is the invariant:

$$\varepsilon_A \equiv u_{\beta,\beta} = E_{\beta\beta} = E_{11} + E_{22} \tag{7.3.16}$$

The invariant represents the change in area per unit area in the plane of the plate. We shall call this quantity the *area strain*. We now introduce the radial displacement u as the only unknown displacement function, see Fig. 7.3.2 which shows a circular slab. Polar coordinates (R, θ) are applied and due to axisymmetry the radial displacement is a function of R alone: $u = u(R)$. The relevant strains are the longitudinal strains:

$$\varepsilon_R = \frac{du}{dR}\,, \quad \varepsilon_\theta = \frac{l - l_o}{l_o} = \frac{2\pi(R+u) - 2\pi R}{2\pi R} = \frac{u}{R} \tag{7.3.17}$$

The area strain is now:

$$\varepsilon_A = \varepsilon_R + \varepsilon_\theta = \frac{du}{dR} + \frac{u}{R} = \frac{1}{R}\frac{d(Ru)}{dR} \tag{7.3.18}$$

The Navier equation for the R-direction is supplied by (7.3.15) when x_α is replaced by R. The result is:

$$\frac{d}{dR}\left[\frac{1}{R}\frac{d(Ru)}{dR}\right] + \frac{1-\nu}{2G}\rho\left(b_R - \ddot{u}_R\right) = 0 \tag{7.3.19}$$

The coordinate stresses in polar coordinates are $\sigma_R, \tau_{R\theta}$, and σ_θ, see Fig. 7.3.2. Due to axisymmetry the shear stress $\tau_{R\theta}$ is zero. When the displacement u has been determined from (7.3.19), the stresses may be determined from Hooke's law (7.3.7), now written as:

$$\sigma_R(R) = \frac{2G}{1-\nu}\left[\frac{du}{dR} + \nu\frac{u}{R}\right]\,, \quad \sigma_\theta(R) = \frac{2G}{1-\nu}\left[\frac{u}{R} + \nu\frac{du}{dR}\right] \tag{7.3.20}$$

Example 7.1. Circular Plate with a Hole

A circular plate of radius b has a concentric hole of radius a, as shown in Fig. 7.3.3. The edge of the hole is subjected to a pressure p, and the outer edge of the slab is subjected to a pressure q. We shall determine the state of stress and the radial displacement of the plate. The Navier equation (7.3.19) is in this case reduced to the equilibrium equation:

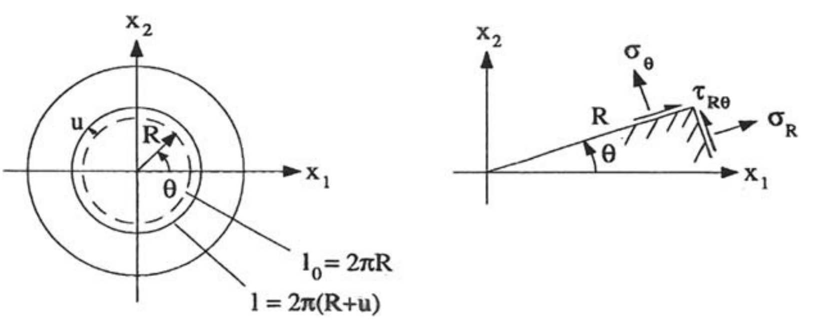


Fig. 7.3.2 Circular plate with axissymmetrical displacement

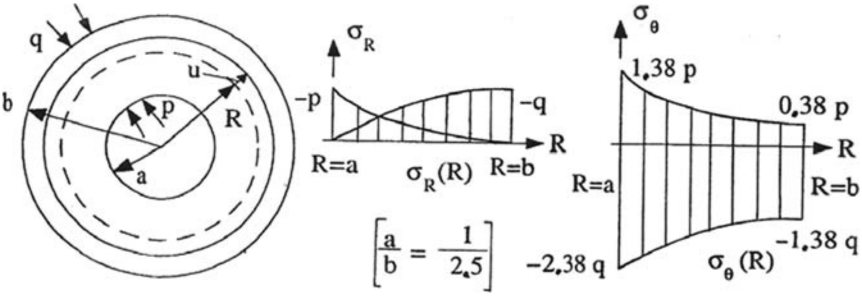


Fig. 7.3.3 Circular plate with a concentric hole. Examples 7.1 and 7.2. Radial stress σ_R and tangential stress σ_θ as functions of the radial distance R

$$\frac{d}{dR}\left[\frac{1}{R}\frac{d\,(Ru)}{dR}\right] = 0$$

Two integrations result in:

$$u = AR + \frac{B}{R}\,, \quad \frac{du}{dR} = A - \frac{B}{R^2}\,, \quad A \text{ and } B \text{ are constants of integration}$$

An expression for the radial stress is now obtained from (7.3.20):

$$\sigma_R(R) = 2G\left[\frac{1+\nu}{1-\nu}A - B\frac{1}{R^2}\right]$$

The boundary conditions give:

$$\sigma_R(a) = -p \;\Rightarrow\; 2G\left[\frac{1+\nu}{1-\nu}A - B\frac{1}{a^2}\right] = -p\,,$$

$$\sigma_R(b) = -q \;\Rightarrow\; 2G\left[\frac{1+\nu}{1-\nu}A - B\frac{1}{b^2}\right] = -q \quad \Rightarrow$$

$$2G\frac{1+\nu}{1-\nu}A = \frac{p\,(a/b)^2 - q}{1-(a/b)^2}\,, \quad 2GB = \frac{(p-q)\,a^2}{1-(a/b)^2}$$

The final solution to the problem is then:

$$\sigma_R(R) = \frac{1}{1-(a/b)^2}\left\{-\left[\left(\frac{a}{R}\right)^2 - \left(\frac{a}{b}\right)^2\right]p - \left[1 - \left(\frac{a}{R}\right)^2\right]q\right\} \tag{7.3.21}$$

$$\sigma_\theta(R) = \frac{1}{1-(a/b)^2}\left\{\left[\left(\frac{a}{R}\right)^2 + \left(\frac{a}{b}\right)^2\right]p - \left[1 + \left(\frac{a}{R}\right)^2\right]q\right\} \tag{7.3.22}$$

$$u(R) = \frac{1}{2G}\frac{a}{1-(a/b)^2}\left\{\left[\frac{1-\nu}{1+\nu}\left(\frac{a}{b}\right)^2\frac{R}{a} + \frac{a}{R}\right]p - \left[\frac{1-\nu}{1+\nu}\frac{R}{a} + \frac{a}{R}\right]q\right\} \tag{7.3.23}$$

The stress distribution in the radial direction and in the circumferential direction are shown in Fig. 7.3.3. For the particular case when $p = q$, we get the reasonable result:

$$\sigma_R = \sigma_\theta = -p \,, \quad u = -\frac{1-\nu}{2G(1+\nu)} pR$$

The strain in the direction normal to the plane of the plate is found from $(7.3.9)_4$:

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_R + \sigma_\theta) = \frac{\nu}{G(1+\nu)} \frac{q - (a/b)^2 p}{1-(a/b)^2} = \text{constant} \tag{7.3.24}$$

Example 7.2. Rotating Circular Plate

The plate in Fig. 7.3.3 rotates with a constant angular velocity ω about its axis. The pressures $p = q = 0$. The state of stress and the radial displacement are to be determined when the boundary conditions alternatively are given by:

Case I. Plate without a hole, $a = 0$. Outer edge, $R = b$, of the plate is stress free.

Case II. Plate with a hole, $a > 0$. Stress free inner and outer edges.

The rotation of the plate results in a normal acceleration a_n towards the axis of rotation. Hence:

$$\ddot{u} = \ddot{u}_R = -a_n = -\omega^2 R$$

The Navier equation (7.3.19) is reduced to:

$$\frac{d}{dR}\left[\frac{1}{R}\frac{d(Ru)}{dR}\right] = -\frac{1-\nu}{2G}\rho\,\omega^2 R$$

Two integrations provide the result:

$$u(R) = AR + \frac{B}{R} - \frac{1-\nu}{16G}\rho\,\omega^2 R^3 \,, \quad A \text{ and } B \text{ are constants of integration} \tag{7.3.25}$$

The stresses are given by the (7.3.20):

$$\begin{aligned}
\sigma_R(R) &= 2G\left[\frac{1+\nu}{1-\nu}A - B\frac{1}{R^2}\right] - \frac{3+\nu}{8}\rho\,\omega^2 R^2 \\
\sigma_\theta(R) &= 2G\left[\frac{1+\nu}{1-\nu}A + B\frac{1}{R^2}\right] - \frac{1+3\nu}{8}\rho\,\omega^2 R^2
\end{aligned} \tag{7.3.26}$$

The constants A and B will be found from the boundary conditions in Case I and Case II respectively.

Case I: Plate without a hole, $a = 0$. Outer edge, $R = b$, is stress free. The boundary conditions are:

$$u(0) = 0 \,, \quad \sigma_R(b) = 0$$

Using the expressions (7.3.25) and (7.3.26) in these conditions, we get two equations from which A and B can be solve. The solution of the equations is:

$$B = 0\,, \quad 2G\frac{1+\nu}{1-\nu}A = \frac{3+\nu}{8}\rho\,\omega^2 b^2\,,$$

The (7.3.24, 7.3.25, and 7.3.20) provide the complete solution to the problem:

$$u(R) = \frac{1-\nu}{16G(1+\nu)}\rho\,\omega^2 b^3\left[(3+\nu)\frac{R}{b} - (1+\nu)\left(\frac{R}{b}\right)^3\right]$$

$$\sigma_R(R) = \frac{3+\nu}{8}\rho\,\omega^2 b^2\left[1 - \left(\frac{R}{b}\right)^2\right], \sigma_\theta(R) = \frac{1}{8}\rho\,\omega^2 b^2\left[(3+\nu) - (1+3\nu)\left(\frac{R}{b}\right)^2\right]$$

$$\sigma_{\max} = \sigma_R(0) = \sigma_\theta(0) = \frac{3+\nu}{8}\rho\,\omega^2 b^2$$

Case II: Plate with a hole. Stress free inner and outer edges, $R = a$ and $R = b$. The boundary conditions are:

$$\sigma_R(a) = 0\,, \quad \sigma_R(b) = 0$$

Using the expression (7.3.26) in these conditions, we get two equations from which the constants of integration A and B can be solve. The solution of the equations is:

$$2G\frac{1+\nu}{1-\nu}A = \frac{3+\nu}{8}\left(a^2+b^2\right)\rho\,\omega^2\,, \quad 2GB = \frac{3+\nu}{8}a^2 b^2 \rho\,\omega^2$$

The (7.3.25, 7.3.26, and 7.3.20) provide the complete solution to the problem:

$$u(R) = \frac{(3+\nu)\rho\,\omega^2 b^3}{16G}\left\{\frac{1-\nu}{1+\nu}\left[1+\left(\frac{a}{b}\right)^2\right]\frac{R}{b} + \left(\frac{a}{b}\right)^2\frac{b}{R} - \frac{1-\nu}{3+\nu}\left(\frac{R}{b}\right)^3\right\}$$

$$\sigma_R(R) = \frac{(3+\nu)\rho\,\omega^2 b^2}{8}\left\{1 + \left(\frac{a}{b}\right)^2 - \left(\frac{a}{R}\right)^2 - \left(\frac{R}{b}\right)^2\right\}$$

$$\sigma_\theta(R) = \frac{(3+\nu)\rho\,\omega^2 b^2}{8}\left\{1 + \left(\frac{a}{b}\right)^2 + \left(\frac{a}{R}\right)^2 - \frac{1+3\nu}{3+\nu}\left(\frac{R}{b}\right)^2\right\}$$

$$\sigma_{\max} = \sigma_\theta(a) = \frac{(3+\nu)\rho\,\omega^2 b^2}{4}\left\{1 + \frac{1-\nu}{3+\nu}\left(\frac{a}{b}\right)^2\right\}$$

If we let the hole radius approach zero, $a \to 0$, we get the result that $\sigma_{\max} = (3+\nu)\rho b^2\omega^2/4$, which is twice the value we found above for $\sigma_{\max}$ in a plate without a hole.

7.3.2 Plane Displacements

A body is in the state of *plane displacements* parallel to the x_1x_2 – plane when:

$$u_\alpha = u_\alpha(x_1, x_2, t)\,, \; u_3 = 0 \qquad (7.3.27)$$

The strains are expressed by:

$$E_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha},_{\beta} + u_{\alpha},_{\beta}\right), \; E_{i3} = 0 \tag{7.3.28}$$

This is also called a state of *plane deformations*, or a state of *plane strains*.

States of plane displacements occur in cylindrical bodies that are held between two rigid and parallel planes, as shown in Fig. 7.3.4, and that are subjected to forces of the type (7.3.1) and (7.3.2). The rigid planes can only transfer normal stresses T_{33}. If the cylindrical body is not held between two rigid planes, but has an appreciable length, we may find the stresses and the displacements in the body by first assuming the body is held between two rigid planes, compute the stress T_{33} from Hooke's law, and then superimpose a solution for the problem where the body is subjected to negative T_{33} stress. This type of problem will be demonstrated in Example 7.3.

The fundamental equations for plane displacements are provided by *Hooke's law for plane displacements*, see Problem 7.4:

$$\begin{aligned}
T_{\alpha\beta} &= 2\mu\left[E_{\alpha\beta} + \frac{\nu}{1-2\nu}E_{\rho\rho}\,\delta_{\alpha\beta}\right], \; T_{33} = \frac{2\nu\mu}{(1-2\nu)}E_{\alpha\alpha} = \nu T_{\alpha\alpha} \\
\sigma_x &= \frac{2G}{1-2\nu}\left[(1-\nu)\,\varepsilon_x + \nu\,\varepsilon_y\right], \; \tau_{xy} = G\gamma_{xy} \\
\sigma_y &= \frac{2G}{1-2\nu}\left[(1-\nu)\,\varepsilon_y + \nu\,\varepsilon_x\right], \; \sigma_z = \frac{2\nu G}{1-2\nu}(\varepsilon_x + \varepsilon_y)
\end{aligned} \tag{7.3.29}$$

$$\begin{aligned}
E_{\alpha\beta} &= \frac{1}{2\mu}\left[T_{\alpha\beta} - \nu T_{\rho\rho}\,\delta_{\alpha\beta}\right] \\
\varepsilon_x &= \frac{1-\nu}{2G}\left[\sigma_x - \frac{\nu}{1-\nu}\sigma_y\right], \; \varepsilon_y = \frac{1-\nu}{2G}\left[\sigma_y - \frac{\nu}{1-\nu}\sigma_x\right], \; \gamma_{xy} = \frac{1}{G}\tau_{xy}
\end{aligned} \tag{7.3.30}$$

The stress tensor, the displacement vector, and the strain tensor must satisfy the Cauchy equations (7.3.4), the strain-displacement relations (7.3.10), and the boundary conditions (7.3.11) and (7.3.12). When the stresses (7.3.29) are substituted into

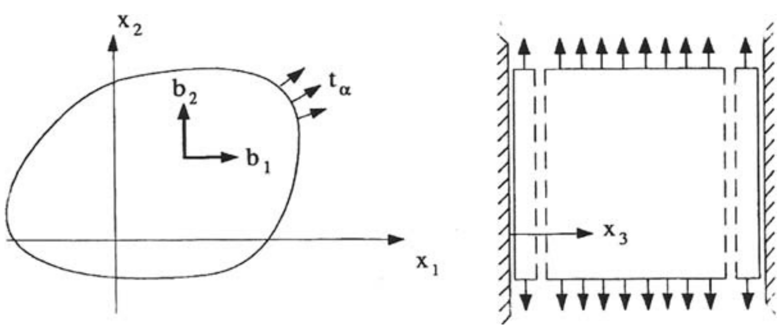


Fig. 7.3.4 Elastic body in plane displacements

the Cauchy equations, the result is the *Navier equations for plane displacements*:

$$u_{\alpha,\beta\beta} + \frac{1}{1-2\nu} u_{\beta,\beta\alpha} + \frac{1}{\mu} \rho \left(b_\alpha - \ddot{u}_\alpha \right) = 0 \tag{7.3.31}$$

We shall find that the fundamental equations (7.3.29, 7.3.30, 7.3.31) for plane displacements mathematically are identical to the similar (7.3.6), (7.3.8), and (7.3.14) for the case of plane stress if we in (7.3.29, 7.3.30, 7.3.31) keep the shear modulus $\mu = \eta/(2(1+\nu))$ unchanged but otherwise replace ν by $\nu/(1+\nu)$. Alternatively we may in the fundamental equations for plane stress keep $\mu = \eta/(2(1+\nu))$ unchanged but otherwise replace ν by $\nu/(1-\nu)$, and the result is the fundamental (7.3.29, 7.3.30, 7.3.31) for plane displacements. Due to the analogy between the two sets of fundamental equations it becomes easy to transfer solutions of problems in plane stress to analogous problems in plane displacements. We may state the rules of transformation as follows.

In the plane displacement equations expressed with the elastic parameters

μ and ν, replace ν by $\dfrac{\nu}{1+\nu}$ $\quad\Rightarrow\quad$ plane stress equations (7.3.32)

In the plane stress equations expressed with the elastic parameters μ and ν

replace ν by $\dfrac{\nu}{1-\nu}$ $\quad\Rightarrow\quad$ plane displacement equations (7.3.33)

The Navier equation for *axisymmetrical displacements u(R)* may be developed along similar lines to the (7.3.19) for plane stress. The result is:

$$\frac{d}{dR}\left[\frac{1}{R}\frac{d(Ru)}{dR}\right] + \frac{(1-2\nu)}{2G(1-\nu)}\rho\left(b_R - \ddot{u}_R\right) = 0 \tag{7.3.34}$$

The relevant stresses are:

$$\sigma_R(R) = \frac{2G(1-\nu)}{(1-2\nu)}\left[\frac{du}{dR} + \frac{\nu}{1-\nu}\frac{u}{R}\right], \; \sigma_\theta(R) = \frac{2G(1-\nu)}{(1-2\nu)}\left[\frac{u}{R} + \frac{\nu}{1-\nu}\frac{du}{dR}\right] \tag{7.3.35}$$

Example 7.3. Thick-Walled Cylinder with Internal and External Pressure

A circular thick-walled cylinder with inner radius a and outer radius b is subjected to an internal pressure p and an external pressure q, see Fig. 7.3.5. The boundary conditions for the radial stress $\sigma_R(R)$ are:

$$\sigma_R(a) = -p\,, \quad \sigma_R(b) = -q \tag{7.3.36}$$

We shall consider three different situations for the plane end surfaces of the cylinder:

1) The end surfaces are fixed as shown in Fig. 7.3.5.
2) The end surfaces are free without stresses.
3) The cylinder is closed with rigid end plates, as indicated to right in Fig. 7.3.5. A force F is necessary to hold the rigid plates in place without axial motion.

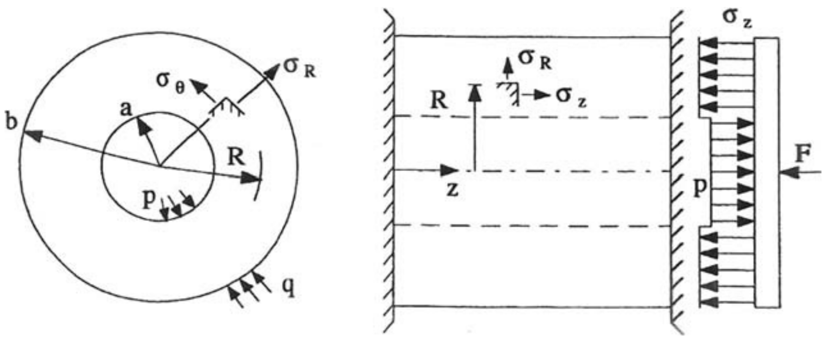


Fig. 7.3.5 Thick walled cylinder with internal pressure p and external pressure q

Let us start by assuming that the end surfaces of the cylinder are prevented from moving in the axial direction. The cylinder will then be in a state of plane displacements. Since we are not considering body forces or accelerations in this problem, the Navier equation (7.3.34) is identical to (7.3.19) for plane stress. The solution to the Navier equation with the boundary conditions (7.3.36) and the general stress formulas (7.3.35), results in the same formulas for the stresses $\sigma_R(R)$ and $\sigma_\theta(R)$ as in Example 7.1. For the radial displacement $u(R)$ however we find:

$$u(R) = \frac{1}{2G}\frac{a}{1-(a/b)^2}\left\{\left[\frac{a}{R}+(1-2\nu)\left(\frac{a}{b}\right)^2\frac{R}{a}\right]p - \left[\frac{a}{R}+(1-2\nu)\frac{R}{a}\right]q\right\} \tag{7.3.37}$$

This expression may be also be obtained directly from formula (7.3.23) by using the rule (7.3.33). The stress on a plane normal to the z-axis is determined from the second formula in the set (7.3.29):

$$\sigma_z \equiv T_{33} = \nu T_{\alpha\alpha} = \nu\left(\sigma_R + \sigma_\theta\right) = -2\nu\frac{q-(a/b)^2 p}{1-(a/b)^2} = \text{ constant} \tag{7.3.38}$$

We shall now assume that the end surfaces of the cylinder are free and without stresses. To the solution above we need only to add a normal stress in the z-direction that is equal to the constant σ_z in formula (7.3.38) for plane displacements, but with opposite sign. This addition does not influence the stresses σ_R and σ_θ. The cylinder will now in fact be in a state of plane stress and the radial displacement $u(R)$ is given by (7.3.23) in Example 7.1. The strain in the z-direction is:

$$\varepsilon_z = -\frac{\sigma_z}{E} = \frac{\nu}{G(1+\nu)}\frac{q-(a/b)^2 p}{1-(a/b)^2} = \text{ constant} \tag{7.3.39}$$

The result is identical to the result (7.3.24) in Example 7.1. The radial and tangential strains get a constant addition equal to:

$$\varepsilon_R = \varepsilon_\theta = -\nu\,\varepsilon_z = -\frac{\nu^2}{G(1+\nu)}\frac{q-(a/b)^2 p}{1-(a/b)^2} = \text{ constant} \tag{7.3.40}$$

Using formula $(7.3.17)_2$ we obtain the additional radial displacement due to this tangential strain:

$$\Delta u(R) = R\,\varepsilon_\theta = -\frac{\nu^2}{G(1+\nu)}\,\frac{q-(a/b)^2\,p}{1-(a/b)^2}R \tag{7.3.41}$$

When this displacement is added to the radial displacement given by (7.3.37), we obtain the displacement (7.3.23) in Example 7.1.

If the cylinder is closed by rigid end plates, as indicated to the right in Fig. 7.3.5, these plates will, under the assumption of plane displacements for the cylinder, be subjected to an axial tension $\sigma_z \cdot \pi(b^2-a^2)$, an axial compression $p \cdot \pi a^2$, and an additional external force F, see Fig. 7.3.5. For the case when $q = 0$, i.e. when the cylinder is subjected to internal pressure p only, the extra force is a compressive force equal to:

$$F = p\cdot\pi a^2 - \sigma_z \cdot \pi\left(b^2-a^2\right) = \pi a^2\,(1-2\nu)\,p$$

This extra force may be eliminated by superposition of a constant tensile stress in the z-direction equal to:

$$\sigma_z = \frac{F}{\pi\,(b^2-a^2)} = \frac{1-2\nu}{1-(a/b)^2}\left(\frac{a}{b}\right)^2 p = \text{constant}$$

The constant tensile stress results in constant strains in the z-direction and in the radial and tangential directions. Using formula $(7.3.17)_2$ we obtain the additional radial displacement due to the tangential strain:

$$\Delta u(R) = R\varepsilon_\theta = R\cdot(-\nu\varepsilon_z) = R\cdot\left(-\nu\frac{\sigma_z}{E}\right) = \frac{\nu\,(1-2\nu)}{2G\,(1+\nu)}\,\frac{1}{1-(a/b)^2}\left(\frac{a}{b}\right)^2 pR \tag{7.3.42}$$

7.3.3 Airy's Stress Function

The choice of stresses as primary unknown functions is only natural in static problems, i.e. when the acceleration $\ddot{\mathbf{u}} = \mathbf{0}$, or in problems where the acceleration is known a priori. In the latter case we introduce an extraordinary body force $(-\ddot{\mathbf{u}})$ and a "corrected body force", $\mathbf{b}-\ddot{\mathbf{u}}$, and the problem is again a static one. The Cauchy equations (7.3.4) are now *equations of equilibrium*:

$$\operatorname{div}\mathbf{T} + \rho\,\mathbf{b} = \mathbf{0} \quad \Leftrightarrow \quad T_{\alpha\beta},_\beta + \rho\, b_\alpha = 0 \tag{7.3.43}$$

Let us assume that we have found three stress components $T_{\alpha\beta}$ that satisfy the two equations of equilibrium (7.3.43). The strain components $E_{\alpha\beta}$ may then be determined from Hooke's law (7.3.8) in the case of plane stress, or from Hooke's law (7.3.30) in the case of plane displacements. Then the displacements should follow from (7.3.10). But we cannot be sure that "any" three components of strain will give two displacement components u_α. The relation (7.3.10) represents three equations for the two unknown functions u_α. From the strain-displacement (7.3.10) we may develop the equation:

$$E_{11,22} + E_{22,11} - 2E_{12,12} = 0 \quad \Leftrightarrow \quad \varepsilon_{x,yy} + \varepsilon_{y,xx} - \gamma_{xy,xy} = 0 \tag{7.3.44}$$

This equation is called the *compatibility equation* and represents a necessary and sufficient condition for the three strain functions $E_{\alpha\beta}$ to give two displacement functions u_α.

Sections 5.3.9 and 5.3.10 present the compatibility equations for a general state of small deformations, and furthermore give the proof for their necessity and sufficiency. The proof assumes that the material region considered is *simply-connected*. This implies that any closed curve in the region may be shrunk to a point. A region containing a piercing hole does not represent a simply connected region. For such regions, in general called *multply-connected regions*, additional conditions have to be imposed. Example 7.9 provides a case where the region is doubly-connected and an extra condition is introduced for the unknown displacement function.

Because we will use stress components as primary unknown functions, we write the compatibility (7.3.44) in terms of the stress components. In the case of plane stress Hooke's law (7.3.8) and the equations of equilibrium (7.3.43) are used to transform the compatibility (7.3.44) into:

$$\nabla^2 T_{\alpha\alpha} + (1+\nu)\,\rho\, b_{\alpha,\alpha} = 0 \quad \text{plane stress} \tag{7.3.45}$$

In the case of plane displacements we apply Hooke's law (7.3.30) and the equations of equilibrium (7.3.43) to express the equation of compatibility (7.3.44) as:

$$\nabla^2 T_{\alpha\alpha} + \frac{1}{1-\nu}\,\rho\, b_{\alpha,\alpha} = 0 \quad \text{plane displacements} \tag{7.3.46}$$

Note that this also may be derived from (7.3.45) by use of the transformation (7.3.33).

In cases where the body forces may be neglected, $b_\alpha = 0$, the equations of equilibrium (7.3.43) and the compatibility equation (7.3.45) or (7.3.46) are reduced to the following set of equations:

$$T_{\alpha\beta,\beta} = 0\,, \quad \nabla^2 T_{\alpha\alpha} = 0 \tag{7.3.47}$$

For any scalar field $\Psi(\mathbf{r})$ coordinate stresses defined by the expressions:

$$T_{11} = \Psi_{,22}\,, \quad T_{22} = \Psi_{,11}\,, \quad T_{12} = -\Psi_{,12} \tag{7.3.48}$$

satisfy the equations of equilibrium, $T_{\alpha\beta,\beta} = 0$, identical. The compatibility equation, $\nabla^2 T_{\alpha\alpha} = 0$, now becomes:

$$\nabla^2\nabla^2\Psi = 0 \quad \Leftrightarrow \quad \nabla^4\Psi = 0 \tag{7.3.49}$$

This equation is a *biharmonic partial differential equation*. The operator ∇^4 is called the *biharmonic operator* and is in Cartesian coordinates:

$$\nabla^4 \equiv \nabla^2 \nabla^2 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 y^2} + \frac{\partial^4}{\partial y^4} \tag{7.3.50}$$

The scalar field $\Psi(\mathbf{r})$ is called *Airy's stress function*, named after George Biddell Airy [1801–1892]. Any scalar field $\Psi(\mathbf{r})$ that is the solution to the biharmonic (7.3.49), i.e. the compatibility equation, gives stresses from (7.3.48) that satisfy the equations of equilibrium in $(7.3.47)_1$ and that provide compatible strains through Hooke's law.

In cases where the body forces b_α may not be neglected, we proceed as follows. First we try to find any particular solution to the equations of equilibrium (7.3.43) and the compatibility equation (7.3.45) or (7.3.46), but without necessarily satisfying any boundary conditions. The complete solution to the problem in question is then given by the sum of the particular solution and a homogeneous solution determined from an Airy's stress function. This total solution must satisfy the boundary conditions of the problem.

A series of simple states of stress may be derived from the stress function:

$$\Psi = Ax^2 + Bxy + Cy^2 + Dx^3 + Ex^2y + Fxy^2 + Gy^3 + Hx^3y + Kxy^3 \tag{7.3.51}$$

$A, B, \ldots, K$ are constants. Each term in this stress function satisfies the compatibility equation (7.3.49).

Example 7.4. Cantilever Beam with Rectangular Cross-Section

The stress function:

$$\Psi = Bxy + Kxy^3$$

provides a satisfactory solution to the beam problem illustrated in Fig. 7.3.6. The height h of the beam is assumed to be greater than the width b of the beam, which leads us to assume a state of plane stress. The beam is loaded by shear stresses on the free end surface at $x = 0$. The resultant of the shear stresses is a known force F, while the distribution of F is not given, and we shall accept the distribution of the shear stresses at the end surface that the solution requires. For convenience we shall use a mixture of x and y and numbers for indices in this example.

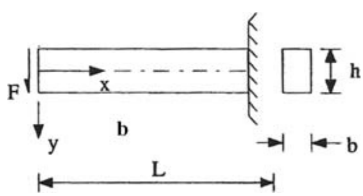


Fig. 7.3.6 Cantilever beam

The stress components become:

$$\sigma_x(x,y) = T_{11} = \frac{\partial^2\Psi}{\partial y^2} = 6Kxy\,, \quad \sigma_y = T_{22} = \frac{\partial^2\Psi}{\partial x^2} = 0\,,$$
$$\tau_{xy}(y) = T_{12} = -\frac{\partial^2\Psi}{\partial x\partial y} = -B - 3Ky^2$$

The constants B and K will now be determined from the boundary conditions:

$$\sigma_x(0,y) = 0 \text{ is satified}\,,\ \tau_{xy}(\pm h/2) = -B - 3K\,(\pm h/2)^2 = 0 \Rightarrow\ B = -3Kh^2/4$$
$$\int_{-h/2}^{h/2} T_{12}\,b\,dy = -Bbh - 3Kb\left[\frac{y^3}{3}\right]_{-h/2}^{h/2} = -F \Rightarrow \quad -Bbh - Kbh^3/4 = -F$$

From these two equations we find the constants B and K.

$$B = \frac{3F}{2bh}\,,\ \ K = -\frac{2F}{bh^3}$$

The stress components are then:

$$\sigma_x(x,y) = -\frac{12F}{bh^3}xy\,,\ \ \sigma_y = 0\,,\ \ \tau_{xy}(y) = -\frac{3F}{2bh}\left[1 - \left(\frac{2y}{h}\right)^2\right]$$

These are the same stresses that are given by the elementary beam theory.

If the real distribution of shear stresses on the free end surface is known and deviates from the result obtained from the stress function, we may assume that the state of stress in the beam is everywhere approximately the one found above from the stress function, except in a small region near the free end. Such an assumption, which we very often have to make in the theory of elasticity, is called the application of the *Saint-Venant's principle*, named after Barré de Saint-Venant [1797–1886].

We now turn to displacements: $u_1(x,y) \equiv u_x(x,y)$ and $u_2(x,y) \equiv u_y(x,y)$. From Hooke's law for plane stress (7.3.9) we get:

$$\varepsilon_x = E_{11} = u_{1,1} \equiv \frac{\partial u_x}{\partial x} = \frac{\sigma_x}{E} = -\frac{12F}{Ebh^3}xy\,,\ \ \varepsilon_y = E_{22} = u_{2,2} \equiv \frac{\partial u_y}{\partial y} = -\nu\frac{\sigma_x}{E}$$
$$= -\frac{12F\nu}{Ebh^3}xy$$
$$\gamma_{xy} = 2E_{12} = u_{1,2} + u_{2,1} \equiv \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = \frac{1}{G}\tau_{xy} = \frac{2(1+\nu)}{E}\tau_{xy}$$
$$= -\frac{3(1+\nu)F}{Ebh^2}\left[1 - \left(\frac{2y}{h}\right)^2\right]$$

Integrations of $u_{1,1}$ and $u_{2,2}$ result in:

$$u_1(x,y) = -\frac{6F}{Ebh^3}x^2y + f_1(y)\,, \quad u_2(x,y) = \frac{6\nu F}{Ebh^3}xy^2 + f_2(x)$$

$f_1(y)$ and $f_2(x)$ are unknown functions. The expressions for u_1 and u_2 are substituted into the expression for $u_{1,2} + u_{2,1}$, and we get:

$$-\frac{6F}{Ebh^3}x^2 + \frac{df_1}{dy} + \frac{6\nu F}{Ebh^3}y^2 + \frac{df_2}{dx} = -\frac{3(1+\nu)F}{Ebh}\left[1 - \left(\frac{2y}{h}\right)^2\right]$$

From this equation we deduce the results:

$$f_1(y) = \frac{(4+2\nu)F}{Ebh^3}y^3 - \left[A + \frac{3(1+\nu)F}{Ebh}\right]y + C\,, \quad f_2(x) = \frac{2F}{Ebh^3}x^3 + Ax + B$$

A, B, and C are constants of integration to be determined from the boundary conditions at the fixed support, where $x = L$. It appears from the expressions for the displacements u_α that it is not possible to demand that $u_\alpha = 0$ over the entire cross section at $x = L$. Such a requirement would indeed not be realistic anyway since we must expect that the material in the support itself will be somewhat deformed due to the stresses transmitted from the beam. We therefore choose first to require: $u_\alpha = 0$ at $x = L, y = 0$. These two conditions give:

$$u_1|_{x=L,y=o} = 0 \quad \Rightarrow \quad C = 0$$

$$u_2|_{x=L,y=o} = 0 \quad \Rightarrow \quad \frac{2FL^3}{Ebh^3} + AL + B = 0 \quad \Rightarrow \quad B = -\frac{2FL^3}{Ebh^3} - AL \qquad (7.3.52)$$

A third boundary condition is obtained by either requiring: 1) $u_{2,1} = 0$ at $x = L, y = 0$, which means that the axis of the beam is parallel to the x-axis, as in the elementary beam theory, or: 2) $u_{1,2} = 0$ at $x = L, y = 0$, which implies that the cross section at the support is parallel to the y-axis where the cross section intersects the beam axis. Figure 7.3.7 shows both alternative boundary conditions at the fixed support. For the two alternative boundary conditions we get, when also the result (7.3.52) is applied:

$$1)\;\; u_{2,1}|_{x=L,y=o} = 0 \quad \Rightarrow \quad \frac{6FL^2}{Ebh^3} + A = 0 \quad \Rightarrow \quad A = -\frac{6FL^2}{Ebh^3}\,, \quad B = \frac{4FL^3}{Ebh^3}$$

$$2)\;\; u_{1,2}|_{x=L,y=o} = 0 \quad \Rightarrow \quad -\frac{6FL^2}{Ebh^3} - \left[A + \frac{3(1+\nu)F}{Ebh}\right] = 0 \quad \Rightarrow$$

$$A = -\frac{6FL^2}{Ebh^3} - \frac{3(1+\nu)F}{Ebh}\,, \quad B = \frac{4FL^3}{Ebh^3} + \frac{3(1+\nu)FL}{Ebh}$$

The displacements according to alternative 1) become:

$$u_1(x,y) = \frac{FL^3}{Ebh^3}\left\{6\left[1 - \left(\frac{x}{L}\right)^2\right]\frac{y}{L} + 2(2+\nu)\left(\frac{y}{L}\right)^3 - 3(1+\nu)\left(\frac{h}{L}\right)^2\frac{y}{L}\right\}$$

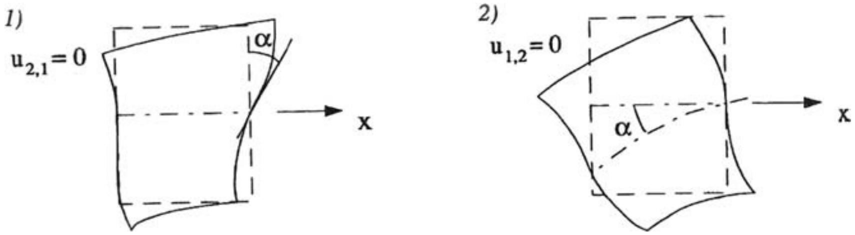


Fig. 7.3.7 Two alternative boundary conditions at the fixed support

$$u_2(x,y) = \frac{FL^3}{Ebh^3}\left\{4+2\left(\frac{x}{L}\right)^3 - 6\left[1-\nu\left(\frac{y}{L}\right)^2\right]\frac{x}{L}\right\}\ ,\quad u_{2,\max} = u_2\,(0,y) = \frac{4FL^3}{Ebh^3}$$

The displacements according to alternative 2) become:

$$u_1(x,y) = \frac{FL^3}{Ebh^3}\left\{6\left[1-\left(\frac{x}{L}\right)^2\right]\frac{y}{L}+2(2+\nu)\left(\frac{y}{L}\right)^3\right\}$$

$$u_2(x,y) = \frac{FL^3}{Ebh^3}\left\{4+3(1+\nu)\left(\frac{h}{L}\right)^2+2\left(\frac{x}{L}\right)^3-6\left[1-\nu\left(\frac{y}{L}\right)^2+\frac{1}{2}(1+\nu)\left(\frac{h}{L}\right)^2\right]\frac{x}{L}\right\}$$

$$u_{2,\max} = u_2\,(0,y) = \frac{FL^3}{Ebh^3}\left[4+3(1+\nu)\left(\frac{h}{L}\right)^2\right]$$

Elementary beam theory gives:

$$u_2(x,0) = \frac{FL^3}{Ebh^3}\left[4+2\left(\frac{x}{L}\right)^3-6\frac{x}{L}\right],\quad u_{2,\max} = u_2\,(0,0) = \frac{4FL^3}{Ebh^3}$$

The displacement $u_2(x,y)$ according to the alternative boundary condition 2) may also be determined by adding to the displacement $u_2(x,y)$ for the alternative boundary condition 1) found above a rigid-body counter-clockwise rotation given by the angle, see Problem 7.8:

$$\alpha = -\,u_{1,2},\big|_{x=L,y=o} = \frac{3\,(1+\nu)\,F}{E\,b\,h}$$

In this example we have found a solution that gives stresses and displacement in the cantilever beam in Fig. 7.3.6 when we accept the approximation regarding the distribution of shear stresses on the free end surface and the uncertainty about the proper displacement conditions at the fixed end surface.

7.3.4 Airy's Stress Function in Polar Coordinates

The general Cauchy equations of motion in polar coordinates are given by (3.2.39) and (3.2.40). By setting the accelerations and body forces equal to zero, we obtain the proper equations of equilibrium. The following formulas for the coordinate stresses in polar coordinates, Fig. 7.3.8, will satisfy these equations of equilibrium:

$$\sigma_R = \frac{1}{R^2}\frac{\partial^2\Psi}{\partial\theta^2} + \frac{1}{R}\frac{\partial\Psi}{\partial R}, \quad \sigma_\theta = \frac{\partial^2\Psi}{\partial R^2}$$
$$\tau_{R\theta} = -\frac{1}{R}\frac{\partial^2\Psi}{\partial R\partial\theta} + \frac{1}{R^2}\frac{\partial\Psi}{\partial\theta} = -\frac{\partial}{\partial R}\left[\frac{1}{R}\frac{\partial\Psi}{\partial\theta}\right] \tag{7.3.53}$$

The Laplace operator and the biharmonic operator in polar coordinates are:

$$\nabla^2 = \frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial\theta^2} \tag{7.3.54}$$

$$\nabla^4 = \nabla^2\nabla^2 = \left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial\theta^2}\right) \tag{7.3.55}$$

Example 7.5. Edge Load on a Semi-Infinite Elastic Plate

The plate is presented in Fig. 7.3.9 and defined by the region $x > 0$. The material is subjected to an edge load q. The load intensity q is given as a force per unit length in the z-direction. We shall find the stresses on planes parallel to the z-axis, which is normal to plane of the figure.

This stress function will provide us with the solution:

$$\Psi = -\frac{q}{\pi}R\,\theta\sin\theta \tag{7.3.56}$$

The polar angle θ is measured from the direction line of the load, as shown in Fig. 7.3.9. The load direction line forms the angle α with respect to the x–axis. By applying the biharmonic operator (7.3.53) to this stress function we shall find that the equation of compatibility (7.3.49 is satisfied. The coordinate stresses become:

$$\sigma_R = \frac{1}{R^2}\frac{\partial^2\Psi}{\partial\theta^2} + \frac{1}{R}\frac{\partial\Psi}{\partial R} = -\frac{2q}{\pi}\frac{1}{R}\cos\theta$$

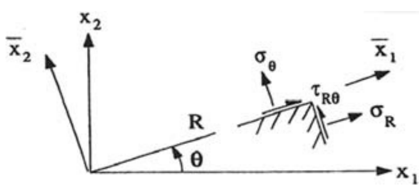
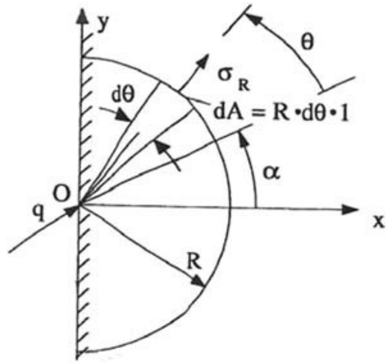


Fig. 7.3.8 Coordinate stresses in polar coordinates

Fig. 7.3.9 Semi-infinite elastic plate subjected to an edge load q



$$\sigma_\theta = \frac{\partial^2 \Psi}{\partial R^2} = 0\,, \quad \tau_{R\theta} = -\frac{\partial}{\partial R}\left[\frac{1}{R}\frac{\partial \Psi}{\partial \theta}\right] = 0$$

The boundary condition: $\mathbf{t} = \mathbf{0}$ on the free surface at $x = 0$, except at the origin O, which is a singular point, is clearly satisfied. In order to show that the state of stress is in equilibrium with the externally applied edge load q, we check the equilibrium of a body formed as a semi cylinder with radius R, see Fig. 7.3.9. Equating to zero the forces in the direction of the edge load q and in the direction normal to q and referring to Fig. 7.3.9, we obtain:

$$q = -\int\limits_A (\sigma_R \cdot \cos\theta)\, dA = \frac{2q}{\pi} \int\limits_{-\pi/2-\alpha}^{\pi/2-\alpha} \cos^2\theta\, d\theta = \frac{2q}{\pi}\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_{-\pi/2-\alpha}^{\pi/2-\alpha} = q$$

$$0 = \int\limits_A (\sigma_R \cdot \sin\theta)\, dA = -\frac{2q}{\pi} \int\limits_{-\pi/2-\alpha}^{\pi/2-\alpha} \cos\theta \sin\theta\, d\theta = -\frac{2q}{\pi}\left[\frac{1}{2}\sin^2\theta\right]_{-\pi/2-\alpha}^{\pi/2-\alpha} = 0$$

We have thus shown that all the necessary requirements to the stress function (7.3.56) are satisfied.

This problem was first investigated and solved by Boussinesq in 1885 and Flamant in 1892, and is therefore often referred to as *the Boussinesq-Flamant problem*.

Example 7.6. Edge Load on a Wedge

The stress function in the previous example also gives the solution to the two problems shown in Fig. 7.3.10 if we only adjust the constant $(-q/\pi)$. Figure 7.3.10 shows two wedges, each having a wedge angle of 2α. With C as an initially unknown constant we first set:

$$\Psi = -CR\theta\sin\theta \quad \Rightarrow \quad \sigma_R = -\frac{2C}{R}\cos\theta, \sigma_\theta = \tau_{R\theta} = 0$$

Equilibrium of the cylinder segments bounded by the radius R requires that:

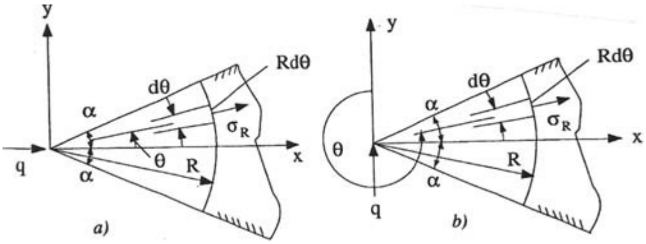


Fig. 7.3.10 Wedges subjected to edge load q

Case a in Fig. 7.3.10a:

$$q = -\int_A (\sigma_R \cdot \cos\theta)\, dA = 2C \int_{-\alpha}^{+\alpha} \cos^2\theta\, d\theta = 2C\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_{-\alpha}^{+\alpha}$$

$$= 2C\left[\alpha + \frac{1}{2}\sin 2\alpha\right] \quad \Rightarrow \quad C = \frac{q}{2\alpha + \sin 2\alpha}$$

Case b in Fig. 7.3.10b:

$$q = -\int_A (\sigma_R \cdot \cos\theta)\, dA = 2C \int_{3\pi/2-\alpha}^{3\pi/2+\alpha} \cos^2\theta\, d\theta = 2C\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_{3\pi/2-\alpha}^{3\pi/2+\alpha}$$

$$= 2C\left[\alpha - \frac{1}{2}\sin 2\alpha\right] \quad \Rightarrow \quad C = \frac{q}{2\alpha - \sin 2\alpha}$$

We now have the following non-zero coordinate stresses in the two cases:

$$\text{Case a: } \sigma_R = -\frac{2q}{2\alpha + \sin 2\alpha}\frac{\cos\theta}{R}, \quad \text{Case b: } \sigma_R = -\frac{2q}{2\alpha - \sin 2\alpha}\frac{\cos\theta}{R}$$

Example 7.7. Circular Cylinder with Edge Loads

The state of stress in a circular cylinder or a thin cylindrical plate of diameter d and which is subjected to two diametrically opposite edge loads q, see Fig. 7.3.11d, is obtained by a superposition of the following three states of stress:

a) Edge load q on a semi-infinite space, Fig. 7.3.11a.
b) Edge load q on a semi-infinite space, Fig. 7.3.11b.
c) Plane isotropic tensile stress $\sigma_o = 2q/\pi d$, Fig. 7.3.11c.

We start by considering the material between two parallel planes a distance d apart, see Fig. 7.3.11 a and b. An edge force q on the upper plane, Fig. 7.3.11a, is balanced by normal and shear stresses on the lower plane in such a manner that the state of stress is given by Example 7.5:

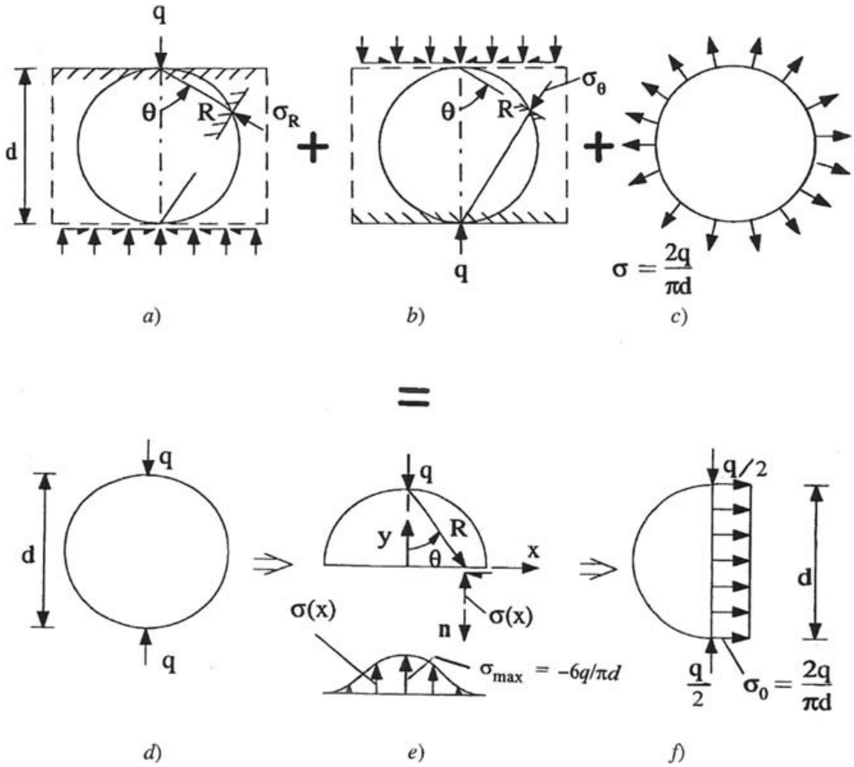


Fig. 7.3.11 Superposition of edge loads

$$\sigma_R = -\frac{2q}{\pi}\frac{1}{R}\cos\theta\ , \quad \sigma_\theta = \tau_{R\theta} = 0$$

In the circular cylindrical surface of diameter $d = R/\cos\theta$ shown in Fig. 7.3.11, the radial stress is constant and equal to $\sigma_R = -2q/\pi d$. An edge force q on the lower plane, Fig. 7.3.11b, is similarly balanced by normal and shear stresses on the upper plane. In the circular cylindrical surface the only non-zero coordinate stress is $\sigma_\theta = -2q/\pi d$. This implies that when the effects of the two edge loads, Fig. 7.3.11 a and b, are superimposed, the state of stress in the cylindrical surface is plane-isotropic. The stress on the cylindrical surface is therefore a constant pressure equal to $2q/\pi d$. If we now add an isotropic state of tensile stress $\sigma_o = 2q/\pi d$, as shown in Fig. 7.3.11c, the cylindrical surface becomes stress free. The superposition of the three load cases in Fig. 7.3.11a, b, and c results in the situation shown in Fig. 7.3.11d: A circular cylinder of diameter d is loaded by two opposite edge forces q.

The computational work to find the stress matrix in any particle in the cylinder shown in Fig. 7.3.11d, becomes very extensive. Therefore we shall confine the computation to the stresses on two characteristic diametrical sections as shown in Fig. 7.3.11e and f.

For the section in Fig. 7.3.11e the unit normal is $\mathbf{n} = \cos\theta \mathbf{e}_R - \sin\theta \mathbf{e}_\theta$. The contribution to the normal stress σ from the upper knife load is then:

$$\sigma = \mathbf{n} \cdot \mathbf{T} \cdot \mathbf{n} = \cos\theta \cdot \sigma_R \cdot \cos\theta = -\frac{2q}{\pi}\frac{\cos^3\theta}{R}$$

We set:

$$\cos\theta = \frac{d/2}{R}\,,\;\; R^2 = \left(\frac{d}{2}\right)^2 + x^2 = \left(\frac{d}{2}\right)^2\left[1+\left(\frac{2x}{d}\right)^2\right]$$

Then:

$$-\frac{2q}{\pi}\frac{\cos^3\theta}{R} = -\frac{2q}{\pi}\frac{(d/2R)^3}{R} = \frac{2q}{\pi}\frac{d^3}{8R^4} = -\frac{q}{\pi d}\left[\frac{2}{1+(2x/d)^2}\right]^2$$

Totally from the three load cases in Figure a, b, and c we obtain:

$$\sigma(x) = \left[-\frac{2q}{\pi}\frac{\cos^3\theta}{R}\right]\cdot 2 + \frac{2q}{\pi d} = \frac{2q}{\pi d}\left\{1-\left[\frac{2}{1+4(x/d)^2}\right]^2\right\}$$

Figure 7.3.11e shows the normal stress distribution. The maximum (compressive) normal stress $\sigma_{\max} = -6q/\pi d$. Due to symmetry the shear stress on the section is zero.

On the section shown in Fig. 7.3.11f only the load case in Fig. 7.3.11c contributes. Thus the normal stress is a constant tension $\sigma_o = 2q/\pi d$, while the shear is zero. This particular result is used to find the ultimate stress of brittle materials having high compressive strength and low tensile strength, as for instance rocks and soils. The result of the load case in Fig. 7.3.11d is also used in the determination of the stress optical constant in the method of *photoelasticity*.

Example 7.8. Rectangular Plate with a Hole. Kirsch's Problem (1898)

A rectangular plate of width b and height h and with a small hole of radius $a << h$ and b, Fig. 7.3.12, is subjected to a normal stress $\sigma_x = \sigma$ on the sides $x = \pm b/2$. The sides $y = \pm h/2$ are stress free. The stresses in the neighborhood of the hole are to be determined.

Sufficiently far from the hole we may assume that the stresses are as for a plate without the hole. Thus:

$$\sigma_x = \sigma\,,\;\; \sigma_y = \tau_{xy} = 0 \text{ for } |x| \gg a\,,\; |y| \gg a$$

Because we choose to use polar coordinates (R,θ), this state of stress may be expressed through the formulas (3.3.37) and (3.3.38) as:

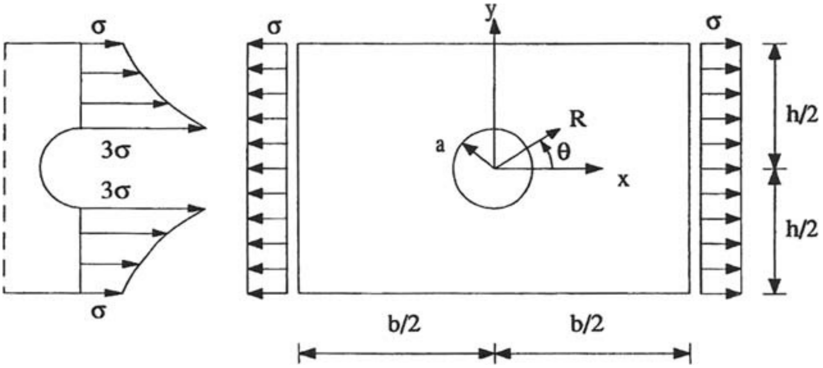


Fig. 7.3.12 Rectangular plate with a hole

$$\begin{aligned}\sigma_R = \tfrac{\sigma}{2}(1+\cos 2\theta)\ ,\quad \sigma_\theta = \tfrac{\sigma}{2}(1-\cos 2\theta)\\ \tau_{R\theta} = -\tfrac{\sigma}{2}\sin 2\theta\end{aligned} \quad \text{for } |x| \gg a\ ,\ |y| \gg a \tag{7.3.57}$$

The following stress function provides approximately the correct stresses in the plate.

$$\Psi = -\frac{\sigma a^2}{2}\ln R + \frac{\sigma}{4}R^2 + \frac{\sigma a^2}{4}\left[2 - \left(\frac{a}{R}\right)^2 + \left(\frac{R}{a}\right)^2\right]\cos 2\theta \tag{7.3.58}$$

The compatibility (7.3.49) is satisfied, and the state of stress is:

$$\sigma_R(R,\theta) = \frac{1}{R}\frac{\partial\Psi}{\partial R} + \frac{1}{R^2}\frac{\partial^2\Psi}{\partial\theta^2} = -\frac{\sigma}{2}\left[1 - \left(\frac{a}{R}\right)^2\right] + \frac{\sigma}{2}\left[1 - 4\left(\frac{a}{R}\right)^2 + 3\left(\frac{a}{R}\right)^4\right]\cos 2\theta$$
$$\sigma_\theta(R,\theta) = \frac{\partial^2\Psi}{\partial R^2} = \frac{\sigma}{2}\left[1 + \left(\frac{a}{R}\right)^2\right] - \frac{\sigma}{2}\left[1 + 3\left(\frac{a}{R}\right)^4\right]\cos 2\theta$$
$$\tau_{R\theta}(R,\theta) = -\frac{\partial}{\partial R}\left[\frac{1}{R}\frac{\partial\Psi}{\partial\theta}\right] = -\frac{\sigma}{2}\left[1 + 2\left(\frac{a}{R}\right)^2 - 3\left(\frac{a}{R}\right)^4\right]\sin 2\theta$$

At the edge of the hole, $R = a$, the stresses are:

$$\sigma_R(a,\theta) = 0\ ,\quad \tau_{R\theta}(a,\theta) = 0\ ,\quad \sigma_\theta(a,\theta) = (1 - 2\cos 2\theta)\,\sigma$$
$$\sigma_{\theta,\max} = \sigma_\theta(a,\pm\pi/2) = 3\sigma\ ,\quad \sigma_{\theta,\min} = \sigma_\theta(a,0) = \sigma_\theta(a,\pi) = -\sigma$$

The state of stress is in accordance with the condition of a stress free surface in the hole. On the sides of the plate, $x = \pm b/2$ and $y = \pm h/2$, we may assume that:

$$\frac{a}{R} \le \frac{2a}{b} \ll 1 \text{ for } x = \pm\frac{b}{2} \quad \text{and} \quad \frac{a}{R} \le \frac{2a}{h} \ll 1 \text{ for y } = \pm\frac{h}{2}$$

The stress formulas reduce to the expressions (7.3.57) for the boundary conditions on the edges. On the section $x = 0 \Leftrightarrow \theta = \pm\pi/2$, the shear stress is zero and the normal stress becomes, see Fig. 7.3.12:

$$\sigma_\theta(R,\pi/2) = \left[1 + \frac{1}{2}\left(\frac{a}{R}\right)^2 + \frac{3}{2}\left(\frac{a}{R}\right)^4\right]\sigma$$

At the hole, $R = a$, the plate has a *stress concentration* $\sigma_\theta = 3\sigma$, independent of the radius a, as long as the condition $a << h$ and b is satisfied. For $h = 4a$, R. C. J. Howland, see [48], has found the solution that: $\sigma_\theta(a,\pi/2) = 4.3\sigma$ and $\sigma_\theta(h/2,\pi/2) = 0.75\sigma$. The stress concentration increases as the hole approaches the edges $y = \pm h/2$.

7.3.5 Axial Symmetry

For problems with a plane state of stress that is symmetrical with respect to a z-axis, we may assume that the Airy stress function is a function only of the radial coordinate R:

$$\Psi = \Psi(R) \tag{7.3.59}$$

The general solution of the equation of compatibility (7.3.49) is:

$$\Psi(R) = A\ln R + BR^2\ln R + CR^2 + D \tag{7.3.60}$$

A, B, C, and D are constants of integration. The coordinate stresses become:

$$\begin{aligned}\sigma_R(R) &= \frac{1}{R}\frac{d\Psi}{dR} = \frac{A}{R^2} + B(1 + 2\ln R) + 2C\\ \sigma_\theta(R) &= \frac{d^2\Psi}{dR^2} = -\frac{A}{R^2} + B(3 + 2\ln R) + 2C\end{aligned} \tag{7.3.61}$$

The shear stress $\tau_{R\theta}$ is zero, which is in accordance with the symmetry condition.

Example 7.9. Circular Plate with a Hole

The problem in Example 7.1, see Fig. 7.3.3, is now revisited. The general solution (7.3.61) contains three unknown constants of integration: A, B, and C. To determine these three constants we have only two boundary conditions:

$$\sigma_R(a) = -p\,,\quad \sigma_R(b) = -q$$

We know that the general solution satisfies the equations of equilibrium and the compatibility (7.3.49). As mentioned in connection with the development of the compatibility (7.3.49), for multi-connected region we have to introduce additional conditions that secure a unique and continuous displacement field. The hole introduces a double-connected region: A closed material curve surrounding the hole, may not be shrunk to zero. For this reason we have to investigate closer the displacements that result in the general expressions (7.3.61) for the stresses. Using the expressions (7.3.17) for the strains, Hooke's law (7.3.9), and the stress expressions (7.3.61), we obtain:

$$\varepsilon_R = \frac{du}{dR} = \frac{1}{E}\left\{\frac{A}{R^2} + B(1+2\ln R) + 2C - \nu\left[-\frac{A}{R^2} + B(3+2\ln R) + 2C\right]\right\} \tag{7.3.62}$$

$$\begin{aligned}\varepsilon_\theta &= \frac{u}{R} = \frac{1}{E}\left\{-\frac{A}{R^2} + B(3+2\ln R) + 2C - \nu\left[\frac{A}{R^2} + B(1+2\ln R) + 2C\right]\right\} \quad \Rightarrow \\ u &= \frac{1}{E}\left\{-\frac{A}{R} + B(3+2\ln R)R + 2CR - \nu\left[\frac{A}{R} + B(1+2\ln R)R + 2CR\right]\right\} \quad \Rightarrow \\ \frac{du}{dR} &= \frac{1}{E}\left\{\frac{A}{R^2} + B(3+2\ln R+2) + 2C - \nu\left[-\frac{A}{R^2} + B(1+2\ln R+2) + 2C\right]\right\}\end{aligned} \tag{7.3.63}$$

If we compare the two expressions (7.3.62) and (7.3.63) for *du/dR*, we see that the constant B must be zero. Thus we have the following general expressions for the stresses and for the radial displacement:

$$\begin{aligned}\sigma_R(R) &= \frac{A}{R^2} + 2C\,, \quad \sigma_\theta(R) = -\frac{A}{R^2} + 2C \\ u(R) &= \frac{1}{E}\left[-(1+\nu)\frac{A}{R} + 2(1-\nu)CR\right]\end{aligned} \tag{7.3.64}$$

These expressions have the same structures as those found in Example 7.1. The boundary conditions in the present example and in Example 7.1 are the same, and the final results must therefore be the same as given in Example 7.1.

Example 7.10. Pure Bending of Curved Beam

Figure 7.3.13 shows a curved beam with a circular axis and constant rectangular cross-section of height $h = b - a$ and width t. The beam is subjected to a bending

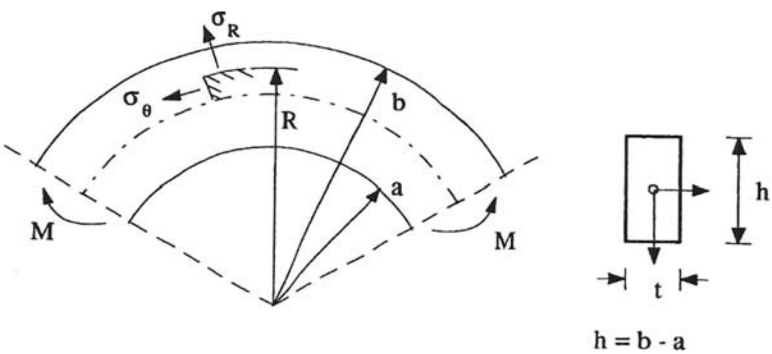


Fig. 7.3.13 Bending of a curved beam

moment M at each end. The stress distribution in the beam is to be determined based on the condition that it is uniform along the axis of the beam.

As a starting point we use the general solution (7.3.61). The boundary conditions are formulated as:

$$\sigma_R(a) = 0\,, \quad \sigma_R(b) = 0\,, \quad \int_a^b \sigma_\theta(R)\,t\,dR = 0\,, \quad \int_a^b \sigma_\theta(R)\,R\,t\,dR = -M \tag{7.3.65}$$

Substituting the general expressions (7.3.61) for the stresses $\sigma_R(R)$ and $\sigma_\theta(R)$ into the first, the second, and the fourth of these boundary conditions, we obtain three equations for the three unknown constants A, B, and C:

$$\frac{A}{a^2} + B(1 + 2\ln a) + 2C = 0\,, \quad \frac{A}{b^2} + B(1 + 2\ln b) + 2C = 0$$
$$\left[A \ln\left(\frac{b}{a}\right) + B\left(b^2 \ln b - a^2 \ln a\right) + C\left(b^2 - a^2\right)\right] t = -M \tag{7.3.66}$$

It may be shown that the third of the boundary conditions (7.3.65) is automatically satisfied when the first two are satisfied. The solution of the three linear equations (7.3.66) for the constants A, B, and C is:

$$A = -\frac{4M}{K} a^2 b^2 \ln\left(\frac{b}{a}\right), \quad B = -\frac{2M}{K}\left(b^2 - a^2\right), \quad C = \frac{M}{K}\left[b^2 - a^2 + 2\left(b^2 \ln b - a^2 \ln a\right)\right]$$
$$K = \left\{\left(b^2 - a^2\right)^2 - 4a^2 b^2 \left[\ln\left(\frac{b}{a}\right)\right]^2\right\} t$$

The stresses are then:

$$\sigma_R = -\frac{4Mb^2}{K}\left[\left(\frac{a}{R}\right)^2 \ln\left(\frac{b}{a}\right) - \ln\left(\frac{b}{R}\right) - \left(\frac{a}{b}\right)^2 \ln\left(\frac{R}{a}\right)\right] \tag{7.3.67}$$

$$\sigma_\theta = -\frac{4Mb^2}{K}\left[\left(\frac{a}{R}\right)^2 \ln\left(\frac{b}{a}\right) - \ln\left(\frac{b}{R}\right) - \left(\frac{a}{b}\right)^2 \ln\left(\frac{R}{a}\right) + 1 - \left(\frac{a}{b}\right)^2\right] \tag{7.3.68}$$

The problem does not specify how the stresses, represented by the bending moment M, are distributed over the end section of the beam. A deviation from the stress distribution provided by the solution for $\sigma_\theta(R)$ given above, will according to the Saint-Venant's principle have little influence on the stress distribution at a short distance from the end sections of the beam.

The elementary beam theory, which really presupposes that the axis of the beam is straight before deformation, gives a linear distribution of $\sigma_\theta(R)$ over the cross-section of the beam. The extremal values for $\sigma_\theta(R)$ according to this theory are:

$$\sigma_{\theta,\max} = \sigma_\theta(a) = +\frac{6M}{th^2}\,, \quad \sigma_{\theta,\min} = \sigma_\theta(b) = -\frac{6M}{th^2}$$

The exact theory represented by formula (7.3.68) gives:

$$\text{For } b=2a: \quad \sigma_{\theta,\max}=\sigma_\theta(a)=+7.73\frac{M}{th^2}\,, \quad \sigma_{\theta,\min}=\sigma_\theta(b)=-4.86\frac{M}{th^2}$$
$$\text{For } b=3a: \quad \sigma_{\theta,\max}=\sigma_\theta(a)=+9.14\frac{M}{th^2}\,, \quad \sigma_{\theta,\min}=\sigma_\theta(b)=-4.38\frac{M}{th^2}$$

7.4 Torsion of Cylindrical Bars

A bar is defined by an axis and a cross-section. A *cylindrical bar* has a straight axis and constant cross-section. When the bar is subjected to loads perpendicular to its axis, the bar is called a *beam*.

The bending of beams having symmetrical and unsymmetrical cross-sections is discussed in books on Strength of Materials. The elementary beam theory provides sufficiently accurate results for the distribution of normal stresses on cross-sections and of the shear stress distribution on simple thin-walled cross-sections. Normally the shear stresses on the cross sections of beams that have a ratio of beam height to beam length less than 1/10, are negligible when compared with the normal stresses on the cross sections. The elementary beam theory also gives satisfactory results for computing beam deflections.

A bar twisted by torques M at the ends, Fig. 7.4.1, is said to be in *pure torsion*. The elementary theory of torsion found in textbooks on Strength of Materials applies only to circular cylindrical bars. This theory is also called the *Coulomb theory of torsion*, after Charles Augustin Coulomb [1789–1857]. In the present section we shall primarily discuss the "non-elementary" theory of torsion of cylindrical bars with arbitrary cross-sections. This theory is called the *Saint-Venant's theory of torsion*. However, it is convenient to start this section by a short presentation of the Coulomb theory.

7.4.1 The Coulomb Theory of Torsion

The circular cylindrical bar in Fig. 7.4.1 is subjected to *torques*, or *torsional moments, M* at the ends. It is most convenient to use the *xyz* notation for the Cartesian

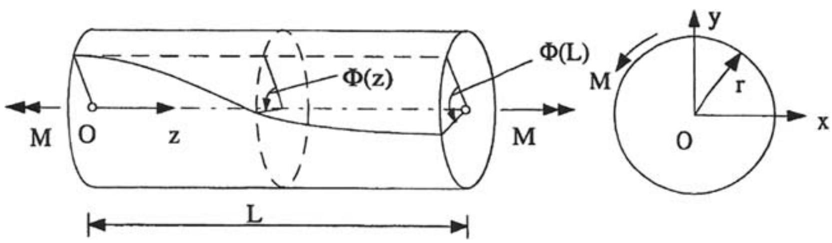


Fig. 7.4.1 Torsion of a circular cylindrical bar

coordinates in this section, but also apply the index-notation. The torques result in a rotation of the end cross-section at $z = L$ relative to the end cross section at $z = 0$. The angle of rotation is called the *angle of twist* and is denoted by Φ. If we on the cylindrical surface of the bar draw contours of cross-sections, we shall see that when the bar is twisted the plane contours remain plane and that the lengths of diameters do not change. This observation leads us to formulate the following *deformation hypothesis*:

During torsion of a circular cylindrical bar plane cross-sections are rotated as rigid planes.

The deformation hypothesis is the basis of the *Coulomb theory of torsion* for circular cylindrical bars. The theory will now be developed.

Since the stress resultant on any cross section must be the same and equal to the torque M, it follows that the angle of twist $\Phi(z)$ is proportional to the distance z from the end of the bar, where $z = 0$. The constant angle of twist per unit length of the rod is denoted by ϕ. Thus:

$$\Phi(z) = \phi z \tag{7.4.1}$$

In order to find the state of strain in the bar we consider an element of the bar between two cross-sections, a distance dz apart, and within a cylindrical surface of radius R, as shown in Fig. 7.4.2. From the figure we derive the only non-zero strain component:

$$\gamma = \phi R \tag{7.4.2}$$

The material is assumed to be linearly elastic, and from Hooke's law it follows that the only non-zero stress component is:

$$\tau \equiv \tau_{\theta z} = G\gamma = G\phi R$$

Since M represents the resultant of the shear stress distribution over the cross-section, it follows that:

$$M = \int_A (\tau \cdot R)\, dA = G\phi \int_A R^2\, dA \tag{7.4.3}$$

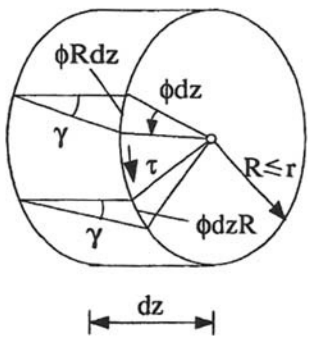


Fig. 7.4.2 Element of bar

The integral on the right hand side is the *polar moment of area* I_p. For a massive circular bar of radius r and a thick-walled pipe of inner radius a and outer radius r the polar moments of area are:

$$\text{Massive bar: } I_p = \frac{\pi r^4}{2}\,, \quad \text{Thick-walled pipe:} I_p = \frac{\pi}{2}\left(r^4 - a^4\right) \tag{7.4.4}$$

From (7.4.2, 7.4.3) we obtain the results:

$$\phi = \frac{M}{GI_p}\,, \quad \tau(R) = \frac{M}{I_p}R\,, \quad \tau_{\max} = \frac{M}{I_p}r \tag{7.4.5}$$

These formulas also apply when the torque varies along the bar: $M = M(z)$. It is not necessary to assume that the angle of twist Φ is small. The theory only requires that ϕ should be small, see (7.4.2). Material lines on the cylindrical surface that are generatrices before deformation, i.e. straight lines parallel to the axis of the bar, are deformed into helices, see Fig. 7.4.1.

7.4.2 The Saint-Venant Theory of Torsion

The analysis of torsion of elastic cylindrical bars of non-circular cross-sections is more complex than for bars having circular cross-section. Figure 7.4.3 shows the result of torsion of a bar with rectangular cross section. Large strains are allowed in the figure to clearly demonstrate the mode of deformation. The deformation in Fig. 7.4.3 may be demonstrated with a rubber shaft, or simply by subjecting a rectangular everyday eraser to torsion. We may draw generatrices and contours of cross-sections on the surface of the bar. When the bar is deformed the generatrices become helices, while the deformed cross-sectional contours indicate that cross-sections are deformed to curved surfaces. The cylindrical outer surface of the bar is stress free and the resultant of the normal stresses on a cross-section is zero. We shall use Cartesian coordinates in the analysis, and with the $z(\equiv x_3)$-axis along the axis of the bar. It is furthermore convenient to mix the xy- and the x_1x_2-notation. Based on the observation about stresses presented above it seems reasonable to assume a state of stress represented by a stress matrix without normal stress components, i.e.:

$$T_{11} = T_{22} = T_{33} = 0 \tag{7.4.6}$$

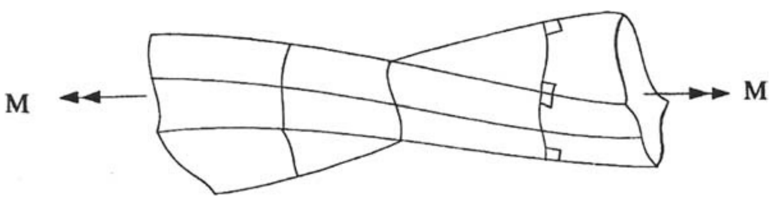


Fig. 7.4.3 Torsion of non-circular cylindrical bar

This stress assumption is realized by assuming the following *deformation hypothesis*:

When a bar is subjected to torsion, plane material cross-sections deform without strains in their surfaces and in such a way that all cross-sections obtain the same warped form. The longitudinal strain in the axial direction is zero.

The deformation hypothesis is the starting point of the *Saint-Venant theory of torsion*. As a consequence of the deformation hypothesis the projections of cross-sections onto the xy-plane rotates as rigid planes, and we may introduce an angle of twist Φ, see Fig. 7.4.4, which is proportional to the distance z from the end surface of the bar:

$$\Phi = \phi z \,, \quad \phi = \text{constant} \tag{7.4.7}$$

Since the longitudinal strain in the axial direction is zero, it follows that the displacement u_3 in the z-direction is only proportional to the angle of twist ϕ per unit length and independent of z. Thus we may set:

$$u_3 = \phi\, \psi(x,y) \tag{7.4.8}$$

The unknown function $\psi(x,y)$ is called the *warping function*. We assume small displacements, which imply that ϕ must be a small angle. The displacements of a particle P in the x- and y-directions may therefore be given as:

$$u_1 = -\Phi y = -\phi z y \,, \quad u_2 = \Phi x = \phi z x \tag{7.4.9}$$

The formulas (7.4.7, 7.4.8) are mathematical expressions of the deformation hypothesis. From Hooke's law and the relations between strains and displacements we find that the state of stress is given by:

$$\begin{aligned} T_{13} &= G\gamma_{13} = G\,(u_{1,3} + u_{3,1}) = G\phi\,(-y + \psi_{,1}) \\ T_{23} &= G\gamma_{23} = G\,(u_{2,3} + u_{3,2}) = G\phi\,(x + \psi_{,2}) \end{aligned} \tag{7.4.10}$$

All other coordinate stresses are zero. For a circular cylindrical bar the warping function $\psi = 0$, and the stresses T_{13} and T_{23} are components of the shear stress $\tau(R)$ given by (7.4.5).

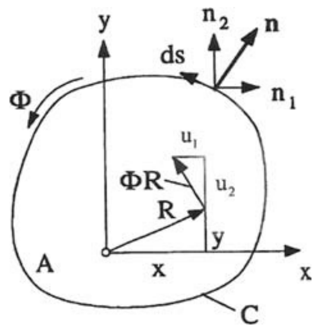


Fig. 7.4.4 Cross-section of the bar

The state of stress, given by the formulas (7.4.10) must satisfy the equilibrium conditions given by the Cauchy equations of motion, of which only one is not trivially satisfied:

$$\operatorname{div}\mathbf{T}=\mathbf{0} \quad \Leftrightarrow \quad T_{ik,k}=0 \quad \Rightarrow \quad T_{31,1}+T_{32,2}=0 \quad \Rightarrow \quad T_{13,1}+T_{23,2}=0 \quad \text{on } A \tag{7.4.11}$$

A is the projected cross-section as shown in Fig. 7.4.4. On the outer, cylindrical surface of the bar, i.e. along the contour C of the surface A, the stress boundary condition is:

$$\mathbf{t}=\mathbf{T}\cdot\mathbf{n}=\mathbf{0} \quad \text{along } C \;\Rightarrow\; T_{31}n_1+T_{32}n_2=0 \;\Rightarrow\; T_{13}n_1+T_{23}n_2=0 \quad \text{along } C \tag{7.4.12}$$

On the end surfaces of the bar, we can only require that the resultant of the stress distribution is represented by the torque M alone. Thus the resultant forces in the x- and y-directions must be zero, and the resultant moment about the z-axis is equal to the torque M:

$$\int_A T_{13}\,dA=\int_A T_{23}\,dA=0\,, \quad \int_A (T_{23}\,x-T_{13}\,y)\,dA=M \tag{7.4.13}$$

The real distribution of the stresses over the end surfaces is dependent upon how the torque M is supplied to the bar. But regardless of how this is done, we shall accept a solution that gives another distribution of stresses over the end surfaces. We only demand that the stress distribution is equivalent to the real one in the sense that the conditions (7.4.13) are satisfied. We may assume that the effect of this discrepancy between the real and computed stress distribution over the end surfaces is negligible except in small regions near the end surfaces. This assumption is supported by the *Saint-Venant's principle*, also referred to in Example 7.4.

The stresses (7.4.10) are now substituted into the equilibrium equation (7.4.11) and the stress boundary condition (7.4.12), and we obtain the following conditions for the warping function ψ:

$$\nabla^2\psi=0 \quad \text{on } A \tag{7.4.14}$$

$$\frac{d\psi}{dn}\equiv\psi_{,1}\frac{dx}{dn}+\psi_{,2}\frac{dy}{dn}\equiv\psi_{,\alpha}\,n_\alpha=yn_1-xn_2 \quad \text{along } C \tag{7.4.15}$$

The solution of (7.4.14, 7.4.15) is a standard problem in *potential theory* called the *Neumann's problem*, after Franz Ernst Neumann [1798–1895]. Equation (7.3.14) is called the *Laplace equation*.

When the warping function has been found from the (7.4.14, 7.4.15), it will be shown that the first and the second of the conditions (7.4.13) are satisfied. First we find, using (7.4.10) and (7.4.14), that:

$$\int_A T_{13}\,dA=G\phi\int_A(-y+\psi_{,1})\,dA=G\phi\int_A\{[x(-y+\psi_{,1})]_{,1}+[x(x+\psi_{,2})]_{,2}\}\,dA$$

Then, using the Gauss integration theorem in a plane, Theorem C.2, followed by application of the condition (7.3.15), we obtain:

$$\int_A T_{13}\, dA = G\phi \oint_C \{[x(-y+\psi_{,1})]\, n_1 + [x(x+\psi_{,2})]\, n_2\}\, ds$$
$$= G\phi \oint_C \{x[(\psi_{,1} n_1 + \psi_{,2} n_2) - (yn_1 - xn_2)]\}\, ds = 0$$

The fulfillment of the second condition in (7.4.13) is shown similarly.

The third of the conditions (7.4.13) provides the following relation between the torque M and the angle of twist ϕ.

$$M = GJ\phi\ , \quad J = \int_A \left[x^2 + y^2 + x\Psi_{,2} - y\Psi, 1\right] dA \tag{7.4.16}$$

The parameter J is called the *torsion constant* of the cross section, and the combination GJ is called the *torsional stiffness* of the bar.

Example 7.11. Elliptical Cross-Section

The simplest non-trivial solution of the Laplace equation (7.4.14) is given by the warping function:

$$\psi = kxy\ , \quad k = \text{constant} \tag{7.4.17}$$

We will now determine the contour curve C that together with this warping function satisfies the boundary condition (7.4.15). The condition (7.4.15) gives:

$$kyn_1 + kxn_2 = yn_1 - xn_2 \quad \text{along } C \tag{7.4.18}$$

From Fig. 7.4.4 we derive the relations:

$$n_1 = \frac{dy}{ds}\ , \quad n_2 = -\frac{dx}{ds} \tag{7.4.19}$$

Equation (7.4.18) is now be reorganized to give:

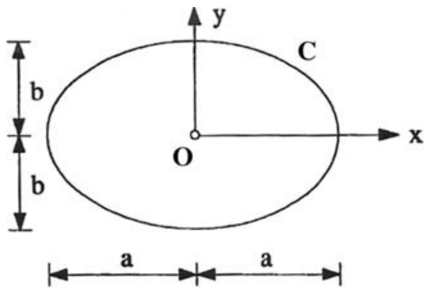
$$\frac{d}{ds}\left[x^2 + \frac{1-k}{1+k}y^2\right] = 0 \quad \text{along } C \quad \Rightarrow \quad x^2 + \frac{1-k}{1+k}y^2 = \text{constant along } C$$

The contour C is thus an ellipse. We write the equation of the ellipse C on the standard form:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1\ , \quad k = -\frac{a^2 - b^2}{a^2 + b^2}$$

a and b are the semiaxes, see Fig. 7.4.5. The function $\psi = kxy$ is thus the warping function for the torsion of a massive bar with elliptical cross-section. The torsion constant J is determined from the integral in equation (7.4.16), and the stress distribution is found from the expressions (7.4.10) when ϕ is supplied from (7.4.16). The results are:

Fig. 7.4.5 Elliptical cross-section



$$J=\frac{\pi a^3 b^3}{a^2+b^2}\,,\quad \tau_{xz}\equiv T_{13}=-\frac{2M}{\pi a b^3}y\,,\quad \tau_{yz}\equiv T_{23}=\frac{2M}{\pi a^3 b}x$$

The combined shear stress on the cross-section is:

$$\tau=\sqrt{T_{13}^2+T_{23}^2}=\frac{2M}{\pi a b^2}\sqrt{\left(\frac{y}{b}\right)^2+\left(\frac{b}{a}\right)^2\left(\frac{x}{a}\right)^2}$$

At the contour C the cross-sectional shear is:

$$\tau=\frac{2M}{\pi a b^2}\sqrt{\left(\frac{y}{b}\right)^2+\left(\frac{b}{a}\right)^2\left[1-\left(\frac{y}{b}\right)^2\right]}=\frac{2M}{\pi a b^2}\sqrt{\left(\frac{b}{a}\right)^2+\left[1-\left(\frac{b}{a}\right)^2\right]\left(\frac{y}{b}\right)^2}$$

When $b<a$ the maximum shear stress acts at the particles $(x,y)=(0,\pm b)$:

$$\tau_{\max}=\frac{2M}{\pi a b^2}\quad \text{on } A \text{ for } x=0 \text{ and } y=\pm b$$

For the case $a=b=r$ we get:

$$J=\pi r^4=I_p$$

and the results (7.4.5) for the circular cross-section.

7.4.3 Prandtl's Stress Function

An alternative mathematical formulation of the torsion problem is obtained by the introduction of *Prandtl's stress function* $\Omega(x,y)$ named after Ludwig Prandtl [1875–1953]. This stress function has the property that the equilibrium equation (7.4.11) is satisfied identically. This is achieved by setting:

$$T_{13}=\Omega_{,2}\,,\quad T_{23}=-\Omega_{,1} \tag{7.4.20}$$

Because this indirectly implies that we use the stresses as primary unknown functions, we must ensure that the resulting strains are compatible. That is, the strains γ_{13} and γ_{23} calculated from the stresses T_{13} and T_{23} by Hooke's law, must be compatible and give unique displacement function $u_3(x,y)$. From the strain-displacement relations (7.2.1) and the assumptions (7.4.9) concerning the displacements, we obtain:

$$\gamma_{13} = u_{1,3} + u_{3,1} = -\phi\, y + u_{3,1} \ , \quad \gamma_{23} = u_{2,3} + u_{3,2} = \phi\, x + u_{3,2}$$

From these equations we derive the following *compatibility condition* for the strains.

$$u_{3,12} - u_{3,21} = (\gamma_{13,2} + \phi) - (\gamma_{23,1} - \phi) \equiv 0 \quad \Rightarrow$$
$$\gamma_{13,2} - \gamma_{23,1} = -2\phi \quad \text{compatibility equation for strains} \qquad (7.4.21)$$

By Hooke's law:

$$T_{13} = G\gamma_{13} \ , \quad T_{23} = G\gamma_{23}$$

and (7.4.20), the compatibility condition (7.4.21) is rewritten to:

$$\nabla^2 \Omega = -2G\phi \text{ on } A \qquad (7.4.22)$$

The stress function $\Omega(x,y)$ must also satisfy a boundary condition on the contour curve C of the cross-section A. This condition will be derived from the stress condition (7.4.12) using the relations (7.4.19).

Substituting the stresses (7.4.20) into the boundary condition (7.4.12), and then using the relations (7.4.19), we obtain:

$$\Omega_{,2}\frac{dy}{ds} + (-\Omega_{,1})\left(-\frac{dx}{ds}\right) = \frac{d\Omega}{ds} = 0 \text{ along } C$$

This implies that $\Omega(x,y)$ must be a constant on the contour C. For the sake of simplicity we set the constant equal to zero. Thus we have obtained the boundary condition:

$$\Omega = 0 \text{ along } C \qquad (7.4.23)$$

We must now show that the first two boundary conditions (7.4.13) are satisfied by the stress function $\Omega(x,y)$ and find what the third of the conditions (7.4.13) leads to. Using the expressions (7.4.20), Gauss theorem in a plane, Theorem C.2, and finally (7.4.23), we obtain:

$$\int_A T_{13}\, dA = \int_A \Omega_{,2}\, dA = \oint_C \Omega n_2\, ds = 0 \ , \quad \int_A T_{23}\, dA = -\int_A \Omega_{,1}\, dA = -\oint_C \Omega n_1\, ds = 0$$

These results prove that the first two boundary conditions (7.4.13) are satisfied

From the third of the conditions (7.4.13) it follows that:

$$M = \int_A (T_{23}x - T_{13}y)\,dA = -\int_A (\Omega_{,1}\,x + \Omega_{,2}\,y)\,dA = -\int_A (\Omega x_\alpha)_{,\alpha}\,dA + 2\int_A \Omega\,dA$$
$$= -\oint_C \Omega x_\alpha n_\alpha\,ds + 2\int_A \Omega\,dA$$

Due to (7.4.23) the result is that:

$$M = 2\int_A \Omega\,dA \tag{7.4.24}$$

We may now conclude that the Prandtl stress function $\Omega(x,y)$ has to satisfy the differential equation (7.4.22) and the boundary condition (7.4.23):

$$\nabla^2\Omega = -2G\phi \text{ on } A\,, \quad \Omega = 0 \text{ along } C \tag{7.4.25}$$

The mathematical problem represented by (7.4.25) is called the *Poisson problem* in Potential Theory.

The warping function $\psi(x,y)$ and the Prandtl stress function $\Omega(x,y)$ are related through:

$$\Omega_{,2} = G\phi\,(-y + \psi_{,1})\,,\ \Omega_{,1} = G\phi\,(-x - \psi_{,2}) \tag{7.4.26}$$

These relations follow by comparing the stress expressions (7.4.10) and (7.4.20). The compatibility condition (7.4.22) may also be derived directly from the relations (7.4.26) and the compatibility equation (7.4.14).

In the next section we will present a practical further application of the use of the Prandtl stress function.

Example 7.12. Elliptical Cross-Section

For torsion of a bar having an elliptical cross-section with semiaxes a and b, Fig. 7.4.5, the Prandtl stress function is:

$$\Omega = -\frac{a^2b^2}{a^2+b^2}G\phi\left[\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 1\right] \tag{7.4.27}$$

The boundary condition (7.4.23) is obviously satisfied, and it is easily shown that the differential equation (7.4.22) also is satisfied. From (7.4.24) we obtain:

$$M = 2\int_A \Omega\,dA = -\frac{2a^2b^2}{a^2+b^2}G\phi\left[\frac{1}{a^2}\int_A x^2\,dA + \frac{1}{b^2}\int_A y^2\,dA - \int_A dA\right]$$

The integrals are found to be:

$$\int_A x^2\,dA = I_y = \frac{\pi a^3 b}{4}\,,\quad \int_A y^2\,dA = I_x = \frac{\pi a b^3}{4}\,,\quad \int_A dA = A = \pi a b$$

I_y and I_x are the second moments of area of the cross-section A about the y-axis and the x-axis respectively. Finally, the torque M becomes:

$$M = \frac{\pi a^3 b^3}{a^2 + b^2} G\phi$$

The stress function $\Omega(x,y)$, the torsion constant J in (7.4.16), and the shear stresses on the cross section, according to the formulas (7.4.20), are now:

$$\Omega = -\frac{M}{\pi ab}\left[\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 1\right], \; J = \frac{\pi a^3 b^3}{a^2 + b^2}$$
$$T_{13} = \Omega_{,2} = -\frac{2M}{\pi a b^3} y, \;\; T_{23} = -\Omega_{,1} = \frac{2M}{\pi a^3 b} x$$

Compare with the results in Example 7.11.

7.4.4 The Membrane Analogy

Figure 7.4.6 shows a flexible membrane attached to a plane stiff border C of a hole with a plane area A. The membrane is subjected to a constant pressure p and is

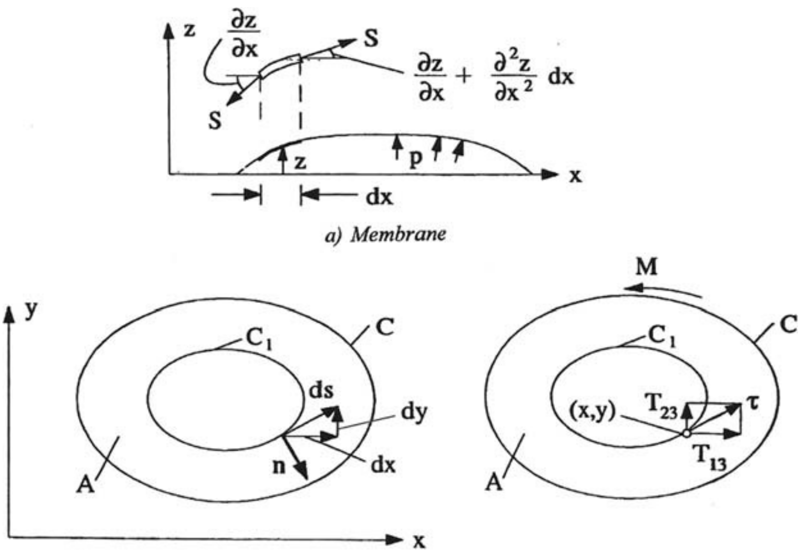


Fig. 7.4.6 **a**) Membrane subjected to a pressure p. **b**) Cross-section of a bar in torsion

stretched to a curved form given by a function $z = z(x,y)$. We assume the membrane force S, given as a force per unit length, is a constant and equal in all directions. The condition of a constant membrane force S is satisfied by a soap film membrane.

We assume that z is small compared to the smallest diameter of the area A. An element of the membrane with the projection $dx \cdot dy$ on the xy-plane is in equilibrium under the action of the pressure p and the membrane force S. The equilibrium equation in the z-direction is:

$$p \cdot (dx \cdot dy) - (S \cdot dy) \cdot \frac{\partial z}{\partial x} + (S \cdot dy) \cdot \left(\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial x^2} dx \right) - (S \cdot dx) \cdot \frac{\partial z}{\partial y} +$$
$$(S \cdot dx) \cdot \left(\frac{\partial z}{\partial y} + \frac{\partial^2 z}{\partial y^2} dy \right) = 0 \quad \Rightarrow$$
$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \equiv \nabla^2 z = -\frac{p}{S} \tag{7.4.28}$$

The boundary condition for the z-function along the border curve C is:

$$z = 0 \quad \text{along } C \tag{7.4.29}$$

The mathematical problem consisting of finding the function $z(x,y)$ from the differential equation (7.4.28) and the boundary condition (7.4.29), is a Poisson problem and thus mathematically analogous to the problem (7.4.25) of finding the Prandtl stress function $\Omega(x,y)$. When we choose for the plane A of the hole a cross-section A of a cylindrical bar subjected to torsion, and the constant p/S equal to the constant $2G\phi\alpha$, where:

$$\alpha = \frac{p}{2G\phi S} \tag{7.4.30}$$

then the membrane function $z(x,y)$ becomes identical to:

$$z(x,y) = \alpha\, \Omega(x,y) \tag{7.4.31}$$

Let C_1 be the contour line of the membrane, such that $z \equiv \alpha\Omega =$ constant along C_1. In direction of the tangent to the line C_1, Fig. 7.4.6:

$$\frac{\partial z}{\partial s} = 0 \quad \Rightarrow \quad \frac{\partial \Omega}{\partial s} = \frac{\partial \Omega}{\partial x}\frac{dx}{ds} + \frac{\partial \Omega}{\partial y}\frac{dy}{ds} = 0 \quad \Rightarrow \quad -T_{23}\frac{dx}{ds} + T_{13}\frac{dy}{ds} = 0$$

This result and Fig. 7.4.6 show that the resultant of the shear stresses on the cross section does not have a component normal to the contour line C_1 in the cross-section. This means that the contour lines of the membrane are vector lines of the shear stresses on the cross-section of the bar. In other words, in every particle on the cross-section the resultant shear stress τ is directed along the contour line $\Omega(x,y) =$ constant through the particle. We may call the contour lines *shear stress lines* in the cross-section. Since the resultant shear stress τ is tangential to the contour line C_1

and thus perpendicular to the unit normal **n** to the contour line, pointing outward, we find from Fig. 7.4.6:

$$\tau = T_{13}\frac{dx}{ds} + T_{23}\frac{dy}{ds} = \frac{\partial\Omega}{\partial y}\left(-\frac{dy}{dn}\right) + \left(-\frac{\partial\Omega}{\partial x}\right)\frac{dx}{dn} \quad \Rightarrow \quad \tau = -\frac{\partial\Omega}{\partial n} = -\frac{1}{\alpha}\frac{\partial z}{\partial n} \tag{7.4.32}$$

The n-coordinate is measured along the normal **n**. The result (7.4.32) shows that the maximum cross-sectional shear stress occurs in particles where the membrane has the steepest slope. If a net of contour lines have been drawn, these points of maximum shear stress will be where the contour lines are closest together.

The formula (7.4.24) shows that the torque M is proportional to the volume V between the membrane and the surface A.

$$M = 2\int_A \Omega dA = \frac{2}{\alpha}\int_A z dA = \frac{2V}{\alpha} \tag{7.4.33}$$

The membrane analogy may be used in an experimental determination of the stress distribution on a cross section A of a bar in torsion. A membrane of soap is often used for this purpose. The constant p/S may be determined by introducing a calibrating membrane over a circular hole near the hole representing the cross-sectional area A. The circular hole, representing a circular cross section, and the hole with cross-sectional area A are covered by a soap film having the same membrane force S, and are subjected to the same pressure p. The known solution of the torsion problem for a circular cylindrical bar is then used to compute the constant p/S.

The membrane analogy may be used for bars with open and closed thin-walled cross sections and for bars with cell like cross sections. We shall not elaborate on the subject, only refer to the literature, e.g. [48].

Example 7.13. Narrow Rectangular Cross-Section

Figure 7.4.7 shows a membrane over a rectangular hole for which the width b is much smaller the height h. We may discard the small regions near the short sides and thus consider the membrane as a cylindrical surface, implying that $\partial z/\partial x \equiv 0$. Then from the formulas (7.4.28) and (7.4.29) we obtain:

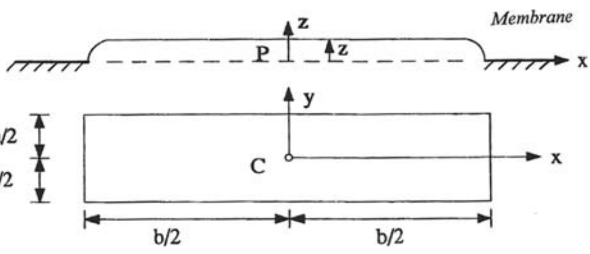


Fig. 7.4.7 Membrane for a narrow rectangular cross-section

$$\frac{d^2z}{dy^2} = -\frac{p}{S}\,,\;\; z = 0 \quad \text{for}\; y = \pm\frac{h}{2} \quad \Rightarrow \quad z(y) = \frac{ph^2}{8S}\left[1 - \left(\frac{2y}{h}\right)^2\right]$$

This membrane provides the solution to the torsion problem for a cylindrical bar with a narrow rectangular cross-section with width b and height h. The Prandtl stress function is:

$$\Omega(y) = \frac{z(y)}{\alpha} = \frac{G\,\phi\,h^2}{4}\left[1 - \left(\frac{2y}{h}\right)^2\right]$$

This function is substituted into formula (7.4.24) for the torque, which then gives:

$$M = 2\int_A \Omega\,dA = \frac{G\,\phi\,h^2}{2}\left[b\int_{-h/2}^{h/2}\left[1 - \left(\frac{2y}{h}\right)^2\right]dy\right] = \frac{G\,\phi\,h^3 b}{3}$$

The angle of twist ϕ per unit length of the bar, the torsion constant J, and the maximum shear stress on the cross-section are then, formulas (7.4.16, 7.4.20):

$$\phi = \frac{3M}{Gh^3b}\,,\;\; J = \frac{M}{G\phi} = \frac{h^3b}{3}\,,\;\; \tau_{\max} = -\left.\frac{d\Omega}{dy}\right|_{y=h/2} = G\,\phi\,h = \frac{3M}{h^2b}$$

7.5 Thermoelasticity

7.5.1 Constitutive Equations

When the strains in an isotropic, linearly elastic material, i.e. a Hookean material, do not originate only from the stresses in the material, Hooke's law given by (7.2.6, 7.2.7, 7.2.8), has to be modified. We shall in this section discuss the important case of a Hookean material subjected to a temperature field $\theta(\mathbf{r},t)$, which may imply expansions and contractions in the material. It is assumed that the changes in the temperature, from a homogeneous temperature θ_o in the reference configuration K_o, are small enough not to influence the elastic properties of the material. Furthermore, we assume that the thermal properties are homogeneous and isotropic. Thermal isotropy implies that when the material experiences *free thermal deformation*, i.e. stress-free deformation, the longitudinal strain is the same in all directions in a particle of the material and given by $\varepsilon^t = \alpha\cdot(\theta - \theta_o)$. The parameter α is the *linear coefficient of thermal expansion* of the material. The state of strain in free thermal deformation is therefore isotropic and form invariant. The strain tensor is:

$$E^t_{ij} = \alpha\cdot(\theta - \theta_o)\,\delta_{ij} \quad \Leftrightarrow \quad \mathbf{E}^t = \alpha\cdot(\theta - \theta_o)\,\mathbf{1} \tag{7.5.1}$$

A volume element of the material subjected to a constant change in temperature $(\theta - \theta_o)$ preserves its shape and the volumetric strain due to the thermal deformation is $\varepsilon_v = 3\alpha(\theta - \theta_o)$. The strains (7.5.1) are in general not compatible, which implies that elastic strains due to stresses are developed to make the total strains, thermal plus elastic, compatible. It may be shown, see Problem 5.8 that the condition for the strains in (7.5.1) to be compatible is that the change in temperature $(\theta - \theta_o)$ is a linear function of the place vector $\mathbf{r}$, or the x-coordinates. The total strains in a body of a Hookean material, represented by the strain tensor $\mathbf{E}$, may be considered to comprise of the sum of three contributions:

1) *thermal strains* according to (7.5.1),
2) *elastic strains* to create compatible strains when the material resists free thermal deformation,
3) *elastic strains* produced by the stresses due to the external forces on the body and the motion of the body.

The two contributions of elastic strains are represented by the elastic strain tensor:

$$\mathbf{E}^e = \mathbf{E} - \mathbf{E}^t = \mathbf{E} - \alpha \cdot (\theta - \theta_o)\mathbf{1} \quad \Leftrightarrow \quad E^e_{ij} = E_{ij} - E^t_{ij} = E_{ij} - \alpha \cdot (\theta - \theta_o)\,\delta_{ij} \tag{7.5.2}$$

Substitution of this strain tensor into Hooke's law, (7.2.6) and (7.2.7), yields:

$$\mathbf{E} = \frac{1+\nu}{\eta}\mathbf{T} - \frac{\nu}{\eta}(\mathrm{tr}\,\mathbf{T})\,\mathbf{1} + \alpha \cdot (\theta - \theta_o)\,\mathbf{1} \quad \Leftrightarrow$$
$$E_{ij} = \frac{1+\nu}{\eta}T_{ij} - \frac{\nu}{\eta}T_{kk}\,\delta_{ij} + \alpha \cdot (\theta - \theta_o)\,\delta_{ij} \tag{7.5.3}$$

The inverse form becomes:

$$\mathbf{T} = \frac{\eta}{1+\nu}\left[\mathbf{E} + \frac{\nu}{1-2\nu}(\mathrm{tr}\,\mathbf{E})\,\mathbf{1}\right] - 3\kappa\alpha \cdot (\theta - \theta_o)\,\mathbf{1} \quad \Leftrightarrow$$
$$T_{ij} = \frac{\eta}{1+\nu}\left[E_{ij} + \frac{\nu}{1-2\nu}E_{kk}\,\delta_{ij}\right] - 3\kappa\alpha \cdot (\theta - \theta_o)\,\delta_{ij} \tag{7.5.4}$$

The constitutive equations (7.5.3) and (7.5.4) represent the *Duhamel-Neumann law*, named after Duhamel [1838] and Neumann [1841].

7.5.2 Plane Stress

In the case of plane stress, $T_{i3} = 0$, the constitutive equations (7.5.3) and (7.5.4) are replaced by:

$$E_{\alpha\beta} = \frac{1+\nu}{\eta}\left[T_{\alpha\beta} - \frac{\nu}{1+\nu}T_{\rho\rho}\,\delta_{\alpha\beta}\right] + \alpha\cdot(\theta-\theta_o)\,\delta_{\alpha\beta}$$
$$E_{33} = -\frac{\nu}{\eta}T_{\rho\rho} + \alpha\cdot(\theta-\theta_o) \tag{7.5.5}$$

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) + \alpha\cdot(\theta-\theta_o)$$
$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) + \alpha\cdot(\theta-\theta_o)\,, \quad \gamma_{xy} = \frac{1}{G}\tau_{xy} \tag{7.5.6}$$
$$T_{\alpha\beta} = \frac{\eta}{1+\nu}\left[E_{\alpha\beta} + \frac{\nu}{1-\nu}E_{\rho\rho}\,\delta_{\alpha\beta}\right] - \frac{\eta}{1-\nu}\alpha\cdot(\theta-\theta_o)\,\delta_{\alpha\beta} \tag{7.5.7}$$

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) - \frac{E}{1-\nu}\alpha\cdot(\theta-\theta_o)$$
$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x) - \frac{E}{1-\nu}\alpha\cdot(\theta-\theta_o)\,, \quad \tau_{xy} = G\gamma_{xy} \tag{7.5.8}$$

The equilibrium equations (7.3.43) and the constitutive equations (7.5.5) may be used to express the compatibility equations (7.3.44) as:

$$\nabla^2 T_{\alpha\alpha} + (1+\nu)\,\rho\, b_{\alpha,\alpha} + \eta\,\alpha\,(\theta-\theta_o)_{,\alpha\alpha} = 0 \tag{7.5.9}$$

From these equations it readily follows that a linear temperature field $(\theta-\theta_o)$ does not contribute to the stresses if the surface of the body is not subjected to displacement conditions.

The Navier equations (7.3.14) represent the Cauchy equations of motion expressed in displacements. When the stresses are expressed by the Duhamel-Neumann equations (7.5.7) rather than Hooke's law, the following modified Navier equations are obtained:

$$u_{\alpha,\beta\beta} + \frac{1+\nu}{1-\nu}\left[u_{\beta,\beta\alpha} - 2\alpha\,(\theta-\theta_0)_{,\alpha}\right] + \frac{1}{\mu}\rho\,(b_\alpha - \ddot{u}_\alpha) = 0 \tag{7.5.10}$$

In an axisymmetrical case these equations are reduced to one equation for the radial displacement $u \equiv u_R(R)$, confer equation (7.3.19):

$$\frac{d}{dR}\left[\frac{1}{R}\frac{d(Ru)}{dR}\right] - (1+\nu)\,\alpha\frac{d(\theta-\theta_o)}{dR} + \frac{1-\nu}{2G}\rho\,(b_R - \ddot{u}_R) = 0 \tag{7.5.11}$$

Example 7.14. Circular Plate Mounted on a Rigid Rod

A circular plate of radius b has a concentric hole of radius a. The plate is mounted on a rigid rod of radius a, see Fig. 7.5.1. The plate is cooled from the temperature θ_o to the temperature $\theta_1 < \theta_o$. We shall determine the state of stress in the plate and the radial displacement.

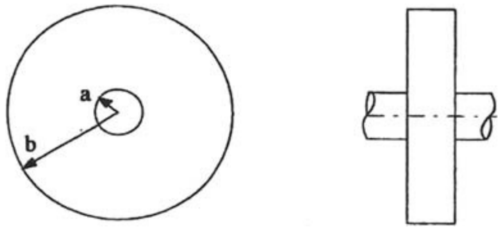


Fig. 7.5.1 Circular plate mounted on a rigid rod

Since body forces are not considered in this problem, and since the term involving the temperature in the compatibility equation (7.5.9) in this case becomes zero, we may choose between the general solution from Example 7.9 using Airy's stress function, or we may solve the Navier equation (7.5.11) as was done in Example 7.1. In the case of a more complex but axisymmetric temperature field $\theta(R)$, it is most convenient to use the Navier equation. The solution procedure is then as follows. The general solution of the Navier equation (7.5.11), with $b_R = \ddot{u} = 0$, is:

$$u(R) = AR + \frac{B}{R} + (1+\nu)\,\alpha\frac{1}{R}\int\left[\theta(R) - \theta_o\right]R\,dR \tag{7.5.12}$$

A and B are constants of integration. In the present example: $\theta(R) - \theta_o = \theta_1 - \theta_o < 0$. Therefore:

$$u(R) = AR + \frac{B}{R} - (1+\nu)\,\alpha\,(\theta_o - \theta_1)\,\frac{R}{2} \tag{7.5.13}$$

The strains are given by the formulas (7.3.17) as:

$$\varepsilon_R = \frac{du}{dR}, \quad \varepsilon_\theta = \frac{u}{R} \tag{7.5.14}$$

The stresses are determined from the constitutive equations (7.5.8), and become:

$$\begin{aligned}
\sigma_R(R) &= 2G\left[\frac{1+\nu}{1-\nu}A - B\frac{1}{R^2}\right] + G(1+\nu)\,\alpha\,(\theta_o - \theta_1)\\
\sigma_\theta(R) &= 2G\left[\frac{1+\nu}{1-\nu}A + B\frac{1}{R^2}\right] + G(1+\nu)\,\alpha\,(\theta_o - \theta_1)
\end{aligned} \tag{7.5.15}$$

The boundary conditions are: 1) $u(a) = 0$ and 2) $\sigma_R(b) = 0$. The final solution to the problem is then:

$$u(R) = -C\left[R - \frac{a^2}{R}\right], \quad C = \alpha\cdot(\theta_o - \theta_1)\left[1 + \frac{1-\nu}{1+\nu}\left(\frac{a}{b}\right)^2\right]^{-1} \tag{7.5.16}$$

$$\sigma_R(R) = 2GC\left(\frac{a}{b}\right)^2\left[1 - \left(\frac{b}{R}\right)^2\right], \quad \sigma_\theta(R) = 2GC\left(\frac{a}{b}\right)^2\left[1 + \left(\frac{b}{R}\right)^2\right] \tag{7.5.17}$$

If we let $a \to 0$, we see that the stresses disappear, in accordance with the fact that the rigid core has vanished and the plate therefore has free thermal deformation, with:

$$u(R) = -\alpha \cdot (\theta_o - \theta_1) R \tag{7.5.18}$$

7.5.3 Plane Displacements

In the case of plane displacements, $u_3 = 0 \Rightarrow E_{33} = 0$, the constitutive equations (7.5.3) and (7.5.4) are reduced to:

$$E_{\alpha\beta} = \frac{1}{2\mu}\left[T_{\alpha\beta} - \nu T_{\rho\rho}\,\delta_{\alpha\beta}\right] + (1+\nu)\,\alpha \cdot (\theta - \theta_o)\,\delta_{\alpha\beta} \tag{7.5.19}$$

$$\varepsilon_x = \frac{1-\nu}{2G}\left(\sigma_x - \frac{\nu}{1-\nu}\sigma_y\right) + (1+\nu)\,\alpha \cdot (\theta - \theta_o)$$

$$\varepsilon_y = \frac{1-\nu}{2G}\left(\sigma_y - \frac{\nu}{1-\nu}\sigma_x\right) + (1+\nu)\,\alpha \cdot (\theta - \theta_o)\,, \quad \gamma_{xy} = \frac{1}{G}\tau_{xy} \tag{7.5.20}$$

$$T_{\alpha\beta} = 2\mu\left[E_{\alpha\beta} + \frac{\nu}{1-2\nu}E_{\rho\rho}\,\delta_{\alpha\beta}\right] - 3\kappa\,\alpha \cdot (\theta - \theta_o)\,\delta_{\alpha\beta}, \quad \kappa = \frac{2\mu\,(1+\nu)}{3\,(1-2\nu)} \tag{7.5.21}$$

$$T_{33} = \frac{2\nu\mu}{1-2\nu}E_{\rho\rho} - 3\kappa\alpha \cdot (\theta - \theta_o) \tag{7.5.22}$$

$$\sigma_x = \frac{2G}{1-2\nu}\left[(1-\nu)\,\varepsilon_x + \nu\,\varepsilon_y\right] - 3\kappa\alpha \cdot (\theta - \theta_o)$$

$$\sigma_y = \frac{2G}{1-2\nu}\left[(1-\nu)\,\varepsilon_y + \nu\,\varepsilon_x\right] - 3\kappa\alpha \cdot (\theta - \theta_o)\,, \quad \tau_{xy} = G\gamma_{xy} \tag{7.5.23}$$

The equilibrium equations (7.3.43) and the constitutive equations (7.5.20) may be used to express the compatibility equations (7.3.44) as:

$$\nabla^2 T_{\alpha\alpha} + \frac{1}{1-\nu}\left[\rho\, b_{\alpha},_{\alpha} + \eta\,\alpha\,(\theta - \theta_o),_{\alpha\alpha}\right] = 0 \tag{7.5.24}$$

As in the case of plane stress we see that a linear temperature field $(\theta - \theta_o)$ does not contribute to the stresses $T_{\alpha\beta}$ if the surface of the body is not subjected to displacement conditions for u_α.

Modified *Navier equations*, which represent the Cauchy equations of motion expressed in displacements, now become:

$$u_{\alpha},_{\beta\beta} + \frac{1}{1-2\nu}\left[u_{\beta},_{\beta\alpha} - 2\,(1+\nu)\,\alpha \cdot\ (\theta - \theta_0),_{\alpha}\right] + \frac{1}{\mu}\rho\,(b_\alpha - \ddot{u}_\alpha) = 0 \tag{7.5.25}$$

7.6 Hyperelasticity

7.6.1 Elastic Energy

In the case of a uniaxial stress history $\sigma(t)$ with the strain $\varepsilon(t)$ in the direction of the stress σ, the stress power per unit volume is $\omega = \sigma\dot{\varepsilon}$. The stress work w per unit volume when the material is deformed from the reference configuration K_o at time t_o to the present configuration K at time t will be:

$$w = \int_{t_o}^{t} \omega\, dt = \int_{t_o}^{t} \sigma\, \dot{\varepsilon}\, dt = \int_{\varepsilon_o}^{\varepsilon} \sigma\, d\varepsilon \tag{7.6.1}$$

ε and ε_o are the strains in K and K_o espectively. Figure 7.6.1 shows that the stress work is represented by the area under the stress curve $\sigma(\varepsilon)$ in the $\sigma\varepsilon$-diagram.

Under the assumption of small deformations the stress power per unit volume for a general state of stress $\mathbf{T}$ and the corresponding state of strain $\mathbf{E}$ is given by $\omega = \mathbf{T} : \dot{\mathbf{E}}$. The stress work done between the configurations K and K_o is given by:

$$w = \int_{t_o}^{t} \omega\, dt = \int_{t_o}^{t} \mathbf{T} : \dot{\mathbf{E}}\, dt = \int_{\mathbf{E}_o}^{\mathbf{E}} \mathbf{T} : d\mathbf{E} \tag{7.6.2}$$

A material is called *hyperelastic, Green elastic*, or *conservative* if the response of the material is such that the stress power and the stress work may be derived from a scalar valued potential $\phi = \phi\,[\mathbf{E}]$ such that:

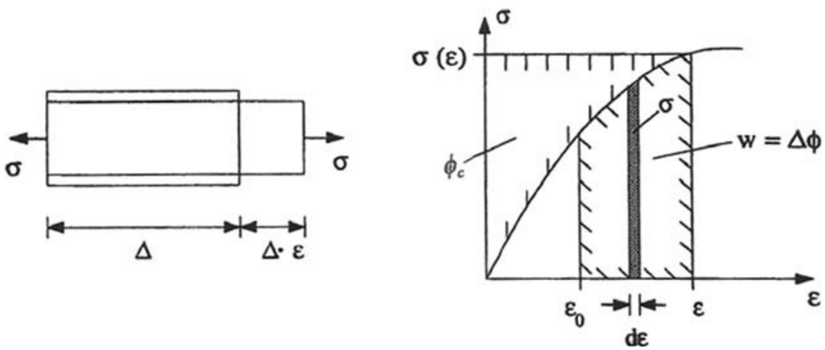


Fig. 7.6.1 Stress work w, elastic energy ϕ, and complementary energy ϕ_c

$$\omega = \dot{\phi} = \frac{\partial \phi}{\partial \mathbf{E}} : \dot{\mathbf{E}} = \frac{\partial \phi}{\partial E_{ij}} \dot{E}_{ij} \tag{7.6.3}$$

$$w = \int_{t_o}^{t} \omega \, dt = [\phi]_{t_o}^{t} = \phi[\mathbf{E}] - \phi[\mathbf{E}_o] \equiv \Delta\phi \tag{7.6.4}$$

The potential $\phi[\mathbf{E}]$ is called *elastic energy* or *strain energy per unit volume*. We choose to set $\phi[\mathbf{0}] = 0$, such that the elastic energy is zero when the material is free of strain. A body with volume V has a total elastic energy:

$$\Phi = \int_V \phi \, dV \tag{7.6.5}$$

The stress power supplied to the body is:

$$P^d = \int_V \omega \, dV = \int_V \dot{\phi} \, dV = \dot{\Phi} \tag{7.6.6}$$

The result for the material derivative of the integral in (7.6.5) is obtained under the assumption of small volumetric strains, which means that we may neglect the fact that the volumes V and dV change with time. The result (7.6.6) shows that the stress power P^d supplied to the body is equal to the time rate of change in the elastic energy of the body. The balance equation of mechanical energy (6.1.12) may now be written as:

$$P = \dot{K} + \dot{\Phi} \tag{7.6.7}$$

Integration with respect to time yields the *work and energy equation*:

$$\int_{t_o}^{t} P \, dV = [K + \Phi]_{t_o}^{t} \quad \Rightarrow \quad W = \Delta(K + \Phi) \tag{7.6.8}$$

W represents the work of the external forces on the body. For a body of a hyperelastic material the work of the external forces is converted to two kinds of energy: *kinetic energy* and *elastic energy*. Positive work on the body is conserved in the body as kinetic energy and elastic energy. When the work is negative, the body delivers energy to the surroundings.

For a hyperelastic material the stress power per unit volume is represented by two expressions:

$$\omega = T_{ij} \dot{E}_{ij} = \frac{\partial \phi}{\partial E_{ij}} \dot{E}_{ij} \tag{7.6.9}$$

Note that in the differentiation of ϕ in (7.6.9) we must treat E_{ij} and E_{ji} for $i \neq j$ as independent variables. In other words, ϕ must be treated as a function of 9 independent variables E_{ij}. Since $\partial\phi/\partial\mathbf{E}$ is not dependent upon the rate of strain $\dot{\mathbf{E}}$, the stress power ω is a linear function of $\dot{\mathbf{E}}$. This implies that the stresses T_{ij} are also independent of $\dot{\mathbf{E}}$. Equation (7.6.9) will therefore imply that:

$$\mathbf{T} = \frac{\partial \phi}{\partial \mathbf{E}} \quad \Leftrightarrow \quad T_{ij} = \frac{\partial \phi}{\partial E_{ij}} \tag{7.6.10}$$

The result may be obtained as follows. First we choose the case:

$$\dot{E}_{11} \neq 0 \text{ all other } \dot{E}_{ij} = 0$$

which results in the stress power:

$$\omega = T_{11}\dot{E}_{11} = \frac{\partial \phi}{\partial E_{11}}\dot{E}_{11} \quad \Rightarrow \quad T_{11} = \frac{\partial \phi}{\partial E_{11}} \tag{7.6.11}$$

Then we choose:

$$\dot{E}_{12} = \dot{E}_{21} \neq 0 \text{ all other } \dot{E}_{ij} = 0$$

which implies the stress power:

$$\omega = T_{12}\dot{E}_{12} + T_{21}\dot{E}_{22} = (T_{12} + T_{21})\,\dot{E}_{12} = \left[\frac{\partial \phi}{\partial E_{12}} + \frac{\partial \phi}{\partial E_{21}}\right]\dot{E}_{12} \quad \Rightarrow$$

$$T_{12} = \frac{1}{2}\left[\frac{\partial \phi}{\partial E_{12}} + \frac{\partial \phi}{\partial E_{21}}\right] \quad \Rightarrow \quad T_{12} = \frac{\partial \phi}{\partial E_{12}} \tag{7.6.12}$$

From the results (7.6.11, 7.6.12) and similar results for other (*ij*)-index pairs the general result (7.6.10) follows. Remember that in the differentiation of ϕ in (7.6.10) we must treat E_{ij} and E_{ji} for $i \neq j$ as independent variables, i.e. ϕ must be treated as a function of 9 independent variables E_{ij}.

According to the constitutive equations (7.6.10) for hyperelastic materials the stresses are functions of the strains: $\mathbf{T} = \mathbf{T}[\mathbf{E}]$. Thus the following implication:

$$Hyperelasticity \Rightarrow Elasticity$$

All material models that have been proposed in the literature for real elastic materials are hyperelastic. In principle however, an elastic material model may be dissipative, i.e. some of the work done on such a material may turn into heat.

Complementary energy per unit volume ϕ_c is defined by the expression:

$$\phi_c = \mathbf{T} : \mathbf{E} - \phi\,[\mathbf{E}] \tag{7.6.13}$$

For uniaxial stress the complimentary energy is:

$$\phi_c = \sigma\,\varepsilon - \phi\,[\varepsilon] \tag{7.6.14}$$

In the $\sigma\varepsilon$ – diagram ϕ_c is represented by the vertically hatched area above the $\sigma(\varepsilon)$ – curve in Fig. 7.6.1. From the definitions (7.6.13) and equation (7.6.10) we obtain the result:

$$\frac{\partial \phi_c}{\partial \mathbf{T}} = \frac{\partial \mathbf{T}}{\partial \mathbf{T}} : \mathbf{E} + \mathbf{T} : \frac{\partial \mathbf{E}}{\partial \mathbf{T}} - \frac{\partial \phi}{\partial \mathbf{E}} : \frac{\partial \mathbf{E}}{\partial \mathbf{T}} = \mathbf{E} + \mathbf{T} : \frac{\partial \mathbf{E}}{\partial \mathbf{T}} - \mathbf{T} : \frac{\partial \mathbf{E}}{\partial \mathbf{T}} = \mathbf{E} \quad \Rightarrow$$

$$\mathbf{E} = \frac{\partial \phi_c}{\partial \mathbf{T}} \quad \Leftrightarrow \quad E_{ij} = \frac{\partial \phi_c}{\partial T_{ij}} \tag{7.6.15}$$

It will be shown in Sect. 7.8 that for linearly hyperelastic materials, for which the stresses are linear functions of the strains or the strains are linear functions of the stresses, i.e.:

$$T_{ij} = S_{ijkl} E_{kl} \quad \Leftrightarrow \quad E_{ij} = K_{ijkl} T_{kl} \tag{7.6.16}$$

the number of independent stiffnesses S_{ijkl} or compliances K_{ijkl} is at most 21.

In Sect. 7.2.2 the following expression was developed for the elastic energy per unit volume for an isotropic, linearly elastic material, i.e. a *Hookean material*:

$$\phi = \frac{1}{2}\mathbf{T} : \mathbf{E} = \mu \mathbf{E} : \mathbf{E} + \frac{1}{2}\left(\kappa - \frac{2}{3}\mu\right)(\mathrm{tr}\,\mathbf{E})^2 \tag{7.6.17}$$

Expressed in terms of stresses we find:

$$\phi = \frac{1}{2}\mathbf{T} : \mathbf{E} = \frac{1}{4\mu}\left[\mathbf{T} : \mathbf{T} - \frac{\nu}{1+\nu}(\mathrm{tr}\,\mathbf{T})^2\right] \tag{7.6.18}$$

The complementary energy per unit volume for a Hookean material becomes:

$$\phi_c = \mathbf{T} : \mathbf{E} - \phi = \phi \tag{7.6.19}$$

The Hookean material model is thus hyperelastic, and elastic energy and complementary energy are equal. In Sect. 7.8 it will be shown that this is true for all linearly hyperelastic materials.

It is reasonable to require that the elastic energy always is positive when $\mathbf{E} \neq 0$, i.e. the elastic energy per unit volume ϕ must be a *positive definite scalar-valued function* of the strain tensor **E**. It is given as Problem 7.21 to show that this requirement implies the following conditions for the elasticities of Hookean materials:

$$E \equiv \eta > 0\,, \quad G \equiv \mu > 0\,, \quad \kappa > 0 \quad \Rightarrow \quad -1 < \nu \leq 0.5 \tag{7.6.20}$$

In the expression (7.2.15) we suggested that the Poisson's ratio ν should be expected to be a number between 0 and 0.5. Negative values of ν appear to be very unrealistic, and no real materials have been found with negative Poisson's ratio.

7.6.2 The Basic Equations of Hyperelasticity

The primary objective of the theory of elasticity is to provide methods for calculating stresses, strains, and displacements in elastic bodies subjected to body forces and prescribed boundary conditions for contact forces and/or displacements on the surface of the bodies. The basic equations of the theory are:

The Cauchy equations of motion:

$$\operatorname{div}\mathbf{T} + \rho\,\mathbf{b} = \rho\,\ddot{\mathbf{u}} \quad \Leftrightarrow \quad T_{ik,k} + \rho\,b_i = \rho\,\ddot{u}_i \tag{7.6.21}$$

Hooke's law for isotropic, linearly elastic materials:

$$\mathbf{T} = 2\mu\left[\mathbf{E} + \frac{\nu}{1-2\nu}(\operatorname{tr}\mathbf{E})\,\mathbf{1}\right] \quad \Leftrightarrow \quad T_{ik} = 2\mu\left[E_{ik} + \frac{\nu}{1-2\nu}E_{jj}\,\delta_{ik}\right] \tag{7.6.22}$$

$$\mathbf{E} = \frac{1}{2\mu}\left[\mathbf{T} - \frac{\nu}{1+\nu}(\operatorname{tr}\mathbf{T})\,\mathbf{1}\right] \quad \Leftrightarrow \quad E_{ik} = \frac{1}{2\mu}\left[T_{ik} - \frac{\nu}{1+\nu}T_{jj}\,\delta_{ik}\right] \tag{7.6.23}$$

The strain-displacement relations:

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T\right) \quad \Leftrightarrow \quad E_{ik} = \frac{1}{2}\left(u_{i,k} + u_{k,i}\right) \tag{7.6.24}$$

The component form of these equations applies only to Cartesian coordinate systems.

If temperature effects have to be included, Hooke's law (7.6.22) should be replaced by the Duhamel-Neumann law (7.5.4). Equations (7.6.21, 7.6.22, 7.6.23, 7.6.24) represent 15 equations for the 15 unknown functions T_{ik}, E_{ik}, and u_i. The boundary conditions are expressed by the contact forces $\mathbf{t}$ and the displacements $\mathbf{u}$ on the surface A of the body. The part of the surface A on which $\mathbf{t}$ is prescribed, is denoted A_σ. On the rest of the surface, denoted A_u, we assume that the displacement $\mathbf{u}$ is prescribed. For static problems ($\ddot{\mathbf{u}} = \mathbf{0}$) the boundary conditions are:

$$\mathbf{t} = \mathbf{T}\cdot\mathbf{n} = \mathbf{t}^* \text{ on } A_\sigma, \quad \mathbf{u} = \mathbf{u}^* \text{ on } A_u \tag{7.6.25}$$

$\mathbf{t}^*$ and $\mathbf{u}^*$ are the prescribed functions, and $\mathbf{n}$ is the unit normal vector on A_σ. We may have cases where the boundary conditions on parts of A are given as combinations of prescribed components of the contact force $\mathbf{t}$ and displacement $\mathbf{u}$. For dynamic problems conditions with respect to time must be added, for instance as initial conditions on the displacement field $\mathbf{u}(\mathbf{r},t)$:

$$\mathbf{u} = \mathbf{u}^{\#}(\mathbf{r},0) \text{ and } \dot{\mathbf{u}} = \dot{\mathbf{u}}^{\#}(\mathbf{r},0) \text{ in } V \tag{7.6.26}$$

V denotes the volume of the body. $\mathbf{u}^{\#}\,(\mathbf{r},0)$ and $\dot{\mathbf{u}}^{\#}\,(\mathbf{r},0)$ are prescribed functions.

In analytical solutions of problems in the theory of elasticity we may use *Saint-Venant's semi-inverse method*. In this method the stresses, displacements and strains are to a certain extent assumed and then completely determined by the basic equations and the boundary conditions. The assumptions are based on reasonable physical considerations, usually related to the state of deformations. Examples on such assumptions are provided by the deformation hypotheses in the elementary beam theory and in the theory of torsion of cylindrical bars. The semi-inverse method may be employed for most problems in continuum mechanics.

It is convenient to transform the basic equations in accordance to which unknown functions we like to choose as primary unknown variables in a problem.

The Displacement Vector u as Primary Unknown Variable

The strain-displacement relation (7.6.24) is introduced in Hooke's law (7.6.22), and the result is:

$$T_{ij} = \mu \left[u_{i,j} + u_{j,i} + \frac{2\nu}{1-2\nu} u_{k,k}\, \delta_{ij} \right] \tag{7.6.27}$$

When these expressions for the stresses are substituted into the Cauchy equations of motion (7.6.21), we obtain the equations of motion in terms of displacements.

$$\nabla^2 \mathbf{u} + \frac{1}{1-2\nu} \nabla (\nabla \cdot\, \mathbf{u}) + \frac{\rho}{\mu} (\mathbf{b} - \ddot{\mathbf{u}}) = \mathbf{0} \quad \Leftrightarrow$$
$$u_{i,kk} + \frac{1}{1-2\nu} u_{k,ki} + \frac{\rho}{\mu} (b_i - \ddot{u}_i) = 0 \tag{7.6.28}$$

These three equations are the *Navier equations* in the general three-dimensional case. When the three u_i – functions are found from the Navier equations, the stresses and the strains may be computed from (7.6.27) and (7.6.24) respectively. The boundary conditions (7.6.25) and the initial conditions (7.6.26) complete the solution to the problem.

In the case of symmetry with respect to a point O the displacement vector $\mathbf{u}$ may be expressed by:

$$\mathbf{u} = u(r)\, \mathbf{e}_r = u_i \mathbf{e}_i\, , \quad u_i = u(r)\, \frac{x_i}{r} \tag{7.6.29}$$

$u(r)$ is the displacement radially from the symmetry point $O, \mathbf{e}_r$ is the unit vector in the radial direction, and u_i are the displacement components in a Cartesian coordinate system Ox. The acceleration vector is now:

$$\ddot{\mathbf{u}} = \ddot{u}(r)\, \mathbf{e}_r = \ddot{u}_i \mathbf{e}_i\, , \quad \ddot{u}_i = \ddot{u}(r)\, \frac{x_i}{r} \tag{7.6.30}$$

The body force will be expressed by a radial component related to the three Cartesian components:

$$\mathbf{b} = b(r)\, \mathbf{e}_r = b_i \mathbf{e}_i\, , \quad b_i = b(r) \frac{x_i}{r} \tag{7.6.31}$$

Referring to spherical coordinates, see Sect. 5.3.3 and the formulas (7.3.17), we find the longitudinal strains:

$$\varepsilon_r = \frac{du}{dr}\, , \quad \varepsilon_\theta = \varepsilon_\phi = \frac{u}{r} \tag{7.6.32}$$

and the volumetric strain:

$$\varepsilon_v = \mathrm{tr}\, \mathbf{E} = u_{i,i} = \varepsilon_r + \varepsilon_\theta + \varepsilon_\phi = \frac{du}{dr} + 2\frac{u}{r} \tag{7.6.33}$$

The displacement field (7.6.29) implies that the radial direction and any direction normal to the radial direction are principal directions of strain. Material line elements in these directions do not rotate due to the deformation, which means that the rotation tensor for small deformations is zero:

$$\tilde{R}_{ik} = \frac{1}{2}\left(u_{i,k} - u_{k,i}\right) = 0 \quad \Rightarrow \quad u_{i,k} = u_{k,i} \quad \Rightarrow$$
$$u_{i,kk} = u_{k,ik} = u_{k,ki} = (\varepsilon_v)_{,i} \tag{7.6.34}$$

Using the result:

$$r^2 = x_i x_i \quad \Rightarrow \quad \frac{\partial r}{\partial x_i} = \frac{x_i}{r} \tag{7.6.35}$$

we can write:

$$(\varepsilon_v)_{,i} = \frac{d\varepsilon_v}{dr}\frac{x_i}{r} \tag{7.6.36}$$

The results (7.6.30, 31, 34, 36) are now used to reduce the Navier equations (7.6.28) to one Navier equation for point symmetric deformation:

$$\frac{d}{dr}\left[\frac{du}{dr} + 2\frac{u}{r}\right] + \frac{1-2\nu}{1-\nu}\frac{\rho}{2G}(b - \ddot{u}) = 0 \tag{7.6.37}$$

Symmetry implies that the state of stress in any particle is expressed by the radial stress σ_r and the stress σ_ϕ on meridian planes. Hooke's law (7.6.22) then gives:

$$\sigma_r = \frac{2G}{1-2\nu}\left[(1-\nu)\frac{du}{dr} + 2\nu\frac{u}{r}\right], \quad \sigma_\phi = \frac{2G}{1-2\nu}\left[\nu\frac{du}{dr} + \frac{u}{r}\right] \tag{7.6.38}$$

Example 7.15. Thick-Walled Spherical Shell with Internal Pressure

A spherical shell with internal radius a and external radius b is subjected to an internal pressure p as shown in Fig. 7.6.2. We want to determine the state of stress $\sigma_r(r)$ and $\sigma_\phi(r)$ and the radial displacement $u(r)$.

The boundary conditions are:

$$\sigma_r(a) = -p\,, \quad \sigma_r(b) = 0 \tag{7.6.39}$$

The Navier equation (7.6.37) in this case becomes:

$$\frac{d}{dr}\left[\frac{du}{dr} + 2\frac{u}{r}\right] = 0 \tag{7.6.40}$$

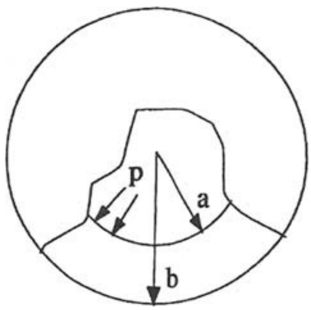


Fig. 7.6.2 Spherical thick-walled shell with internal pressure p

The general solution of this equation is:

$$u(r) = C_1\, r + \frac{C_2}{r^2} \tag{7.6.41}$$

The constants of integration C_1 and C_2 will be determined from the boundary conditions (7.6.39), but first we have to obtain expressions for the stresses using the formulas (7.6.38):

$$\begin{aligned}
\sigma_r(r) &= \frac{2G}{1-2\nu}\left[(1+\nu)\,C_1 - 2\,(1-2\nu)\,\frac{C_2}{r^3}\right] \\
\sigma_\phi(r) &= \frac{2G}{1-2\nu}\left[(1+\nu)\,C_1 + (1-2\nu)\,\frac{C_2}{r^3}\right]
\end{aligned} \tag{7.6.42}$$

When these stresses are substituted into the boundary conditions (7.6.39), we obtain two equations for the unknown constants of integration C_1 and C_2. The solution of the two equations is:

$$C_1 = \frac{1-2\nu}{1+\nu}\,\frac{(a/b)^3}{1-(a/b)^3}\,\frac{p}{2G}\,, \quad C_2 = \frac{a^3}{1-(a/b)^3}\,\frac{p}{4G}$$

The complete solution to the problem is then:

$$\begin{aligned}
u(r) &= \frac{b\,(a/b)^3}{1-(a/b)^3}\left[\frac{1-2\nu}{1+\nu}\,\frac{r}{b} + \frac{1}{2}\left(\frac{b}{r}\right)^2\right]\frac{p}{2G} \\
\sigma_r(r) &= -\frac{1}{1-(a/b)^3}\left[\left(\frac{a}{r}\right)^3 - \left(\frac{a}{b}\right)^3\right]p \quad \leq 0\,, \\
\sigma_\phi(r) &= \frac{1}{1-(a/b)^3}\left[\frac{1}{2}\left(\frac{a}{r}\right)^3 + \left(\frac{a}{b}\right)^3\right]p \quad \geq 0
\end{aligned}$$

The maximum compressive and tensile stresses occur at the inner surface of the shell:

$$\sigma_{\min} = \sigma_r(a) = -p\,, \quad \sigma_{\max} = \sigma_\phi(a) = \frac{\frac{1}{2} + \left(\frac{a}{b}\right)^3}{1-(a/b)^3}\,p$$

If the thickness of the shell $t = b - a$ is very small, such that $a \approx b$, we find that:

$$\sigma_\phi = \frac{a}{2t}\,p \gg \sigma_r\,, \quad t = b - a$$

Confer equation (3.3.16) for a thin–walled shell in Example 3.6.

An interesting special case is obtained if we let the external radius b become very large. Then we get at the inner surface of the shell:

$$\sigma_r(a) = -p\,, \quad \sigma_\phi(a) = \frac{p}{2}$$

which are the stresses in the spherical boundary surface to a spherical cavity in a large elastic body.

In this example it would have been natural to apply the basic equations expressed in spherical coordinate. Due to symmetry we could also have derived the Navier equation (7.6.37) indirectly from the equation of motion of an infinitesimal shell element. But the main purpose of the example has been to demonstrate the application of the basic equations as they have been presented above, and furthermore to illustrate the semi-inverse method, which in this case consists of starting with the displacement (7.6.29).

The Stresses T_{ij} as Primary Unknown Variables

This choice is only natural for static problems: $\ddot{\mathbf{u}} = \mathbf{0}$, or for problems where the acceleration $\ddot{\mathbf{u}}$ is prescribed. In the latter cases we may replace the body force $\mathbf{b}$ by a corrected body force $(\mathbf{b} - \ddot{\mathbf{u}})$, where - $\ddot{\mathbf{u}}$ represents an *extraordinary body force*, also called an *inertia force*. In what follows we assume that $\ddot{\mathbf{u}} = \mathbf{0}$. The Cauchy equations of motion (7.6.21) are three equations for six unknown stresses. Let us assume that we have found a solution for the stresses T_{ij} from of these equations. The strains E_{ij} may then be determined from the Hooke's law (7.6.23) and the displacements u_i should follow from (7.6.24). But it is not certain that the six functions for the strains are compatible and therefore provide unique displacement functions. The six (7.6.24) represent six equations for the three unknown displacement functions u_i. The compatibility equations (5.3.41) must also be satisfied. Since we have chosen the stresses as the primary unknowns, it is natural to express the compatibility equations in terms of stresses. Hooke's law (7.6.23) is used to express the coordinate strains E_{ij} as functions of the coordinate stresses T_{ij}. These relations are substituted into the compatibility equations (5.3.40) and the Cauchy equations (7.6.21) are used to simplify in the process. The result is the *compatibility equations expressed in stresses*, also called the *Beltrami-Michell equations*:

$$
\begin{aligned}
&\nabla^2\mathbf{T} + \frac{1}{1+\nu}\nabla(\nabla \mathrm{tr}\,\mathbf{T}) + \frac{\nu}{1-\nu}[\nabla\cdot\ (\rho\,\mathbf{b})]\,\mathbf{1} + \nabla(\rho\,\mathbf{b}) + (\nabla(\rho\,\mathbf{b}))^T = \mathbf{0} \quad \Leftrightarrow \\
&T_{ij,kk} + \frac{1}{1+\nu}T_{kk,ij} + \frac{\nu}{1-\nu}(\rho\, b_k)_{,k}\delta_{ij} + (\rho\, b_i)_{,j} + (\rho\, b_j)_{,i} = 0
\end{aligned}
\tag{7.6.43}
$$

In principle the problem may now be solved from the three (7.6.21) and the three (7.6.43) with the boundary conditions (7.6.25). A method to simplify the problem is to introduce three stress functions, which satisfy identically the Cauchy equations (7.6.21), and which reduce the compatibility equations (7.6.43) to three equations. Very few three-dimensional problems have been solved by this solution procedure. For two-dimensional problems the (7.6.43) are replaced by (7.3.45) or (7.3.46), and Airy's stress function is introduced to simplify the solution procedure.

7.6.3 The Uniqueness Theorem

The uniqueness theorem. The basic equations (7.6.21, 7.6.22, 7.6.23, 7.6.24) with the boundary conditions (7.6.25) and the initial conditions (7.6.26) have only one solution in the general dynamic case. In a static case the solution is unique if the displacement boundary $A_u \neq 0$, and unique except from an undetermined rigid-body displacement if $A_u = 0$.

The theorem insures that an obtained solution to an elasticity problem is *the only solution* of that problem. The proof of the theorem depends on the elastic energy ϕ per unit volume being a positive definite quantity.

Proof. Let us assume that the following two solutions satisfy the basic equations (7.6.21, 7.6.22, 7.6.23, 7.6.24):

$$\mathbf{u}^{(1)}(\mathbf{r},t) \text{ and } \mathbf{T}^{(1)}(\mathbf{r},t)\,, \quad \mathbf{u}^{(2)}(\mathbf{r},t) \text{ and } \mathbf{T}^{(2)}(\mathbf{r},t) \tag{7.6.44}$$

Because the basic equations are linear, the displacement state and the stress state:

$$\mathbf{u} = \mathbf{u}^{(1)} - \mathbf{u}^{(2)}\,, \quad \mathbf{T} = \mathbf{T}^{(1)} - \mathbf{T}^{(2)} \tag{7.6.45}$$

will also satisfy equation (7.6.21), now with $\mathbf{b} = \mathbf{0}$, i.e. such that: $\mathrm{div}\mathbf{T} = \rho\ddot{\mathbf{u}}$. The displacement $\mathbf{u}$ will also satisfy the boundary conditions and the initial conditions:

$$\mathbf{u}(\mathbf{r},t) = \mathbf{0} \text{ on } A_u \quad \Rightarrow \quad \dot{\mathbf{u}}(\mathbf{r},t) = \mathbf{0} \text{ on } A_u \tag{7.6.46}$$

$$\mathbf{u}(\mathbf{r},0) = \mathbf{0} \text{ and } \dot{\mathbf{u}}(\mathbf{r},0) = \mathbf{0} \text{ in } V \tag{7.6.47}$$

The stress tensor $\mathbf{T}$ will satisfy:

$$\mathbf{t}(\mathbf{r},t) = \mathbf{T} \cdot \mathbf{n} = \mathbf{0} \text{ on } A_\sigma \tag{7.6.48}$$

From the conditions (7.6.46) and (7.6.48) it follows that:

$$\mathbf{t} \cdot \dot{\mathbf{u}} = 0 \text{ on } A \tag{7.6.49}$$

The equation of balance of mechanical energy (6.1.12) gives for the solution (7.6.45), with no body force $\mathbf{b} = \mathbf{0}$ and the boundary condition (7.6.49):

$$\dot{K} + \dot{\Phi} = 0 \tag{7.6.50}$$

where the stress power has been expressed by the total elastic energy Φ, i.e. $P^d = \dot{\Phi}$. The integration of equation (7.6.50) leads to:

$$K + \Phi = C$$

C is a constant of integration, which due to the condition (7.6.47) is equal to zero. In general $K \geq 0$ and $\Phi \geq 0$. Hence we may conclude that:

$$K(t) = \Phi(t) = 0 \tag{7.6.51}$$

The scalars K and Φ are expressed as integrals in which the integrands are positive definite forms with respect to the variables $\mathbf{v} \equiv \dot{\mathbf{u}}$ and $\mathbf{E}$, respectively. The result (7.6.51) therefore implies that:

$$\dot{\mathbf{u}}(\mathbf{r},t) = \mathbf{0} \text{ and } \mathbf{E}(\mathbf{r},t) = \mathbf{0} \quad \text{in } V \tag{7.6.52}$$

The integration of $\dot{\mathbf{u}}$ in (7.6.52) with respect to time and with equation $(7.6.47)_1$ as initial condition yields:

$$\mathbf{u}(\mathbf{r},t) = \mathbf{0} \quad \text{in } V \tag{7.6.53}$$

It now follows from equation (7.6.51) that elastic energy per unit volume is zero, and then from the equations (7.6.52) and (7.6.22) that:

$$\mathbf{T}(\mathbf{r},t) = \mathbf{0} \quad \text{in } V \tag{7.6.54}$$

The results (7.6.53, 7.6.54) show that the two solutions assumed in equation (7.6.44) are identical. This proves the uniqueness theorem.

In a static case time does not enter any of the relevant equations. The displacement $\mathbf{u}$ in equation (7.6.45) must satisfy the boundary condition:

$$\mathbf{u}(\mathbf{r}) = \mathbf{0} \quad \text{on } A_u \tag{7.6.55}$$

and result in stresses satisfying the equilibrium equation and the boundary condition:

$$\operatorname{div}\mathbf{T} = \mathbf{0} \text{ in } V, \quad \mathbf{t}(\mathbf{r}) = \mathbf{T} \cdot \mathbf{n} = \mathbf{0} \text{ on } A_\sigma \tag{7.6.56}$$

We may now write:

$$\mathbf{t} \cdot \mathbf{u} = \mathbf{0} \text{ on } A \quad \Rightarrow \quad \int_A \mathbf{t} \cdot \mathbf{u}\, dA = \int_A \mathbf{u} \cdot \mathbf{T} \cdot \mathbf{n}\, dA = 0$$

which by Gauss' theorem C.3 yields:

$$\int_A \mathbf{u} \cdot \mathbf{T} \cdot \mathbf{n}\, dA = \int_A u_i T_{ij} n_j\, dA = \int_V (u_i T_{ij})_{,j}\, dV = 0 \quad \Rightarrow$$
$$\int_V u_i T_{ij,j}\, dV + \int_V u_{i,j} T_{ij}\, dV = 0 \tag{7.6.57}$$

The first integral vanishes due to the equilibrium equation (7.6.56). The integrand in the second integral may according to equation (7.6.17) be changed to:

$$u_{i,j}T_{ij} = \frac{1}{2}(u_{i,j}T_{ij} + u_{j,i}T_{ji}) = T_{ij}E_{ij} = \mathbf{T} : \mathbf{E} = 2\phi$$

Then equation (7.6.57) has been reduced to:

$$2\int_V \phi \, dV = 0 \tag{7.6.58}$$

In general $\phi \geq 0$, which by equation (7.6.58) implies that $\phi = 0$ in V, which again shows that:

$$\mathbf{E} = \mathbf{0} \ \text{in } V \quad \Rightarrow \quad \mathbf{T} = \mathbf{0} \text{ in } V \tag{7.6.59}$$

Thus the two displacement fields $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ result in the same strains and stresses. The difference between the two displacement fields may only be a small rigid-body displacement, which does not result in strains and changes in the stresses. If the displacement boundary condition applies, i.e. $A_u \neq 0$, the rigid-body displacement must vanish. These arguments prove the uniqueness theorem in a static case.

7.7 Stress Waves in Elastic Materials

In this section we discuss some relatively simple but fundamental aspects of propagation of *stress pulses* or *stress waves* in isotropic, linearly elastic materials. We may also consider these pulses or waves as displacement, deformation, or strain pulses or waves, according to which of these quantities we are interested in. In the general presentation we do not distinguish between pulses and waves, and since the materials are assumed to be elastic, we use the common name *elastic waves*.

Sound in gasses and liquids is elastic waves and will be discussed in Sect. 8.3.3 in the chapter on Fluid Mechanics.

Our analysis starts with longitudinal elastic waves in a bar. Then we treat plane waves in an infinite elastic body. A general discussion of elastic waves in bodies of infinite extent concludes the chapter.

7.7.1 Longitudinal Waves in Cylindrical Bars

Figure 7.7.1 shows an elastic bar of length L and cross-sectional area A. The material has density ρ and modulus of elasticity E. The right end of the bar, at $x = L$, is subjected to an axial force $F(t)$. The force may be given as an impact representing an impulsive force $\hat{F}$ or may be oscillatory, for instance of the form:

$$F(t) = F_o \sin \omega t \tag{7.7.1}$$

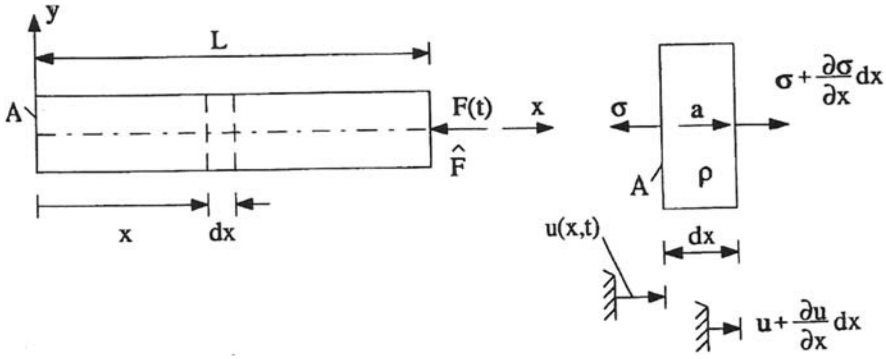


Fig. 7.7.1 Cylindrical bar subjected to a force F(t) or an impulsive force $\hat{F}$ at the end $x = L$

The normal stresses over the cross-section at the end of the bar, $x = L$, will propagate through the rod as what we may call a wave $\sigma(x,t)$ towards the other end of a bar, at $x = 0$. At the end $x = 0$ the stress wave is reflected as a new, reflected wave $\sigma_r(x,t)$. The reflected wave propagates in the positive x-direction towards the end at $x = L$. How the incoming wave $\sigma(x,t)$ is reflected at the end $x = 0$ depends on the boundary condition, i.e. whether the end is free, as indicated in Fig. 7.7.1, or the bar is attached to another body.

As a basis for a simplified analysis of the stress wave propagation problem we make the following assumptions:

1. The cross sections of the bar moves as planes. The displacement in the axial direction is given by:
$$u = u(x,t) \tag{7.7.2}$$
2. The state of stress is uniaxial and given by the normal stress over the cross section of the bar:
$$\sigma = \sigma(x,t) \tag{7.7.3}$$

In addition we assume that the material is linearly elastic with a modulus of elasticity E, and that the deformations are small. Thus we may state:

$$\sigma = E\varepsilon, \quad \varepsilon = \frac{\partial u}{\partial x} \quad \Rightarrow \quad \sigma = E\,\frac{\partial u}{\partial x} \tag{7.7.4}$$

Due to the *Poisson effect*, i.e. the transverse strain due to the Poisson ratio $\nu > 0$, the axial motion will also create motion in the directions normal to the axis of the bar. This motion will result in shear and normal stresses on surfaces parallel to the axis of the bar, and also result in warped cross-sections. We shall discard such secondary effects in the following development of the theory and comment on them at the end of the presentation.

Now we consider a short element of the bar of length dx and mass $dm = \rho \cdot (A \cdot dx)$. The acceleration of the element is $\ddot{u}$. The element is subjected to normal stresses on the two cross-sections. The1. axiom of Euler is applied to the element:

$$\mathbf{f} = m\ddot{\mathbf{u}} \quad \Rightarrow \quad \left[\frac{\partial \sigma}{\partial x} dx\right] \cdot A = [\rho \cdot (A\,dx)] \cdot \ddot{u} \quad \Rightarrow \quad \frac{\partial \sigma}{\partial x} = \rho\,\ddot{u}$$

which by using equation (7.7.4) is transformed to:

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad c = \sqrt{\frac{E}{\rho}} \tag{7.7.5}$$

The partial differential equation (7.7.5) is called a *one-dimensional wave equation*, and the parameter c is called *wave velocity*. The reason for this will be demonstrated below.

The general solution of the wave equation (7.7.5) is given as:

$$u(x,t) = f(ct + x) + g(ct - x) \tag{7.7.6}$$

$f(\alpha)$ and $g(\alpha)$ are two arbitrary functions of one variable α. The solution $u(x,t) = f(ct + x)$ may be interpreted as shown in Fig. 7.7.2. The graph of u in a xu-diagram has constant shape independent of time. The figure shows the graphs at the times t_1 and $t_2 > t_1$. The distance $x_1 - x_2$ between two corresponding points on the graphs is determined by the condition:

$$f(ct_1 + x_1) = f(ct_2 + x_2) \quad \Rightarrow \quad ct_1 + x_1 = ct_2 + x_2 \quad \Rightarrow \quad x_1 - x_2 = c\,(t_2 - t_1)$$

From this result it follows that the displacement $u(x,t) = f(ct + x)$ propagates in the negative x-direction with the velocity c. We therefore call $u(x,t) = f(ct + x)$ a *plane longitudinal displacement wave with the wave velocity* c. We introduce the notation:

$$f'(\alpha) \equiv \frac{df}{d\alpha} \tag{7.7.7}$$

The corresponding strain wave and stress wave are now:

$$\varepsilon(x,t) = \frac{\partial u}{\partial x} = f'(ct + x), \quad \sigma(x,t) = E\varepsilon = E f'(ct + x) \tag{7.7.8}$$

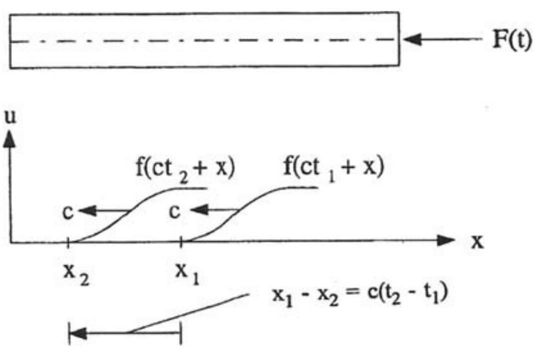


Fig. 7.7.2 Displacement wave $u = f(ct + x)$ in the negative x-direction

The function $f(\alpha)$ is determined by the force $F(t)$ applied at the end $x = L$ of the bar.

$$F(t) = A\sigma(L,t) = AE\varepsilon(L,t) = AEf'(ct+L) \quad \Rightarrow \quad f'(\alpha) = \frac{1}{AE}F([\alpha - L]/c)$$
$$\Rightarrow \quad \sigma(x,t) = E\varepsilon(x,t) = Ef'(ct+x) = \frac{1}{A}F(t+[x-L]/c) \tag{7.7.9}$$

For the harmonic force function (7.7.1), we obtain:

$$\sigma(x,t) = Ef'(ct+x) = \frac{F_o}{A}\sin\left(\left[\frac{2\pi c}{\lambda}\right][ct+x-L]\right) \tag{7.7.10}$$

λ is the *wave length* and defined by:

$$\lambda = \frac{2\pi c}{\omega} \tag{7.7.11}$$

If we choose the initial condition $u(L,0) = 0$, the displacement wave becomes:

$$u(x,t) = f(ct+x) = \frac{F_o}{EA}\frac{\lambda}{2\pi}\left[1 - \cos\left(\left[\frac{2\pi}{\lambda}\right][ct+x-L]\right)\right] \tag{7.7.12}$$

The particle velocity at the cross section x is:

$$v(x,t) = \frac{\partial u}{\partial t} = cf'(ct+x) = \frac{c}{E}\,\sigma(x,t) \tag{7.7.13}$$

The part $u(x,t) = g(ct-x)$ of the general solution (7.7.6) of the wave equation (7.7.5) represents a plane longitudinal displacement wave propagating in the positive x-direction with the wave velocity c given by the formula in (7.7.5). This wave may be established by a reflection at the end $x = 0$ of the incoming displacement wave $u(x,t) = f(ct+x)$. We shall consider two different types of end conditions at $x = 0$:

1) Reflection from a free end at $x = 0$, and
2) Reflexion from a fixed end at $x = 0$.

Reflection from a Free End at $x = 0$

When a displacement wave $u(x,t) = f(ct+x)$ moving in the negative x-direction reaches the free end at $x = 0$, the reflection $u(x,t) = g(ct-x)$ is a displacement wave in the positive x-direction. The total displacement will be the sum of the two waves:

$$u(x,t) = f(ct+x) + g(ct-x) \tag{7.7.14}$$

At the free end, $x = 0$, the resulting stress must vanish at all times:

$$\sigma(0,t) = 0 \quad \Rightarrow \quad \varepsilon(0,t) = \frac{\partial u}{\partial x} = 0 \quad \Rightarrow \quad f'(ct) - g'(ct) = 0 \text{ at all times } t \tag{7.7.15}$$

This result implies that the functions $f(\alpha)$ and $g(\alpha)$ must be equal except for a constant:

$$g(\alpha) = f(\alpha) + C$$

In order to have the natural initial condition that $u(0,0) = 0$, we subtract a constant $f(0)$ from both $f(\alpha)$ and $g(\alpha)$ in equation (7.7.14), and set:

$$u(x,t) = f(ct+x) + f(ct-x) - 2f(0) \tag{7.7.16}$$

Figure 7.7.3 shows the incoming wave $f(ct+x)$, the reflected wave $f(ct-x)$, and the total displacement $u(x,t)$ from equation (7.7.16). In the xu-plane in Fig. 7.7.3 the graphs of the functions $f(ct+x)$ and $f(ct-x)$ are mirror images of each other with respect to the u-axis. The displacement (7.7.16) gives the *axial strain* and *axial stress*:

$$\varepsilon(x,t) = \frac{\partial u}{\partial x} = f'(ct+x) - f'(ct-x) \tag{7.7.17}$$

$$\sigma(x,t) = E\varepsilon = E f'(ct+x) - E f'(ct-x) \tag{7.7.18}$$

The development of the stress function $\sigma(x,t)$ is illustrated in Fig. 7.7.3, where the stress wave $E\ f(ct+x)$ is assumed to represent tension, i.e. a *tension wave*. The reflected stress wave $E\ f(ct-x)$ will then represent compressive stress, that is a *compression wave*. In general:

A tension/compression wave is reflected from a free end as a compression/tension wave.

A heavy impulsive blow to one end of a bar or a detonation of an explosive at the end will initiate a wave of high compressive stress. The reflection of this wave

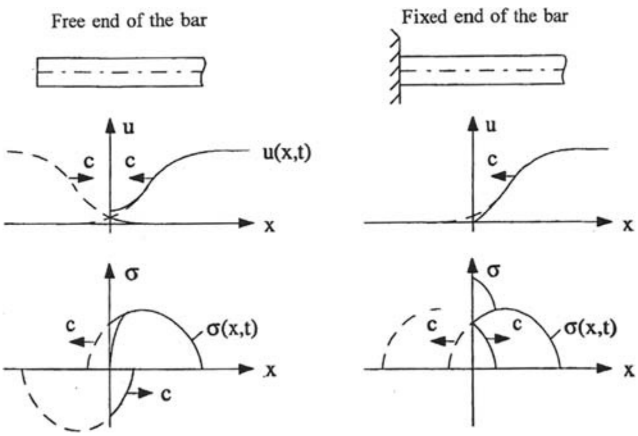


Fig. 7.7.3 Reflection of displacement wave and stress wave from a free and from a fixed end

is a tension wave. Interference of the two waves may result in a total stress of high tension values a short distance from the free end. A bar of a material with a tensile strength that is lower than the compressive strength may therefore experience a tensile fracture a short distance from the free end. We shall return to this phenomenon in Sect. 7.7.7.

Reflection from a Fixed End at $x = 0$

When a displacement wave $f(ct+x)$ reaches a fixed end at $x = 0$, the reflected displacement wave $g(ct-x)$ will be determined by the condition that the combined displacement $u(x,t)$ must be zero at the end:

$$u(0,t) = f(ct) + g(ct) = 0 \text{ at any time } t \tag{7.7.19}$$

This implies that $g(\alpha) = -f(\alpha)$, and hence the total displacement is:

$$u(x,t) = f(ct+x) - f(ct-x) \tag{7.7.20}$$

Figure 7.7.3 illustrates the incoming wave, the reflected wave, and the total displacement. The resulting *axial strain* and *axial stress* are:

$$\varepsilon(x,t) = \frac{\partial u}{\partial x} = f'(ct+x) + f'(ct-x) \tag{7.7.21}$$

$$\sigma(x,t) = E\varepsilon = E f'(ct+x) + E f'(ct-x) \tag{7.7.22}$$

From the expression (7.7.21), and as illustrated in Fig. 7.7.3, it is seen that a stress wave representing tension/compression is reflected as a tensile/compressive stress wave. In general:

A tension/compression wave is reflected from a fixed end as a tension/compression wave.

An interesting consequence of this result will be demonstrated in Sect. 7.7.2.

Critical Comments to the Simplified Theory Presented Above

It was mentioned in the introduction to the theory developed above that the Poisson effect will influence the wave propagation. The assumptions (7.7.2) and (7.7.3), on which the simplified theory is based, neglect the *Poisson effect*. H. Kolsky discusses in "Stress waves in solids" [23] the exact theory for displacement waves in a circular cylindrical rod. The displacement wave is represented by a trigonometric function:

$$u(x,t) = u_o \sin\left(\frac{2\pi}{\lambda}[ct-x]\right) \tag{7.7.23}$$

λ is the wave length. The diameter of the rod is denoted by $2r$. If the ratio $\lambda/r > 10$, the simplified theory presented above is sufficiently accurate, and the wave velocity c is given by the formula $(7.7.5)_2$ and thus independent of the wave length. For $1 < \lambda/r < 10$ the exact theory has to be considered, according to which the wave velocity decreases with the wave length. For wave lengths less than the radius of the rod, $\lambda/r < 1$, the wave velocity again becomes practically constant, but now approximately equal to c_r, which is the wave velocity of the so-called *Rayleigh waves*. The Rayleigh waves are surface waves and are discussed in Sect. 7.7.8. These waves represent motion in the vicinity of free surfaces, and with one displacement component in the direction of the wave propagation and one component in the direction normal to the surface. For steel with $\nu = 0.3,\ E = 210\,\text{GPa}$, and $\rho = 7.83 \cdot 10^3\,\text{kg/m}^3$ we will find:

$$c = \sqrt{\frac{E}{\rho}} = 5179\ \text{m/s},\ c_r = 2952\ \text{m/s}$$

In order to have a numerical example with some relevant numbers we shall consider sound waves with a frequency f between 10 and 10 000 Hz propagating down a steel bar. For steel the velocity of propagation is $c = 5179\,\text{m/sec}$. From the relationship $f = c/\lambda$ we find that the wave length λ lies between 0.52 m and 520m: $0.52\,\text{m} < \lambda < 520\,\text{m}$. Based on the condition that $\lambda_{\text{min}}/r > 10$, we obtain the result $r_{\text{min}} = 52\,\text{mm}$. We may thus conclude that the simplified wave propagation theory presented above, may be used for sound waves of frequencies less than 10 000 Hz in steel bars when the bar diameter is less than 104 mm.

Due to the induced motion perpendicular to the direction of propagation and internal damping, the wave length of displacement waves will increase while the intensity decreases along the direction of propagation. The effect of internal damping will be further discussed in Sect. 9.5 on stress waves in viscoelastic materials.

7.7.2 The Hopkinson Experiment

J. Hopkinson published in 1872 the results of an experiment that showed how the interference of stress waves may lead to fracture. The experiment is discussed by G.I. Taylor in the paper: The Testing of materials at high rates of loading [47]. Figure 7.7.4 shall illustrate the experiment. A steel wire with cross sectional area A and modulus of elasticity E is fixed at one end. The axis of the wire is vertical. The lower end of the wire is attached to a plate with negligible mass, but sufficiently heavy to keep the wire straight. A body of mass m can slide freely on the wire. The body is released from rest from a height h above the plate. The wire used by Hopkinson had static fracture strength of 500 N. The weight of the body could be varied between 31 N and 182 N. The experiment showed that the minimum height h_{min} resulting in fracture, was independent of the mass of the falling body and that the location of the fracture for this height occurred near the upper fixed end.

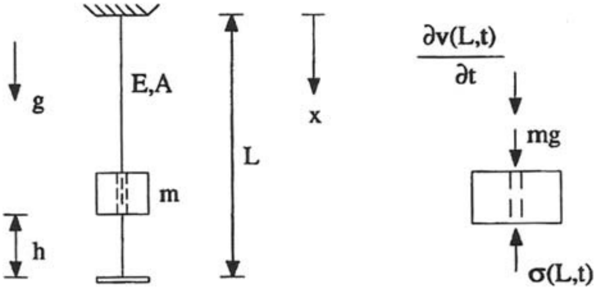


Fig. 7.7.4 The Hopkinson experiment

The Hopkinson theory is as follows. For any given height h the falling body hits the plate at a velocity v_o, which may be determined from the work and energy equation for a rigid body:

$$W = \Delta K \quad \Rightarrow \quad mg \cdot h = \frac{1}{2} m v_o^2 \quad \Rightarrow \quad v_o = \sqrt{2gh} \tag{7.7.24}$$

The velocity v_o is independent of the mass of the body. When the body hits the plate, say at time $t = 0$, the lower end of the wire is suddenly given the velocity v_o. If we denote the particle velocity of the wire at a distance x from the upper end by $v(x,t)$, then:

$$v(L,0) = v_o \tag{7.7.25}$$

From (7.7.13) and the formula for the wave velocity c in (7.7.5) it follows that the velocity wave $v(x,t)$ introduces a tensile stress wave in the wire:

$$\sigma(x,t) = \rho\, c\, v(x,t) \tag{7.7.26}$$

The velocity field $v(x,t)$ and thus the stress field $\sigma(x,t)$ may be determined from the law of motion for the body on the plate, see Fig. 7.7.4:

$$f = m\,a \quad \Rightarrow \quad m\,g - \sigma(L,t) \cdot A = m \frac{\partial v(L,t)}{\partial t} \tag{7.7.27}$$

Using equation (7.7.26), we get:

$$\frac{\partial v(L,t)}{\partial t} + \frac{\rho A c}{m} v(L,t) = g \tag{7.7.28}$$

The solution of equation (7.7.28) that satisfies the condition (7.7.25), is:

$$v(L,t) = \{v_o \exp(-[\rho A/m]\, ct) + (mg/\rho Ac)\,[1 - \exp(-[\rho A/m]\, ct)]\}\, H(t) \tag{7.7.29}$$

The function $H(t)$ is the *Heaviside unit step function*, Oliver Heaviside [1850–1925]:

$$H(t) = \begin{cases} 0 \text{ for } t \leq 0 \\ 1 \text{ for } t > 0 \end{cases} \tag{7.7.30}$$

From the results (7.7.29) and (7.7.26) we get the expression for the stress wave in the wire:

$$\begin{aligned} \sigma(x,t) &= \{\rho c v_o \exp(-[\rho A/m][ct + x - L])\} H(t + (x-L)/c) \\ &\quad + \{(mg/A)[1 - \exp(-[\rho A/m][ct + x - L])]\} H(t + (x-L)/c) \end{aligned} \tag{7.7.31}$$

This is a tensile wave with a front stress $\rho c v_o$. The wave reflected from the fixed upper end of the bar represents also tension. A maximum tensile stress $2\rho c v_o$ will occur at the upper fixed end. Successive reflections from both ends will superimpose and a very complex stress picture develops in the wire. Due to internal material damping and other effects discussed at the conclusion of Sect. 7.7.1, the system wire/rigid body comes to rest after a relatively short time. In the analysis of the experiments performed by J. Hopkinson a theoretical value of maximum tensile stress $4.33\rho c v_o$ occurs after the 3. reflection from the upper fixed end. B. Hopkinson (1905) repeated his father's experiments, with better equipment and a somewhat different objective. In these experiments the theoretical value of the maximum tensile stress is $2.15\rho c v_o$ and occurs after the 2. reflection from the upper end. These results are based only on the first term on the right hand side of equation (7.7.31) and reflections of this stress wave.

7.7.3 Plane Elastic Waves

We assume that a relatively small region in an infinite body of isotropic elastic material is subjected to a mechanical disturbance of some sort. The disturbed region may be considered to be a point source of a displacement field propagating into the undisturbed material as a displacement wave. Sufficiently far away from the source the displacement propagates as a plane wave. Figure 7.7.5 illustrates the situation. In the neighborhood of the x_3-axis and a distance from the source we assume the displacement field:

$$\mathbf{u} = \mathbf{u}(x_3, t) \quad \Leftrightarrow \quad u_i = u_i(x_3, t) \tag{7.7.32}$$

This field really represents three plane waves: a *longitudinal wave* $u_3(x_3,t)$ and two *transverse waves* $u_1(x_3,t)$ and $u_2(x_3,t)$. We shall see that the wave velocities are different for the two types of waves.

The equations of motion in this case are the Navier equations (7.6.28). If we neglect the body forces, the equations are reduced to:

$$u_{i,kk} + \frac{1}{1-2\nu} u_{k,ki} = \frac{\rho}{\mu} \ddot{u}_i \tag{7.7.33}$$

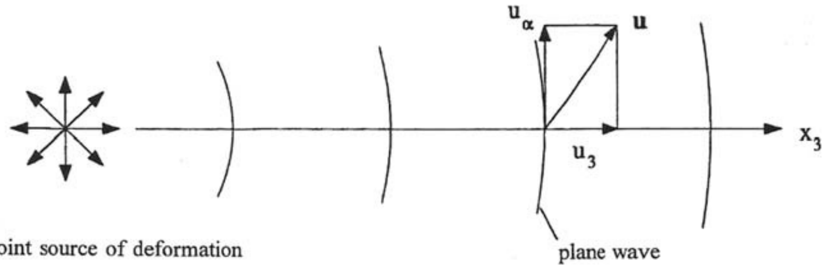


Fig. 7.7.5 Plane displacement wave from a distant point source

The displacement field (7.7.32) is substituted into the Navier equations (7.7.33), and the result is three one-dimensional wave equations:

$$c_l \frac{\partial^2 u_3}{\partial x_3^2} = \frac{\partial^2 u_3}{\partial t^2}, \quad c_l = \sqrt{\frac{2(1-\nu)}{1-2\nu}\frac{\mu}{\rho}} = \sqrt{\frac{1-\nu}{(1+\nu)(1-2\nu)}\frac{\eta}{\rho}} = \sqrt{\frac{\kappa+4\mu/3}{\rho}} \tag{7.7.34}$$

$$c_t \frac{\partial^2 u_\alpha}{\partial x_3{}^2} = \frac{\partial^2 u_\alpha}{\partial t^2}, \quad c_t = \sqrt{\frac{\mu}{\rho}} = \sqrt{\frac{1}{2(1+\nu)}\frac{\eta}{\rho}} \tag{7.7.35}$$

$\mu(=G)$ is the shear modulus, $\eta(=E)$ is the modulus of elasticity, and κ is the bulk modulus. The wave equations (7.7.34) and (7.7.35) have general solutions of the form (7.7.6).

The displacement $u_3(x_3,t)$ represents a motion in the direction of the propagation x_3, and c_l is the *wave velocity of longitudinal waves*. The displacements $u_\alpha(x_3,t)$ represent motions in the directions normal to the direction of propagation, and c_t is the *wave velocity of transverse waves*. Because $0 \le \nu \le 0.5$ we will find that in general:

$$c_t < c_l \tag{7.7.36}$$

For steel with $\nu = 0.3$, $E = 210\,\text{GPa}$, and $\rho = 7.83 \cdot 10^3\,\text{kg/m}^3$, we find $c_t = 3212\,\text{m/s}$, $c_l = 6009\,\text{m/s}$, and $c = 5179\,\text{m/s}$ (from 7.7.5).

The longitudinal wave implies volume changes: $\varepsilon_v = u_{3,3} = E_{33}$, and the wave is therefore called a *dilatational wave* or *volumetric wave*. Because:

$$\text{rot}\,(u_3\,\mathbf{e_3}) \equiv \text{curl}\,(u_3\,\mathbf{e_3}) \equiv \nabla \times (u_3\,\mathbf{e_3}) = \mathbf{0}$$

the displacement $u_3(x_3,t)$ is also called an *irrotational wave*. A physical consequence of an irrotational wave is that the principal directions of strain do not rotate. For the displacement field $u_3(x_3,t)$ the x_i-directions are the principal strain directions. According to Hooke's law expressed by the equations (7.6.27) the stresses are determined by:

$$T_{33} = \frac{2(1-\nu)\mu}{1-2\nu} u_{3,3}, \quad T_{11} = T_{22} = \frac{\nu}{1-\nu} T_{33} \tag{7.7.37}$$

The transverse waves $u_1(x_3,t)$ and $u_2(x_3,t)$ are isochoric, i.e. $\varepsilon_v = 0$, and are therefore also called *dilatation free waves* or *equivoluminal waves*. The stresses are according to Hooke's law (7.6.27):

$$T_{13} = \mu u_{1,3}, \quad T_{23} = \mu u_{2,3} \tag{7.7.38}$$

Because these waves also represent shear stresses on planes normal to the direction of propagation, they are also called *shear waves*. Other names are *distortional waves* and *rotational waves*.

7.7.4 Elastic Waves in an Infinite Medium

We consider a body of an isotropic, linearly elastic material. The extension of the body is so large that the influence from boundary surfaces to other media may be neglected. It is assumed that the material region bounded by a closed surface A is in motion, while the material outside A is undisturbed and at rest. The surface A will spread out into the previously undisturbed region. The motion of the surface is now called a *wave*, and the surface A is called the *wave front*. We will find that the mathematical points on A have a constant velocity in the direction normal to A. This *wave velocity* c is independent of the shape of the wave front A. A further investigation will show that A represents two surfaces: 1) a longitudinal wave front for particle motions normal to the wave front, and which represents volume changes and has the wave velocity c_l, and 2) a transverse wave front for motions parallel to the wave front, and which represents shear stresses on the wave front and rotation, and has the wave velocity c_t.

7.7.5 Seismic Waves

An earth quake initiates three elastic displacement waves. The fastest wave is a longitudinal wave, a volumetric wave, called the *primary wave*, the *P-wave*. The second fastest wave is a transverse wave, a shear wave, called the *secondary wave*, the *S-wave*. Both waves propagate from the earth quake region in all directions, and their intensities, or energy per unit area, decrease with the square of the distance from the earth quake. These two waves therefore are registered by relatively weak signal on a seismograph far away from the epicenter of the earth quake. The third and generally the strongest wave propagates along the surface of the earth and represents a combination of a Rayleigh wave and a *Love wave*.

Rayleigh waves, presented below in Sect. 7.7.8, are surface waves with displacement components in the direction of propagation and in the direction normal to the surface, the latter component being the strongest. The intensity of the Rayleigh waves will in principle decrease with the distance from the surface. The propagation velocity c_r is somewhat smaller than c_t, approximately 10% smaller, depending upon the Poisson ratio of the medium. In the neighborhood of the free surface of a homogeneous material the Rayleigh wave is the only possible surface wave. The wave is named after Lord Rayleigh, 3rd baron (John William Strutt) [1842–1919].

Love waves are surface waves, which near the surface of the earth are equally important as the Rayleigh waves, and represent displacements in the surface and in the direction normal to the direction of propagation. Love waves were introduced by A. E. H. Love [1863–1940]. This type of surface waves is also called *SH-waves* (surface - horizontal). The Love waves may be explained by considering a region of the upper surface of the earth to have different material properties than the rest of the earth surface. The velocity of the Love waves is somewhat smaller than c_t for the earth surface and lies between the transverse wave velocity c'_t for the earth crust and c_t. A condition for the existence of Love waves is that $c'_t < c_t$. For a harmonic wave the wave velocity depends on the wavelength. This means that non-harmonic waves are distorted as they travel through the material, a phenomenon called dispersion.

The surface waves must travel a longer distance than the *P*-waves and the *S*-waves, and are therefore registered somewhat later. The energy per unit area of the surface waves decreases proportionally to the travelled distance. This fact explains that the surface waves give a relatively stronger seismic signal than the primary and secondary waves.

7.7.6 Reflection of Elastic Waves

When a plane volumetric wave meets a plane free surface, it is reflected as a plane volume wave and a plane shear wave, see Fig. 7.7.6. It may be shown that angle α_2 between the surface normal and the direction of propagation of the reflected dilatational wave is equal to the angle α_1 between the surface normal and the direction of the incoming volumetric wave. For the angle β_2 of the reflected shear wave we will find:

$$\frac{\sin\beta_2}{\sin\alpha_1} = \frac{c_t}{c_l} < 1 \tag{7.7.39}$$

The intensity of the shear wave approaches zero as the angle α_1 approaches 0°. The intensity of the reflected volumetric wave is equal to the intensity of the incoming wave when the angle $\alpha_1 = 0$, otherwise it is less. Similarly as for reflection of waves from a free end of a bar, we find that a tensile/compressive wave is reflected as a compressive/tensile wave.

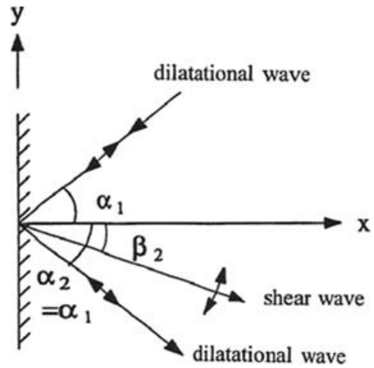


Fig. 7.7.6 Reflection of a dilatation wave from a free surface

When a plane shear wave meets a plane free surface, the displacement component parallel to the surface is reflected as a shear wave and such that the angle β_2 of the reflected wave is equal to the angle β_1 of the incoming wave. The wave intensity is unchanged but the phase, that is the direction of shear strain/stress, is opposite. In Fig. 7.7.7 this displacement component is in the in z-direction that is normal to the figure plane. The displacement component parallel to the xy – plane in Fig. 7.7.7 is reflected as two waves: a shear wave and a dilatational wave. The angle of the reflected shear wave is equal to the angle of the incoming wave: $\beta_2 = \beta_1$. The angle α_2 of the reflected dilatational wave is given by:

$$\frac{\sin\alpha_2}{\sin\beta_1} = \frac{c_l}{c_t} > 1 \tag{7.7.40}$$

7.7.7 Tensile Fracture Due to Compression Wave

Figure 7.7.8 shows a thick steel plate. At the position P on the top face of the plate an explosive charge is detonated, resulting in a compressive stress wave of very high

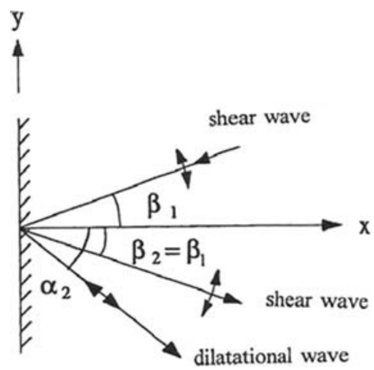


Fig. 7.7.7 Reflection of shear waves from a free surface

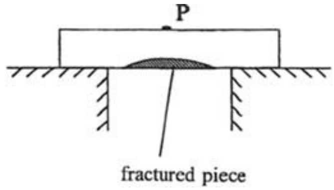


Fig. 7.7.8 Fracture due to reflected tensile wave

intensity. From the free bottom face of the plate the compression wave is reflected as a tension wave. The two waves combine, and a short distance inside the plate from the bottom face the tensile stresses may become larger than the tensile strength of the material, resulting in a fracture surface, with the result that a piece of the plate, indicated by the hatched area in Fig. 7.7.8 is detached from the plate and falls off

7.7.8 Surface Waves. Rayleigh Waves

In the vicinity of a free surface of an elastic body elastic waves are propagated in a special way. Figure 7.7.9 shows a free surface of a body at $y = 0$. We assume that the body, for $y \geq 0$, has plane displacements represented by the displacement field:

$$u_3 = 0, \quad u_\alpha = u_\alpha(x, y, t) \tag{7.7.41}$$

It is further assumed that the deformation is concentrated near the free surface, and we take as a condition for the displacement field that:

$$u_\alpha \to 0 \text{ as } y \to \infty \tag{7.7.42}$$

Using the formulas (7.7.34, 7.7.35) for the wave velocities, we may write the Navier equations (7.7.33) as:

$$c_t{}^2 u_{\alpha,\beta\beta} + \left(c_l{}^2 - c_t{}^2\right) u_{\beta,\beta\alpha} = \ddot{u}_\alpha \tag{7.7.43}$$

The condition of a stress free surface at $y = 0$ provides the following boundary conditions for the displacement field u_α, where Hooke's law for plane displacements (7.3.29) has been used:

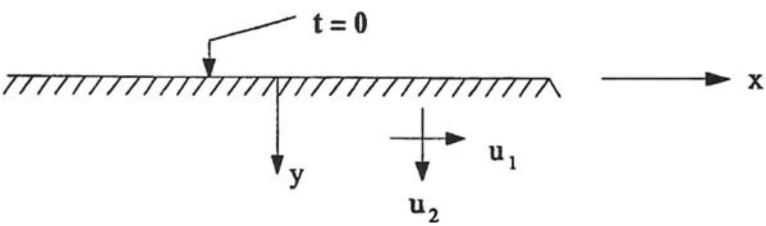


Fig. 7.7.9 Elastic body with free surface

$$T_{22}|_{y=0} = 0 \quad \Rightarrow \quad (1-\nu)\,u_{2,2} + \nu u_{1,1} = 0 \text{ at } y = 0 \tag{7.7.44}$$

$$T_{12}|_{y=0} = 0 \quad \Rightarrow \quad u_{1,2} + u_{2,1} = 0 \text{ at } y = 0 \tag{7.7.45}$$

The solution of the differential equation (7.7.43) satisfying the boundary conditions (7.7.42, 44, 45), is obtained as the following displacement waves, called *Rayleigh waves*:

$$u_1(x,y,t) = A\left[\exp(-ay) - \frac{2ab}{k^2+b^2}\exp(-by)\right]\sin\left[k\,(c_r t - x)\right] \tag{7.7.46}$$

$$u_2(x,y,t) = A\frac{a}{k}\left[\exp(-ay) + \frac{2k^2}{k^2+b^2}\exp(-by)\right]\cos\left[k\,(c_r t - x)\right] \tag{7.7.47}$$

A and k are undetermined constants, and a and b are given by the formulas:

$$a = k\sqrt{1-\left(\frac{c_r}{c_l}\right)^2}, \quad b = k\sqrt{1-\left(\frac{c_r}{c_t}\right)^2} \tag{7.7.48}$$

The wave speed c_r is determined from the cubic equation:

$$\left(\frac{c_r}{c_l}\right)^6 - 8\left(\frac{c_r}{c_t}\right)^4 + 24\left(\frac{c_r}{c_t}\right)^2 - 16\left(\frac{c_r}{c_l}\right)^2 - 16\left[1-\left(\frac{c_t}{c_l}\right)^2\right] = 0 \tag{7.7.49}$$

For the Poisson's ratio $\nu = 0$, we obtain $c_l = \sqrt{2}\,c_t$ and the solution of (7.7.49) only gives one acceptable value for the wave speed: $c_r = 0.874\ c_t$. *The* other two roots of equation (7.7.49) give complex values for a and b. For incompressible materials, $\nu = 0.5$, we find $c_r = 0.955\ c_t$.

7.8 Anisotropic Materials

In this section we shall discuss anisotropic, linearly elastic materials having simple symmetries. The presentation will cover some types of elastic crystals, wood materials, biological materials like bones, and fiber reinforced materials.

It is convenient in the presentation to introduce a special notation for coordinate stresses and coordinate strains referred to a Cartesian coordinate system Ox:

$$\begin{aligned} &T_1 \equiv T_{11} \equiv \sigma_x, \quad T_2 \equiv T_{22} \equiv \sigma_y, \quad T_3 \equiv T_{33} \equiv \sigma_z \\ &T_4 \equiv T_{23} \equiv \tau_{yz}, \quad T_5 \equiv T_{31} \equiv \tau_{zx}, \quad T_6 \equiv T_{12} \equiv \tau_{xy} \end{aligned} \tag{7.8.1}$$

$$\begin{aligned} &E_1 \equiv E_{11} \equiv \varepsilon_x, \quad E_2 \equiv E_{22} \equiv \varepsilon_y, \quad E_3 \equiv E_{33} \equiv \varepsilon_z \\ &E_4 \equiv 2E_{23} \equiv \gamma_{yz}, \quad E_5 \equiv 2E_{31} \equiv \gamma_{zx}, \quad E_6 \equiv 2E_{12} \equiv \gamma_{xy} \end{aligned} \tag{7.8.2}$$

The material response of a fully anisotropic, linearly elastic material is defined by the constitutive equations:

$$T_\alpha = S_{\alpha\beta} E_\beta \quad \Leftrightarrow \quad T = SE \tag{7.8.3}$$

when Greek indices represent the numbers 1 to 6. T and E are (6×1) vector matrices, and S is a (6×6) *elasticity matrix* or *stiffness matrix*.

$$S \equiv \left(S_{\alpha\beta}\right) \equiv \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{pmatrix} \tag{7.8.4}$$

The 36 elements of S are called *elasticities* or *stiffnesses*. It will be shown that if the material is hyperelastic, the stiffness matrix S is symmetric: $S^T = S$, such that only 21 stiffnesses are independent for full anisotropy.

The constitutive equation of a linear elastic material may also be presented as:

$$E_\alpha = K_{\alpha\beta} T_\beta \quad \Leftrightarrow \quad E = KT \tag{7.8.5}$$

$K = [K_{\alpha\beta}]$ is a (6×6) *compliance matrix* or *flexibility matrix*. The elements $K_{\alpha\beta}$ are called *compliances* or *flexibilities*. It follows from (7.8.3) and (7.8.5) that S and K are inverse matrices:

$$K = S^{-1} \tag{7.8.6}$$

If the material is hyperelastic only 21 compliances are independent for full anisotropy.

The stiffness matrix and the compliance matrix of an isotropic, linearly elastic material, i.e. a Hookean material, are:

$$S = \frac{\mu}{1-2\nu} \begin{pmatrix} 2(1-\nu) & 2\nu & 2\nu & 0 & 0 & 0 \\ 2\nu & 2(1-\nu) & 2\nu & 0 & 0 & 0 \\ 2\nu & 2\nu & 2(1-\nu) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{pmatrix} \tag{7.8.7}$$

$$K = \frac{1}{2\mu(1+\nu)} \begin{pmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{pmatrix} \tag{7.8.8}$$

Note that S and K are symmetric matrices. We shall show in the following section that this a general property of anisotropic, hyperelastic materials.

7.8.1 Hyperelasticity

For a hyperelastic material the stress tensor may, according to (7.6.10), be derived from the elastic energy ϕ per unit volume:

$$\mathbf{T} = \frac{\partial \phi}{\partial \mathbf{E}} \quad \Leftrightarrow \quad T_{ij} = \frac{\partial \phi}{\partial E_{ij}} \quad \Leftrightarrow \quad T_\alpha = \frac{\partial \phi}{\partial E_\alpha} \tag{7.8.9}$$

Note that if we differentiate ϕ with respect to the coordinate strains E_{ij} we must treat E_{ij} as 9 independent variables; confer the development of (7.6.10). For a linearly hyperelastic material (7.8.3) and (7.8.9) imply:

$$T_\alpha = S_{\alpha\beta} E_\beta = \frac{\partial \phi}{\partial E_\alpha} \tag{7.8.10}$$

Because:

$$\frac{\partial^2 \phi}{\partial E_\beta \, \partial E_\alpha} = \frac{\partial^2 \phi}{\partial E_\alpha \, \partial E_\beta}$$

it follows from (7.8.10) that:

$$S_{\alpha\beta} = S_{\beta\alpha} \quad \Leftrightarrow \quad S^T = S \tag{7.8.11}$$

Thus, the stiffness matrix S is symmetric. This property reduces the number of independent stiffnesses from 36 for a fully anisotropic, linearly elastic material to 21 for a fully anisotropic, linearly hyperelastic material.

For a hyperelastic material the strain tensor may, according to (7.6.15), be derived from the complementary energy ϕ_c per unit volume.

$$\mathbf{E} = \frac{\partial \phi_c}{\partial \mathbf{T}} \quad \Leftrightarrow \quad E_{ij} = \frac{\partial \phi_c}{\partial T_{ij}} \quad \Leftrightarrow \quad E_\alpha = \frac{\partial \phi_c}{\partial T_\alpha} \tag{7.8.12}$$

For a linearly hyperelastic material (7.8.5) and (7.8.12) imply:

$$K_{\alpha\beta} = K_{\beta\alpha} \quad \Leftrightarrow \quad K^T = K \tag{7.8.13}$$

The compliance matrix is symmetric. Since the stiffness matrix S is symmetric, this result also follows from the relation (7.8.6).

Partial integrations of equations (7.8.10) and (7.8.12), with the boundary conditions:

$$\phi = \phi_c = 0 \quad \text{for} \quad E_\alpha = T_\alpha = 0 \tag{7.8.14}$$

result in the following expressions for the elastic energy and the complementary energy per unit volume for *linearly hyperelastic materials*, defined by the (7.8.4) and (7.8.8):

$$\phi = \phi_c = \frac{1}{2}E_\alpha S_{\alpha\beta} E_\beta = \frac{1}{2}E^T S E = \frac{1}{2}E^T T = \frac{1}{2}T_\alpha K_{\alpha\beta} T_\beta = \frac{1}{2}T^T K T = \frac{1}{2}T^T E \tag{7.8.15}$$

The stiffnesses $S_{\alpha\beta}$ and the compliances $K_{\alpha\beta}$ are determined by testing under uniaxial stress, biaxial stress, and pure shear stress. In these tests so-called *engineering parameters* are introduced: moduli of elasticity, Poisson's ratios, normal stress couplings, and shear moduli. Section 7.8.3, in an example with biomaterial, and Sect. 7.9 on composite materials, present some engineering parameters and relate them to the stiffnesses $S_{\alpha\beta}$ and the compliances $K_{\alpha\beta}$.

7.8.2 Materials with one Plane of Symmetry

If the structure of the material in a particle is symmetric with respect to a plane through the particle, such that the mirror image of the structure with respect to the plane is identical to the structure itself, the number of stiffnesses, and compliances, is reduced from 21 to 13.

Crystals with one symmetry plane are called *monoclinic crystals*. Materials having fibrous structure and with one characteristic fiber direction are symmetric about the plane normal to the fibers.

Figure 7.8.1a shows a volume element of a material with one plane of symmetry normal to the $\mathbf{e}_3$-direction. I Figure 7.8.1b the element is rotated 180° about the $\mathbf{e}_3$-direction. Two material lines symmetrically placed with respect to the plane of symmetry are drawn in the element. A state of strain $\mathbf{E}$ will give the same state of stress $\mathbf{T}$ whether it is introduced to the element before or after the rotation. In other words: the material is insensitive to a rotation 180° about the normal to the plane

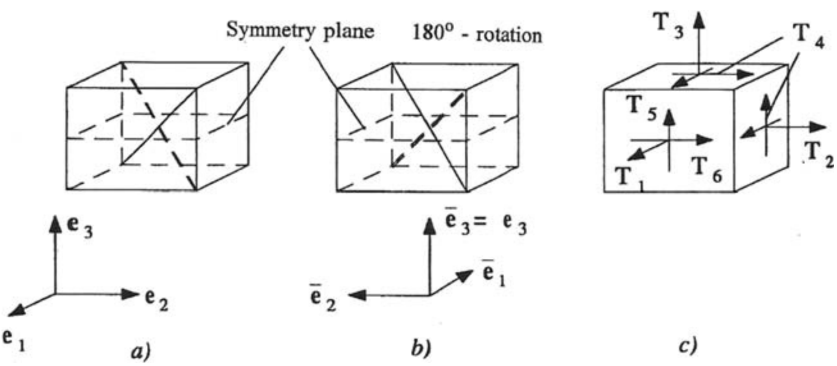


Fig. 7.8.1 Material with one plane of symmetry

of symmetry before the material is subjected to the strain $\mathbf{E}$. With respect to the coordinate directions $\mathbf{e}_i$ the coordinate stresses, see Fig. 7.8.1c, and the coordinate strains are:

$$T = \{T_1, T_2, T_3, T_4, T_5, T_6\}, \quad E = \{E_1, E_2, E_3, E_4, E_5, E_6\} \tag{7.8.16}$$

With respect to the coordinate directions $\bar{\mathbf{e}}_i$ the same state of strain $\mathbf{E}$ and stress $\mathbf{T}$ applied to the element in Fig. 7.8.1a is represented by the coordinate stresses and the coordinate strains:

$$\bar{T} = \{T_1, T_2, T_3, -T_4, -T_5, -T_6\}, \quad \bar{E} = \{E_1, E_2, E_3, -E_4, -E_5, -E_6\} \tag{7.8.17}$$

The constitutive equations expressed with respect to the directions $\mathbf{e}_i$ and $\bar{\mathbf{e}}_i$ are:

$$T_\alpha = S_{\alpha\beta} E_\beta, \quad \bar{T}_\alpha = \bar{S}_{\alpha\beta} \bar{E}_\beta \tag{7.8.18}$$

Since the configurations in Fig. 7.8.1a and b are equivalent as far as elastic properties are concerned, the stiffness matrices related to the directions $\mathbf{e}_i$ and $\bar{\mathbf{e}}_i$ must be identical:

$$\bar{S} = S \quad \Leftrightarrow \quad \bar{S}_{\alpha\beta} = S_{\alpha\beta} \tag{7.8.19}$$

We now choose special values for special Greek indices:

$$\lambda = 4 \text{ and } 5, \ \rho, \gamma = 1, 2, 3, 6$$

The constitutive equations (7.8.18) are expressed as:

$$\begin{aligned} &T_\alpha = S_{\alpha\beta} E_\beta \quad \Rightarrow \\ &T_\lambda = S_{\lambda\rho} E_\rho + S_{\lambda 4} E_4 + S_{\lambda 5} E_5, \quad T_\gamma = S_{\gamma\rho} E_\rho + S_{\gamma 4} E_4 + S_{\gamma 5} E_5 \end{aligned} \tag{7.8.20}$$

$$\begin{aligned} &\bar{T}_\alpha = \bar{S}_{\alpha\beta} \bar{E}_\beta \quad \Rightarrow \\ &-T_\lambda = S_{\lambda\rho} E_\rho + S_{\lambda 4}(-E_4) + S_{\lambda 5}(-E_5), \ T_\gamma = S_{\gamma\rho} E_\rho + S_{\gamma 4}(-E_4) + S_{\gamma 5}(-E_5) \end{aligned} \tag{7.8.21}$$

When (7.8.20) is compared with (7.8.21), we find that:

$$S_{\lambda\rho} = S_{\gamma 4} = S_{\gamma 5} = 0 \quad \Rightarrow \quad S_{4\rho} = S_{5\rho} = S_{\rho 4} = S_{\rho 5} = 0 \quad \text{for } \rho = 1, 2, 3, 6 \tag{7.8.22}$$

Thus the stiffness matrix for materials having one symmetry plane is:

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ & S_{22} & S_{23} & 0 & 0 & S_{26} \\ & & S_{33} & 0 & 0 & S_{36} \\ & symmetry & & S_{44} & S_{45} & 0 \\ & & & & S_{55} & 0 \\ & & & & & S_{66} \end{pmatrix} \tag{7.8.23}$$

The stiffness matrix contains 13 *independent stiffnesses*. The corresponding compliance matrix likewise contains 13 *independent compliances*.

7.8.3 Three Orthogonal Symmetry Planes. Orthotropy

If an elastic material has a structure that is mirror symmetrical about two orthogonal planes, the number of independent elasticities is reduced from 13, for one plane of symmetry, to 9. We shall see that two orthogonal symmetry planes imply three orthogonal planes of symmetry.

Let the two planes of symmetry be normal to the $\mathbf{e}_3$- and $\mathbf{e}_2$-directions in Fig. 7.8.1. We then find, using the same arguments that gave the results in equations (7.8.22), that:

$$\begin{aligned} &S_{4\rho} = S_{5\rho} = S_{6\rho} = S_{\rho 4} = S_{\rho 5} = S_{\rho 6} = 0 \quad \text{for } \rho = 1,2,3 \\ &S_{45} = S_{54} = S_{46} = S_{64} = S_{65} = S_{56} = 0 \end{aligned} \tag{7.8.24}$$

The stiffness matrix is now:

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ & S_{22} & S_{23} & 0 & 0 & 0 \\ & & S_{33} & 0 & 0 & 0 \\ & symmetry & & S_{44} & 0 & 0 \\ & & & & S_{55} & 0 \\ & & & & & S_{66} \end{pmatrix} \tag{7.8.25}$$

The matrix contains 9 *independent stiffnesses*. The corresponding compliance matrix has 9 *independent compliances*.

It follows from the result (7.8.25) that the material is symmetric also with respect to a plane normal to the $\mathbf{e}_1$-direction. Hence, two orthogonal planes of symmetry imply three orthogonal planes of symmetry. This type of symmetry is called *orthotropy*.

Materials having fiber structures and one distinct fiber direction may be orthotropic. If the fibers are directed in three orthogonal directions, the material is orthotropic. Wood gives an example of an approximate orthotropic material. Many crystals are orthotropic.

The elements of the stiffness matrix S in (7.8.25) and the elements $K_{\alpha\beta} = K_{\beta\alpha}$ in the compliance matrix:

$$K = \begin{pmatrix} K_{11} & K_{12} & K_{13} & 0 & 0 & 0 \\ & K_{22} & K_{23} & 0 & 0 & 0 \\ & & K_{33} & 0 & 0 & 0 \\ & symmetry & & K_{44} & 0 & 0 \\ & & & & K_{55} & 0 \\ & & & & & K_{66} \end{pmatrix} \tag{7.8.26}$$

may be determined experimentally from uniaxial tests and pure shear tests. For uniaxial stress T_1 in the direction $\mathbf{e}_1$ normal to one of the symmetry planes the relation $E = KT$ results in three longitudinal strains:

$$\begin{aligned} &E_1 = K_{11} T_1 = \frac{1}{\eta_1} T_1 \\ &E_2 = K_{21} T_1 = -\nu_{21} E_1 = -\frac{\nu_{21}}{\eta_1} T_1, \quad E_3 = K_{31} T_1 = -\nu_{31} E_1 = -\frac{\nu_{31}}{\eta_1} T \end{aligned} \tag{7.8.27}$$

η_1 is a *modulus of elasticity*, and ν_{21} and ν_{31} are *Poisson's ratios*. Similar expressions are formed for a uniaxial stress T_2 in the direction $\mathbf{e}_2$ and a uniaxial stress T_3 in the direction $\mathbf{e}_3$. These expressions introduces the *moduli of elasticity* η_2 and η_3 and the *Poisson's ratios* $\nu_{12}, \nu_{32}, \nu_{13}$, and ν_{23} Because K must be a symmetric matrix, the following relations must be satisfied:

$$\frac{\nu_{21}}{\eta_1} = \frac{\nu_{12}}{\eta_2}, \quad \frac{\nu_{31}}{\eta_1} = \frac{\nu_{13}}{\eta_3}, \quad \frac{\nu_{32}}{\eta_2} = \frac{\nu_{23}}{\eta_3} \tag{7.8.28}$$

Pure states of shear T_4, T_5, or T_6 with respect to two orthogonal directions $\mathbf{e}_i$ and $\mathbf{e}_j$ will only result in shear strains:

$$E_4 = K_{44} T_4 = \frac{1}{\mu_4} T_4, \quad E_5 = K_{55} T_5 = \frac{1}{\mu_5} T_5, \quad E_6 = K_{66} T_6 = \frac{1}{\mu_6} T_6 \tag{7.8.29}$$

where μ_4, μ_5, and μ_6 are *shear moduli*. The compliance matrix in (7.8.26) may now be represented by:

$$K = \begin{pmatrix} 1/\eta_1 & -\nu_{12}/\eta_2 & -\nu_{13}/\eta_3 & 0 & 0 & 0 \\ & 1/\eta_2 & -\nu_{23}/\eta_3 & 0 & 0 & 0 \\ & & 1/\eta_3 & 0 & 0 & 0 \\ & symmetry & & 1/\mu_4 & 0 & 0 \\ & & & & 1/\mu_5 & 0 \\ & & & & & 1/\mu_6 \end{pmatrix} \tag{7.8.30}$$

The 9 independent compliances in the expression (7.8.26) are now replaced by 9 *engineering parameters*: 3 moduli of elasticity, 3 shear moduli, and 3 Poisson's ratios. The 3 remaining Poisson's ratios are found from the three equations (7.8.28). When the compliance matrix K has been found, the stiffness matrix S may be found by inversion. An example of values of elastic parameters for fiber reinforced composites is presented in Sect. 7.9.

Example 7.16. Elastic Parameters of Human Femur and Bovine Femur

Bone as an elastic material is considered to be orthotropic. The table below shows elastic parameters for human femur (thighbone) found from standard material testing (Reilly, D. T. and Burstein, A. H., 1975), and elastic parameters from bovine (ox) femur found from ultra sound testing (Burris, 1983). The $\mathbf{e}_1$-direction represents the long axis of the bone, $\mathbf{e}_2$ is in the radial direction, and $\mathbf{e}_3$ is in the circumferential direction.

Elastic parameters of human femur and bovine femur

	η_1 [GPa]	η_2 [GPa]	η_3 [GPa]	μ_4 [GPa]	μ_5 [GPa]	μ_6 [GPa]	ν_{12}	ν_{13}	ν_{23}	ν_{21}	ν_{31}	ν_{32}
Human	11.5	11.5	17.0	3.3	3.3	3.6	0.58	0.46	0.46	0.58.	0.31	0.31
Bovine	10.79	12.24	18.90	5.96	4.47	3.38	0.51	0.42	0.33	0.45	0.24	0.22

The formulas (7.8.28) have been used to obtain some of the parameters in the table.

7.8.4 Transverse Isotropy

A material is transverse isotropic if an *axis of symmetry* exists with respect to the elastic properties through every particle. A symmetry axis implies that every plane through the axis is a plane of symmetry. Materials with fiber structure may exhibit such properties, For example is this approximately true for wood, for which the direction of the grains may be considered an axis of symmetry. Transverse isotropy implies orthotropy, but the reverse is not true.

The number of independent stiffnesses or compliances is 5 for transverse isotropy. To show this, we start with the stiffness matrix (7.8.25) for orthotropy. The axis of symmetry must be parallel with one of the $\mathbf{e}_i$ - directions. A 90°-rotation of the material about the symmetry axis does not change the apparent elastic structure of the material with respect to a fixed reference, represented for instance by the directions of the coordinate axes $\mathbf{e}_i$. With $\mathbf{e}_3$ parallel to the symmetry axis, we may argue that:

$$\begin{aligned} S_{11} &= S_{22}, \quad S_{13} = S_{23}, \quad S_{44} = S_{55} \\ K_{11} &= K_{22}, \quad K_{13} = K_{23}, \quad K_{44} = K_{55} \\ \eta_1 &= \eta_2, \quad \nu_{13} = \nu_{23}, \quad \mu_4 = \mu_5, \quad \nu_{21} = \nu_{12} \end{aligned} \tag{7.8.31}$$

The last result follows from formula $(7.8.28)_1$.

For plane strain or plane stress with respect to a plane normal to the symmetry axis, the material responds isotropically. This means that principal axes of stress and of strain are coinciding. A state of pure shear stress T_6, see Fig. 7.8.2, results according to (7.8.29) to a state of pure shear strain:

$$E_6 = K_{66} T_6 = \frac{1}{\mu_6} T_6 \tag{7.8.32}$$

Principal stresses and principal strains are obtained from the formulas (3.3.9) and (5.3.36), see Fig. 7.8.2:

$$\sigma_1 = -\sigma_2 = T_6, \quad \varepsilon_1 = -\varepsilon_2 = \frac{1}{2} E_6 \tag{7.8.33}$$

The principal directions $\bar{\mathbf{e}}_1$ and $\bar{\mathbf{e}}_2$ are rotated 45° with respect to the directions $\mathbf{e}_1$ and $\mathbf{e}_2$. The compliance matrix (7.8.30) also applies for the principal directions $\bar{\mathbf{e}}_1$, $\bar{\mathbf{e}}_2$, and $\mathbf{e}_3$. We may therefore use the matrix (7.8.30) and the formulas (7.8.31, 7.8.32 ,7.8.33) to obtain:

$$\left.\begin{array}{c} \varepsilon_1 = \frac{1}{\eta_1}\sigma_1 - \frac{\nu_{12}}{\eta_2}\sigma_2 = \frac{1+\nu_{12}}{\eta_1} T_6 \equiv \frac{1}{2} E_6 = K_{11} T_6 + K_{12}(-T_6) = (K_{11} - K_{12}) T_6 \\ E_6 = \frac{1}{\mu_6} T_6 \equiv K_{66} T_6 \end{array}\right\} \Rightarrow$$

$$\mu_6 = \frac{\eta_1}{2(1+\nu_{12})} \quad \Leftrightarrow \quad K_{66} = 2(K_{11} - K_{12}) \tag{7.8.34}$$

Similarly we find:

$$S_{66} = \frac{1}{2}(S_{11} - S_{12}) \tag{7.8.35}$$

The four conditions for the elements of the matrices K and S provided by the formulas (7.8.31, 34, 35) result in the following forms for the compliance matrix and the stiffness matrix in the case of transverse isotropy:

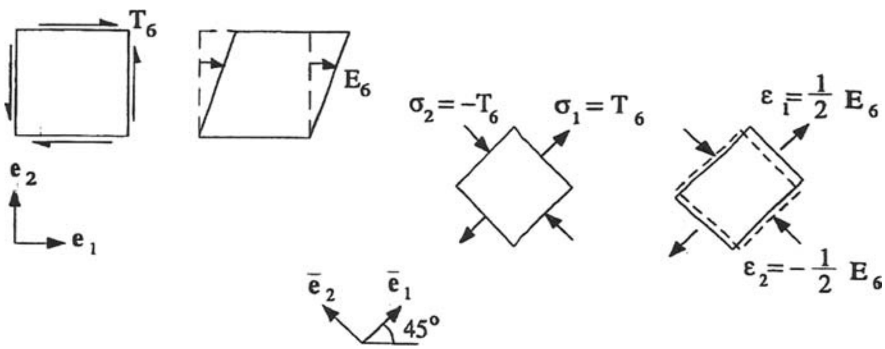


Fig. 7.8.2 Pure shear stress and pure shear strain

$$K = \begin{pmatrix} 1/\eta_1 & -\nu_{12}/\eta_1 & -\nu_{13}/\eta_3 & 0 & 0 & 0 \\ & 1/\eta_1 & -\nu_{13}/\eta_3 & 0 & 0 & 0 \\ & & 1/\eta_3 & 0 & 0 & 0 \\ & symmetry & & 1/\mu_4 & 0 & 0 \\ & & & & 1/\mu_4 & 0 \\ & & & & & 2(1+\nu_{12})/\eta_1 \end{pmatrix} \tag{7.8.36}$$

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ & S_{11} & S_{13} & 0 & 0 & 0 \\ & & S_{33} & 0 & 0 & 0 \\ & symmetry & & S_{44} & 0 & 0 \\ & & & & S_{44} & 0 \\ & & & & & \frac{1}{2}(S_{11}-S_{12}) \end{pmatrix} \tag{7.8.37}$$

The compliance matrix K has 5 *independent compliances*, represented by 5 independent engineering parameters, and the stiffness matrix S has 5 *independent stiffnesses*. Note that the engineering parameters given in Example 7.16 show that the human femur is a transverse isotropic elastic material.

7.8.5 Isotropy

For an isotropic elastic material all axes are symmetry axes. Starting from the matrices (7.8.36) and (7.8.37) we may set:

$$\eta_3 = \eta_1 = \eta_2 = \eta, \quad \nu_{12} = \nu_{13} = \nu, \quad \mu_4 = \mu = \frac{\eta}{2(1+\nu)}$$
$$S_{33} = S_{11}, \quad S_{44} = \frac{1}{2}(S_{11} - S_{12}), \quad S_{13} = S_{12} \tag{7.8.38}$$

These conditions reduce the number of independent compliances to two and independent stiffnesses to two. The stiffness matrix S has the symmetry shown by the matrix (7.8.7). Inverting the compliance matrix (7.8.36), we find:

$$S_{11} = \frac{1-\nu}{(1+\nu)(1-2\nu)}\eta = \frac{2(1-\nu)}{(1-2\nu)}\mu, \quad S_{12} = \frac{\nu}{(1+\nu)(1-2\nu)}\eta = \frac{2\nu}{(1-2\nu)}\mu$$
$$S_{44} = \frac{1}{2}(S_{11} - S_{12}) = \mu = \frac{\eta}{2(1+\nu)} \tag{7.8.39}$$

When the results (7.8.39) and (7.8.38) are substituted into (7.8.37), we obtain the stiffness matrix (7.8.7) for a Hookean material.

7.9 Composite Materials

A composite material, a *composite* for short, is a macroscopic composition of two or more materials. The material properties of a composite may to some extent be calculated from the knowledge of the material properties of its components. Plywood and reinforced concrete are two typical examples of composites.

It is customary to distinguish between three main types of composites:

1. *Fiber composites*
2. *Laminates*
3. *Particular composites*.

Materials of type 1 consist of *fibers* of one material baked into another base material called a *matrix*. Reinforced concrete belongs to this type. A laminate is made of layers, which may have different properties in different directions. A particular composite is mixture of particles of one material in a base material. In the present section we shall concentrate the presentation to fiber reinforced two-dimensional layers, called *laminas*, see Fig. 7.9.1, and a combination of these laminas into a *plate laminate*, see Fig. 7.9.2.

Fibers of a material are much stronger than the bulk material. Glass, for example, may have 300 times higher fiber strength than the base material. A fiber consists of crystals of the material arranged parallel to the fiber axis. The fiber diameter is of the same order of magnitude as the crystal diameter. The fiber therefore has fewer internal defects than the material. Very short fibers are called *whiskers*, and they are even stronger then ordinary fibers.

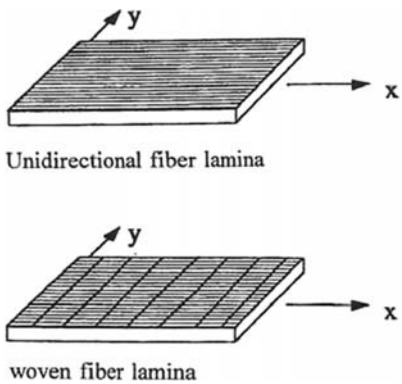


Fig. 7.9.1 Lamina

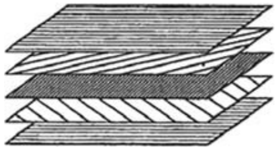


Fig. 7.9.2 Plate laminate

7.9.1 Lamina

Figure 7.9.1 shows two typical laminas: a *unidirectional fiber lamina* and a *woven fiber lamina*. Both types have one direction of dominating strength, represented by the x-direction. Fibers are normally linearly elastic, while the matrix material often shows a viscoelastic or visco-elasto-plastic response. In the present exposition we shall assume that the lamina as a composite material is a linearly elastic material. Furthermore, we shall concentrate the attention to laminas with unidirectional fiber reinforcement, called *unidirectional laminas*. However, all the general results we obtain also apply to laminas with woven fiber reinforcement.

Figure 7.9.3 shows an element of a unidirectional lamina oriented after the *lamina axes* x and y, with the x-axis in the fiber direction. The coordinate stresses with respect to the lamina axes, and for the state of plane stress, are denoted T_x, T_y, and $T_s (= T_{xy})$. The corresponding coordinate strains are E_x, E_y, and $E_s (= \gamma_{xy})$.

A unidirectional lamina represents an orthotropic, linearly elastic material. In the present case of plane stress only 4 independent stiffnesses or 4 independent compliances are relevant. With respect to the *lamina axes* x and y, see Fig. 7.9.3, we introduce the constitutive equations:

$$\bar{T} = \bar{S}\bar{E} \quad \Leftrightarrow \quad \begin{pmatrix} T_x \\ T_y \\ T_s \end{pmatrix} = \begin{pmatrix} S_{xx} & S_{xy} & 0 \\ S_{xy} & S_{yy} & 0 \\ 0 & 0 & S_{ss} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_s \end{pmatrix} \tag{7.9.1}$$

$$\bar{E} = \bar{K}\bar{T} \quad \Leftrightarrow \quad \begin{pmatrix} E_x \\ E_y \\ E_s \end{pmatrix} = \begin{pmatrix} K_{xx} & K_{xy} & 0 \\ K_{xy} & K_{yy} & 0 \\ 0 & 0 & K_{ss} \end{pmatrix} \begin{pmatrix} T_x \\ T_y \\ T_s \end{pmatrix} \tag{7.9.2}$$

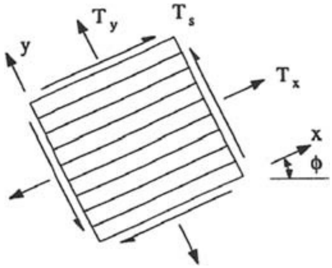


Fig. 7.9.3 Lamina axes

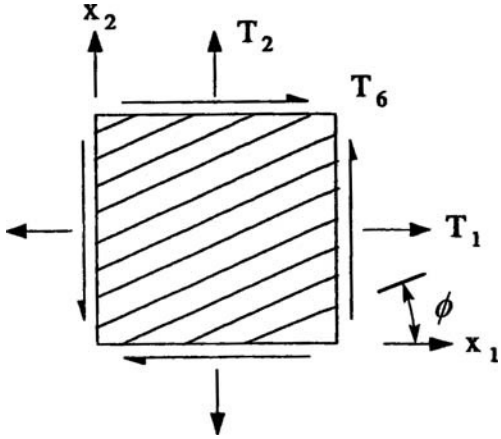


Fig. 7.9.4 Laminate axes

The compliance matrix and the stiffness matrix expressed in *engineering parameters* are according to the form (7.8.30):

$$\bar{K} = \begin{pmatrix} \frac{1}{\eta_x} & -\frac{\nu_x}{\eta_x} & 0 \\ -\frac{\nu_x}{\eta_x} & \frac{1}{\eta_y} & 0 \\ 0 & 0 & \frac{1}{\mu} \end{pmatrix}, \quad \bar{S} = \begin{pmatrix} \alpha\,\eta_x & \alpha\,\nu_x\,\eta_y & 0 \\ \alpha\,\nu_x\,\eta_y & \alpha\,\eta_y & 0 \\ 0 & 0 & \mu \end{pmatrix}, \quad \alpha = \frac{1}{1-\nu_x\,\nu_y} \tag{7.9.3}$$

In the formulas for $\bar{K}$ and $\bar{\mathrm{S}}$ the symmetry of the matrices has been invoked. The new symbols in the formulas are:

$$\begin{aligned} \eta_x &= \frac{1}{K_{xx}} && \text{longitudinal modulus of elasticity} \\ \eta_y &= \frac{1}{K_{yy}} && \text{transverse modulus of elasticity} \\ \nu_x &= -\frac{K_{yx}}{K_{xx}} && \text{longitudinal Poisson's ratio} \end{aligned} \tag{7.9.4}$$

We shall demonstrate how the engineering parameters may be estimated when the elastic properties of the matrix and the fibers are known. The matrix is assumed to be an isotropic elastic material having the modulus of elasticity η_m and the Poisson's ratio ν_m. The fibers are also isotropic elastic with the modulus of elasticity η_f and Poisson's ratio ν_f.

Figure 7.9.5 shows a cubic element of the composite. The length of the edges is L. The volume of the element is then $V = L^3$ and each side has the area $A = L^2$. The volume fraction of matrix and of fibers are c_m and c_f respectively, such that:

$$c_m + c_f = 1$$

Uniaxial stress in the fiber direction, T_x, results in longitudinal strains E_x and E_y. Matrix and fibers get the same strains E_x but have unequal stresses T_{xm} and T_{xf}. The

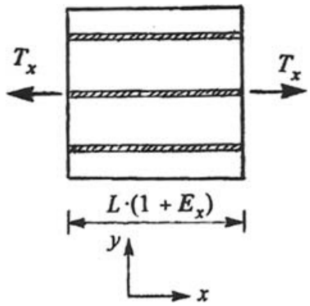


Fig. 7.9.5 Unidirectional lamina

area A of the sides subjected to the stresses T_x, T_{xm}, and T_{xf} is the sum of the area A_m of matrix and the area A_f of fibers. It follows that:

$$A = L^2, \; A_m = c_m L^2 = c_m A, \text{ and } A_f = c_f L^2 = c_f A$$

The modulus of elasticity η_x may be determined as follows. The normal force N_x on the area A is expressed by:

$$N_x = T_x A = T_{xf} A_f + T_{xm} A_m = \quad \Rightarrow \quad (\eta_x E_x) A = \left(\eta_f E_x\right) A_f + (\eta_m E_x)\, A_m \quad \Rightarrow$$
$$\eta_x = \eta_f c_f + \eta_m c_m \tag{7.9.5}$$

The other engineering parameters can only be estimated. Their determination depends on the way the fibers are distributed in the direction normal to the fibers. Figure 7.9.6 shows two extreme cases. In the case in Fig. 7.9.6a we can derive the results:

$$\nu_x = \nu_f c_f + \nu_m c_m, \quad \eta_y = \frac{\eta_f \eta_m}{\eta_f c_m + \eta_m c_f}, \quad \mu = \frac{\mu_f \mu_m}{\mu_f c_m + \mu_m c_f} \tag{7.9.6}$$

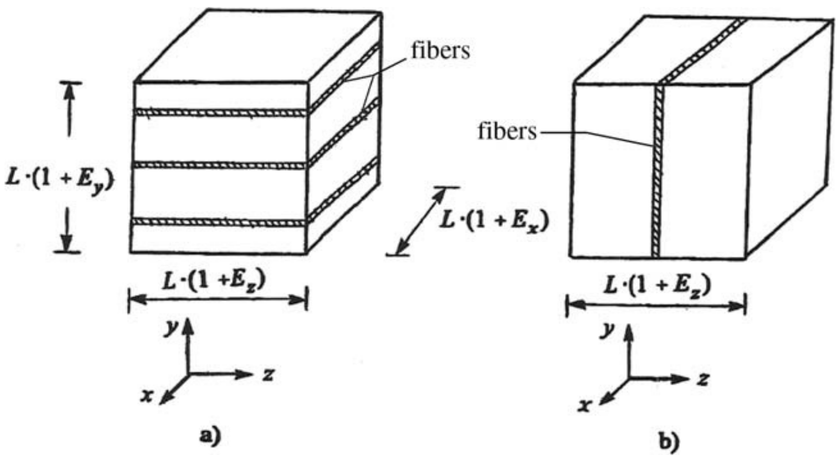


Fig. 7.9.6 Extreme distribution of fibers in the x-direction

where:

$$\mu_f = \frac{\eta_f}{2\left(1+\nu_f\right)}, \quad \mu_m = \frac{\eta_m}{2\left(1+\nu_m\right)} \tag{7.9.7}$$

It may be shown that the formula for η_y represents a lower limit for this modulus of elasticity.

Example 7.17. Elastic Parameters for Fiber Reinforced Epoxy

The table below shows the elastic parameters for three composite materials with epoxy matrix and 0.6 volume fraction of unidirectional fibers of three different fiber materials. The values are obtained from the book: "Mechanical behavior of materials" by Norman E. Dowling [11]. The numbers in parenthesis indicate the elastic properties of the fiber material. Epoxy is isotropic with the modulus of elasticity $\eta = 3.5\,\text{GPa}$ and Poisson's ratio $\nu = 0.33$. Using the formulas (7.9.5) and (7.9.6), we find that the η_x - and ν_x-values are in accordance with the table, while the other parameters do not correspond to the table-values.

parameter	E-glass (GPa)	Kevlar 49 (GPa)	Graphite, T-300 (GPa)
η_x	45 (72.3)	76 (124)	132 (218)
η_y	12	5.5	10.3
μ	4.4	2.1	6.5
ν_y	0.067 (0.22)	0.025 (0.35)	0.020 (0.20)
ν_x	0.25 (0.22)	0.34 (0.35)	0.25 (0.20)

7.9.2 From Lamina Axes to Laminate Axes

When we want to construct a laminate using lamina with different fiber orientations, we need to transform lamina stresses and strains to the stresses T_1, T_2, and $T_6(= T_{12})$, and the strains E_1, E_2, and $E_6(= 2E_{12})$ related to the *laminate axes* x_i, see Fig. 7.9.4. The strain matrix E is the same for all laminas, while the stress matrix T varies with the orientation angle ϕ. For each lamina we define the stiffness matrix S and the compliance matrix K through the relations:

$$T = SE \quad \Leftrightarrow \quad \begin{pmatrix} T_1 \\ T_2 \\ T_6 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_6 \end{pmatrix} \tag{7.9.8}$$

$$E = KT \quad \Leftrightarrow \quad \begin{pmatrix} E_1 \\ E_2 \\ E_6 \end{pmatrix} = \begin{pmatrix} K_{11} & K_{12} & K_{16} \\ K_{12} & K_{22} & K_{26} \\ K_{16} & K_{26} & K_{66} \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_6 \end{pmatrix} \tag{7.9.9}$$

The coordinate transformation from the lamina axes x and y to the laminate axes x_1 and x_2 in Fig. 7.9.3 and Fig. 7.9.4, is given by:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \Leftrightarrow \quad x = Q\bar{x} \tag{7.9.10}$$

The transformation formulas for coordinate stresses and coordinate strains are:

$$T_{\alpha\beta} = Q_{\alpha\gamma} Q_{\beta\lambda} \bar{T}_{\gamma\lambda} \quad \Leftrightarrow \quad T = Q_\sigma \bar{T}, \quad E_{\alpha\beta} = Q_{\alpha\gamma} Q_{\beta\lambda} \bar{E}_{\gamma\lambda} \quad \Leftrightarrow \quad E = Q_\varepsilon \bar{E} \tag{7.9.11}$$

T, $\bar{T}$, E, and $\bar{E}$ are vector matrices for the stresses and the strains in (7.9.1) and (7.9.8), and:

$$Q_\sigma = \begin{pmatrix} \cos^2\phi & \sin^2\phi & -\sin 2\phi \\ \sin^2\phi & \cos^2\phi & \sin 2\phi \\ \frac{1}{2}\sin 2\phi & -\frac{1}{2}\sin 2\phi & \cos 2\phi \end{pmatrix}, \quad Q_\varepsilon = \begin{pmatrix} \cos^2\phi & \sin^2\phi & -\frac{1}{2}\sin 2\phi \\ \sin^2\phi & \cos^2\phi & \frac{1}{2}\sin 2\phi \\ \sin 2\phi & -\sin 2\phi & \cos 2\phi \end{pmatrix} \tag{7.9.12}$$

The difference between the matrices Q_ε and Q_σ is due to the fact that $T_{12} = T_6$ and $E_{12} = (1/2)E_6$. The two matrices Q_ε and Q_σ are related through:

$$Q_\varepsilon = Q_\sigma^{-T} \quad \Leftrightarrow \quad Q_\varepsilon^T = Q_\sigma^{-1} \quad \Leftrightarrow \quad Q_\varepsilon^{-1} = Q_\sigma^T \tag{7.9.13}$$

This may be shown as follows. The elastic energy per unit volume may be expressed alternatively by:

$$\frac{1}{2} T^T E = \frac{1}{2} \bar{T}^T \bar{E} \tag{7.9.14}$$

Then using the formulas (7.9.14) and (7.9.11), we obtain:

$$\bar{T}^T \bar{E} = T^T E = \bar{T}^T Q_\sigma^T Q_\varepsilon \bar{E} \quad \Rightarrow \quad Q_\sigma^T Q_\varepsilon = 1 \quad \Rightarrow \quad (7.9.13)$$

Now we are ready to develop relations between the stiffness matrices and the compliance matrices related to the lamina axes and laminate axes. Using the formulas (7.9.8), (7.9.11), (7.9.1), and (7.9.13), we find:

$$SE = T = Q_\sigma \bar{T} = Q_\sigma \bar{S} \bar{E} = Q_\sigma \bar{S} Q_\varepsilon^{-1} E = Q_\sigma \bar{S} Q_\sigma^T E \tag{7.9.15}$$

Since S and $Q_\sigma \bar{S} Q_\sigma^T$ both are independent of E, it follows from the result (7.9.15) that:

$$S = Q_\sigma \bar{S} Q_\sigma^T \tag{7.9.16}$$

Using similar arguments, we get:

$$K = Q_\varepsilon \bar{K} Q_\varepsilon^T \tag{7.9.17}$$

7.9.3 Engineering Parameters Related to Laminate Axes

The engineering parameters for a lamina related to the laminate axes are defined similarly to the engineering parameters related to the lamina axes. The lamina is subjected in turn to uniaxial stress in the x_1- and the x_2-directions, and to the shear stress T_6. Then:

$$\begin{aligned}
&T_1 \neq 0, \quad T_2 = T_6 = 0 \quad \Rightarrow \\
&\eta_1 = T_1/E_1 = 1/K_{11} \quad \text{modulus of elasticity} \\
&\nu_{21} = -E_2/E_1 = -K_{21}/K_{11} \quad \text{Poisson's ratio} \\
&\nu_{61} = E_6/E_1 = K_{61}/K_{11} \quad \text{shear coupling coefficient}
\end{aligned} \tag{7.9.18}$$

$$\begin{aligned}
&T_2 \neq 0, \quad T_1 = T_6 = 0 \quad \Rightarrow \\
&\eta_2 = T_2/E_2 = 1/K_{22} \quad \text{modulus of elasticity} \\
&\nu_{12} = -E_1/E_2 = -K_{12}/K_{22} \quad \text{Poisson's ratio} \\
&\nu_{62} = E_6/E_2 = K_{62}/K_{22} \quad \text{shear coupling coefficient}
\end{aligned} \tag{7.9.19}$$

$$\begin{aligned}
&T_6 \neq 0, \quad T_1 = T_2 = 0 \quad \Rightarrow \\
&\mu_6 = T_6/E_6 = 1/K_{66} \quad \text{shear modulus} \\
&\nu_{16} = E_1/E_6 = K_{16}/K_{66} \quad \text{normal stress coupling coefficient} \\
&\nu_{26} = E_2/E_6 = K_{26}/K_{66} \quad \text{normal stress coupling coefficient}
\end{aligned} \tag{7.9.20}$$

We use the fact that K is symmetric. Per definitions (7.9.18 ,7.9.19, 7.9.20):

$$K_{12} = -\frac{\nu_{12}}{\eta_2},\ K_{21} = -\frac{\nu_{21}}{\eta_1},\ K_{16} = \frac{\nu_{16}}{\mu_6},\ K_{61} = \frac{\nu_{61}}{\eta_1},\ K_{26} = \frac{\nu_{26}}{\mu_6},\ K_{62} = \frac{\nu_{62}}{\eta_2}$$

Hence:

$$\frac{\nu_{12}}{\eta_2} = \frac{\nu_{21}}{\eta_1},\ \frac{\nu_{16}}{\mu_6} = \frac{\nu_{61}}{\eta_1},\ \frac{\nu_{26}}{\mu_6} = \frac{\nu_{62}}{\eta_2} \tag{7.9.21}$$

From the above it follows that:

$$\bar{K} = \begin{pmatrix} \frac{1}{\eta_1} & -\frac{\nu_{21}}{\eta_1} & \frac{\nu_{61}}{\eta_1} \\ -\frac{\nu_{21}}{\eta_1} & \frac{1}{\eta_2} & \frac{\nu_{62}}{\eta_2} \\ \frac{\nu_{61}}{\eta_1} & \frac{\nu_{62}}{\eta_2} & \frac{1}{\mu_6} \end{pmatrix} \tag{7.9.22}$$

7.9.4 Plate Laminate of Unidirectional Laminas

We shall consider a laminate made of many laminas bound together by the same material as in the matrix of the laminas. The plate is symmetric with respect to its middle surface, see Fig. 7.9.7, and is loaded by forces in the middle surface.

The forces acting on an element of the plate are called *stress resultants* and are given as forces N_1, N_2, and N_6 per unit length along the edges of the element.

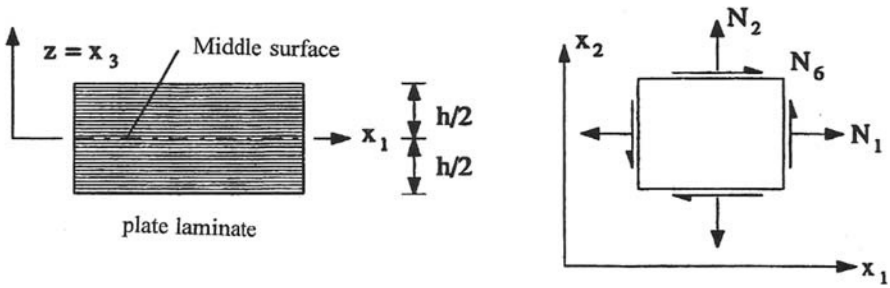


Fig. 7.9.7 Laminate and stress resultants

$$N_1 = \int_{-h/2}^{h/2} T_1\,dz, \quad N_2 = \int_{-h/2}^{h/2} T_2\,dz, \quad N_6 = \int_{-h/2}^{h/2} T_6\,dz \tag{7.9.23}$$

If the middle surface is plane, the laminate is a plate. A laminate with a curved middle surface for which the curvature is small relative to the inverse thickness $1/h$, and with stress resultants given by the formulas (7.9.23), is a *membrane shell*. Problem 7.25 presents an example of a cylindrical laminated membrane shell.

We assume that the strains E_1, E_2, and E_6 are constant over the thickness of the laminate. But due to the different orientations of the laminas of the laminate, the stresses T_1, T_2, and T_6 may vary through the thickness. Let the strains E_1, E_2, and E_6 and the stress resultants N_1, N_2, and N_6 be related through the matrix equations:

$$N \equiv \begin{pmatrix} N_1 \\ N_2 \\ N_6 \end{pmatrix} = AE \quad \Leftrightarrow \quad E = BN, B = A^{-1} \tag{7.9.24}$$

The 3×3-matrices A and B are called the *plate stiffness matrix* and the *plate flexibility matrix*, respectively. Introducing the relation $T = SE$ into (7.9.23), we obtain:

$$A = \int_{-h/2}^{h/2} S\,dz \tag{7.9.25}$$

Since S is symmetric it follows that A and B are symmetric matrices. The integral in formula (7.9.25) may be replaced by a sum. Let the laminate have n different orientations of laminas, specified by n angles $\phi_i, i = 1,2,\ldots,n$. The laminas with orientation angle ϕ_i have the stiffness matrix S_i. The total height is h_i of the laminas with orientation angle ϕ_i. Then from the formula (7.9.25) we get the result:

$$A = \sum_{i=1}^{n} S_i h_i \tag{7.9.26}$$

7.10 Large Deformations

In general an *elastic material*, or more precisely a *Cauchy-elastic material*, is defined by the constitutive equation of the form:

$$\mathbf{T} = \mathbf{T}[\mathbf{F}, \mathbf{r}_o] \tag{7.10.1}$$

The Cauchy stress tensor $\mathbf{T}$ is given by a tensor-valued function of the deformation gradient tensor $\mathbf{F}$ and the position vector $\mathbf{r}_o$ of the particle in the reference configuration K_o. This definition does not assume small deformations or small strains. The deformation gradient $\mathbf{F}$ may be polar decomposed into a stretch tensor $\mathbf{U}$ and the rotation tensor $\mathbf{R}$, i.e. $\mathbf{F} = \mathbf{RU}$. Due to the geometrical interpretation of polar decomposition in Fig. 5.5.3 it is reasonable to require of the constitutive equation (7.10.1) that the two states of deformation represented by $\mathbf{F} = \mathbf{RU}$ and $\mathbf{F} = \mathbf{U}$ respectively result in states of stress that are equal apart from the a rotation of the principal axes of stress. We may therefore expect of the tensor-valued function in (7.10.1) that:

$$\mathbf{T} = \mathbf{T}[\mathbf{F}, \mathbf{r}_o] = \mathbf{R}\,\mathbf{T}[\mathbf{U}, \mathbf{r}_o]\,\mathbf{R}^T \tag{7.10.2}$$

The response function $\mathbf{T}[\mathbf{F}, \mathbf{r}_o]$ may thus be determined from experiments with pure strain, $\mathbf{F} = \mathbf{U} = \mathbf{U}^T$. The argument used to obtain the result (7.10.2) is really an application of a fundamental principle in the general theory of constitutive equations, which is presented in Sect. 11.5 as: *the principle of material objectivity*.

In the case of small deformations between the reference configuration K_o and the present configuration K we have from the formulas (5.5.69) that:

$$\mathbf{R} = \mathbf{1} + \tilde{\mathbf{R}}, \quad \mathbf{R}^T = \mathbf{1} - \tilde{\mathbf{R}}, \quad \mathbf{U} = \mathbf{1} + \mathbf{E} \tag{7.10.3}$$

We assume that the state of stress in K_o is given by $\mathbf{T}_o = \mathbf{T}[\mathbf{1}, \mathbf{r}_o]$. The two first terms in a Taylor series expansion of the response function in (7.10.2) provide the following approximation to the response function, valid for small deformations.

$$\mathbf{T}[\mathbf{U}, \mathbf{r}_o] = \mathbf{T}_o + \mathbf{S}[\mathbf{r}_o] : \mathbf{E} \tag{7.10.4}$$

The *stiffness tensor* $\mathbf{S}(\mathbf{r}_o)$ is defined by:

$$\mathbf{S}[\mathbf{r}_o] = \left.\frac{\partial \mathbf{T}}{\partial \mathbf{U}}\right|_{\mathbf{U}=\mathbf{1}} = \left.\frac{\partial \mathbf{T}}{\partial \mathbf{E}}\right|_{\mathbf{E}=\mathbf{0}}, \quad S_{ijkl}[X] = \left.\frac{\partial T_{ij}}{\partial U_{kl}}\right|_{U=1} = \left.\frac{\partial T_{ij}}{\partial E_{kl}}\right|_{E=0} \tag{7.10.5}$$

When $\mathbf{R}$ from the formulas (7.10.3) and $\mathbf{T}[\mathbf{U}, \mathbf{r}_o]$ from (7.10.4) are substituted into the result (7.10.2), this result is obtained:

$$\mathbf{T} = \mathbf{T}_o + \tilde{\mathbf{R}}\,\mathbf{T}_o - \mathbf{T}_o\,\tilde{\mathbf{R}} + \mathbf{S} : \mathbf{E} \quad \Leftrightarrow \quad T_{ij} = T_{oij} + \tilde{R}_{ik}\,T_{okj} - T_{oik}\,\tilde{R}_{kj} + S_{ijkl}\,E_{kl} \tag{7.10.6}$$

The formula is applicable in incremental solutions in non-linear problems where the non-linearity is due to non-linear elastic material properties and/or to large deformations. The response equation (7.10.6) may also be used in stability investigations.

7.10.1 Isotropic Elasticity

For an isotropic material with a stress free reference configuration the response function (7.10.1) may be replaced by a isotropic tensor-valued function of the *left deformation tensor* $\mathbf{B} = \mathbf{F}\mathbf{F}^T$:

$$\mathbf{T} = \mathbf{T}[\mathbf{B}, \mathbf{r}_o] \tag{7.10.7}$$

In order to understand this we use the polar decomposition theorem to write: $\mathbf{F} = \mathbf{V}\mathbf{R}$, where $\mathbf{V}$ is the left stretch tensor. Now, to deform the material from the reference configuration K_o to the present configuration K we may first let the material in the neighborhood of the particle under consideration be rotated according to the rotation tensor $\mathbf{R}$, and then subject the material to pure strain through the stretch tensor $\mathbf{V}$. Because the material is isotropic, the rotation $\mathbf{R}$ does not influence the stresses resulting from the deformation gradient $\mathbf{F}$. This implies that we may replace $\mathbf{F}$ by $\mathbf{V}$ as the argument tensor in the tensor-valued function (7.10.1).

$$\mathbf{T} = \mathbf{T}[\mathbf{F}, \mathbf{r}_o] = \mathbf{T}[\mathbf{V}, \mathbf{r}_o] \tag{7.10.8}$$

Since $\mathbf{B} = \mathbf{V}^2$, we may now consider the stress to be a tensor-valued function of $\mathbf{B}$ and the result is the function (7.10.7), although the mathematical functions (7.10.8) and (7.10.7) are not the same. Because the material is isotropic, the function $\mathbf{T}[\mathbf{B}, \mathbf{r}_o]$ must be isotropic with respect to the argument tensor $\mathbf{B}$, i.e. if the deformation $\mathbf{B}$ result in the stress tensor $\mathbf{T}$, a $\mathbf{Q}$ – rotated deformation $\mathbf{Q}\mathbf{B}\mathbf{Q}^T$ will result in the stress tensor $\mathbf{Q}\mathbf{T}\mathbf{Q}^T$. According to the results (4.6.17) and (4.6.27) the constitutive equation (7.10.7) may be represented by the two alternative forms:

$$\mathbf{T} = \gamma_o \mathbf{1} + \gamma_1 \mathbf{B} + \gamma_2 \mathbf{B}^2 \tag{7.10.9}$$

$$\mathbf{T} = \phi_o \mathbf{1} + \phi_1 \mathbf{B} + \phi_{-1} \mathbf{B}^{-1} \tag{7.10.10}$$

γ_i and ϕ_i are scalar-valued functions of the principal invariants I_B, II_B, and III_B, or of the principal values of the deformation tensor $\mathbf{B}$.

From the definitions of the displacement gradient $\mathbf{H} = \mathbf{F} - \mathbf{1}$ and the strain tensor:

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T + \mathbf{H}^T \mathbf{H}\right) \tag{7.10.11}$$

we obtain:

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = (\mathbf{1} + \mathbf{H})\left(\mathbf{1} + \mathbf{H}^T\right) \quad \Rightarrow \quad \mathbf{B} = \mathbf{1} + 2\mathbf{E} + \mathbf{H}\mathbf{H}^T - \mathbf{H}^T\mathbf{H} \tag{7.10.12}$$

Because we assume that the material is stress free in the reference configuration K_o, a linearization of the general constitutive equation of an isotropic elastic material (7.10.9) may be presented as the following form of Hooke's law:

$$\mathbf{T} = \lambda\,(\operatorname{tr}\mathbf{E})\,\mathbf{1} + 2\mu\,\mathbf{E} \tag{7.10.13}$$

Confer equation (7.2.19). The parameters λ and μ are the *Lamé-constants*. λ is presented by formula (7.2.21) and μ is identical to the shear modulus.

7.10.2 Hyperelasticity

The stress power of a body with volume V is by definition:

$$P^d = \int_V \omega \, dV = \int_V \mathbf{T} : \mathbf{D} \, dV \tag{7.10.14}$$

The stress work done on the body when it moves from the reference configuration K_o at the time t_o to the present configuration K at the time t is:

$$W = \int_{t_o}^{t} P^d \, dt \tag{7.10.15}$$

A material is called *hyperelastic*, or *Green-elastic*, if the stress work may be derived from a scalar potential $\Phi(t)$, called the *elastic energy* in the body:

$$\Phi(t) = \int_V \psi \rho \, dV \tag{7.10.16}$$

The potential ψ, which is the *specific elastic energy*, i.e. is *elastic energy per unit mass*, is a scalar-valued function of the deformation tensor $\mathbf{C} = \mathbf{F}^T \mathbf{F}$:

$$\psi = \psi[\mathbf{C}, \mathbf{r}_o] \tag{7.10.17}$$

The rotation part $\mathbf{R}$ of $\mathbf{F}$ does obviously not contribute to the elastic energy ψ.

For a hyperelastic material the stress work on the body may now be expressed by:

$$W = \Phi(t) - \Phi(t_o) \tag{7.10.18}$$

such that:

$$P^d = \int_V \omega \, dV = \int_V \mathbf{T} : \mathbf{D} \, dV = \dot{\Phi} = \int_V \dot{\psi} \rho \, dV \tag{7.10.19}$$

It follows that the stress power per unit volume ω is equal to $\dot{\psi}\rho$:

$$\omega = \mathbf{T} : \mathbf{D} = \dot{\psi} \rho \tag{7.10.20}$$

From (7.10.17) it follows that:

$$\dot{\psi} = \frac{\partial \psi}{\partial \mathbf{C}} : \dot{\mathbf{C}} = \frac{\partial \psi}{\partial C_{ij}} \dot{C}_{ij} \tag{7.10.21}$$

An expression for $\dot{\mathbf{C}}$ is found using (5.5.28): $\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}$, where $\mathbf{L}$ is the velocity gradient tensor:

$$\dot{\mathbf{C}} = \dot{\mathbf{F}}^T \mathbf{F} + \mathbf{F}^T \dot{\mathbf{F}} = \mathbf{F}^T \mathbf{L}^T \mathbf{F} + \mathbf{F}^T \mathbf{L}\mathbf{F} = 2\mathbf{F}^T \mathbf{D}\,\mathbf{F}$$

Then (7.10.21) may be further developed:

$$\dot{\psi} = \frac{\partial \psi}{\partial C_{ij}} \dot{C}_{ij} = \frac{\partial \psi}{\partial C_{ij}} \left(2F_{ki} D_{kl} F_{lj}\right) \quad \Rightarrow \quad \dot{\psi} = \left(2\mathbf{F}\, \frac{\partial \psi}{\partial \mathbf{C}}\, \mathbf{F}^T\right) : \mathbf{D} \qquad (7.10.22)$$

This expression is substituted into (7.10.20):

$$\omega = \mathbf{T} : \mathbf{D} = \left(2\rho\, \mathbf{F}\, \frac{\partial \psi}{\partial \mathbf{C}}\, \mathbf{F}^T\right) : \mathbf{D} \qquad (7.10.23)$$

Because the tensor in the parenthesis is symmetric and independent of the rate of deformation tensor $\mathbf{D}$, and $\mathbf{D}$ may be chosen arbitrarily, we may conclude from the equation (7.10.23) that:

$$\mathbf{T} = 2\rho\, \mathbf{F}\, \frac{\partial \psi}{\partial \mathbf{C}}\, \mathbf{F}^T \qquad (7.10.24)$$

Note: When ψ is differentiated with respect to the symmetric tensor $\mathbf{C}$, the components C_{ij} must be treated as 9 independent quantities. Confer the commentaries to (7.6.10) and to (7.8.9). The result (7.10.24) represents the *general constitutive equation of a hyperelastic material*. The equation may be given an alternative form by introducing the strain tensor $\mathbf{E}$:

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{1}) \qquad (7.10.25)$$

the *second Piola-Kirchhoff stress tensor* $\mathbf{S}$ from (5.6.15):

$$\mathbf{S} = J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T} \qquad (7.10.26)$$

and the *continuity equation in a particle* (5.5.65):

$$\rho\, J = \rho_o \qquad (7.10.27)$$

From the constitutive equation (7.10.24) we then obtain the alternative form:

$$\mathbf{S} = \rho_o\, \frac{\partial \psi}{\partial \mathbf{E}} \qquad (7.10.28)$$

In the case of small deformation, we may set $\mathbf{S} = \mathbf{T}$. By introducing the elastic energy per unit volume:

$$\phi = \phi\,[\mathbf{E}, \mathbf{r}_o] = \rho_o\, \psi \qquad (7.10.29)$$

equation (7.10.28) may be expressed as:

$$\mathbf{T} = \frac{\partial \phi}{\partial \mathbf{E}} \tag{7.10.30}$$

Confer equation (7.6.10).

For isotropic materials we may express the elastic energy of the body as an isotropic scalar-valued function of the left deformation tensor $\mathbf{B}$:

$$\psi = \psi[\mathbf{B}, \mathbf{r}_o] = \psi(I_B, II_B, III_B) \tag{7.10.31}$$

I_B, II_B, and III_B are the principal invariants of $\mathbf{B}$. Now:

$$\dot{\psi} = \frac{\partial \psi}{\partial \mathbf{B}} : \dot{\mathbf{B}} \tag{7.10.32}$$

The expression for $\dot{\mathbf{B}}$ is found using (5.5.28), i.e. $\dot{\mathbf{F}} = \mathbf{LF}$:

$$\dot{\mathbf{B}} = \dot{\mathbf{F}}\,\mathbf{F}^T + \mathbf{F}\,\dot{\mathbf{F}}^T = \mathbf{L}\,\mathbf{F}\,\mathbf{F}^T + \mathbf{F}\mathbf{F}^T\,\mathbf{L}^T = \mathbf{L}\,\mathbf{B} + \mathbf{B}\,\mathbf{L}^T$$

Hence:

$$\dot{\psi} = \frac{\partial \psi}{\partial \mathbf{B}} : \dot{\mathbf{B}} = \frac{\partial \psi}{\partial B_{ij}} \dot{B}_{ij} = \frac{\partial \psi}{\partial B_{ij}} \left(L_{ik} B_{kj} + B_{ik} L_{jk} \right) = \left(2 \frac{\partial \psi}{\partial \mathbf{B}} \mathbf{B} \right) : \mathbf{L} \tag{7.10.33}$$

The latter equality is due to the symmetric property of the tensor $\mathbf{B}$. Now, from (7.10.20) and (7.10.33) we obtain:

$$\omega = \mathbf{T} : \mathbf{D} = \mathbf{T} : \mathbf{L} = \dot{\psi}\rho = \left(2\rho \frac{\partial \psi}{\partial \mathbf{B}} \mathbf{B} \right) : \mathbf{L} \tag{7.10.34}$$

Because $\mathbf{L}$ may be chosen arbitrarily, we conclude from the expressions above that:

$$\mathbf{T} = 2\rho \frac{\partial \psi}{\partial \mathbf{B}} \mathbf{B} \tag{7.10.35}$$

Alternative forms of this result are represented by (7.10.9) and (7.10.10).

The *Mooney-Rivlin material* is an example of an isotrop, incompressible, non-linear hyperelastic material used as a material model for rubber. The model is defined by the specific elastic energy:

$$\psi = \frac{1}{2} \frac{\mu}{\rho} \left(\frac{1}{2} + \alpha \right) (I_B - 3) + \frac{1}{2} \frac{\mu}{\rho} \left(\frac{1}{2} - \alpha \right) (II_B - 3) \tag{7.10.36}$$

The material parameters μ and α are elasticities. From (7.10.35) we obtain the stress tensor:

$$\mathbf{T} = \mu \left(\frac{1}{2} + \alpha \right) \mathbf{B} - \mu \left(\frac{1}{2} - \alpha \right) \mathbf{B}^{-1} - p\,\mathbf{1} \tag{7.10.37}$$

Because the material is incompressible, an isotropic stress $-p\mathbf{1}$, representing an indeterminate pressure p, has to be added to the stress tensor. The special case with $\alpha = 1/2$ (7.10.37) defines the model called the *neo-Hookean material*.

Problems

Problem 7.1. Develop the form (7.2.8) of Hooke's law from the form (7.2.7).

Problem 7.2. Develop the decomposition (7.2.18) from Hooke's law, (7.2.8).

Problem 7.3. Develop Hooke's law for plane stress (7.3.6, 7.3.7, 7.3.8, 7.3.9) from the general law (7.2.6, 7.2.7, 7.2.8).

Problem 7.4. Develop Hooke's law for plane displacement (7.3.29, 7.3.30) from the general law (7.2.6, 7.2.7, 7.2.8).

Problem 7.5. A 45°–90° strain rosette, see Problem 5.5, is fixes to the surface of a machine part of steel. For a certain load the following strains are recorded:

$$\varepsilon_x = 560 \cdot 10^{-6},\ \varepsilon_y = 120 \cdot 10^{-6},\ \varepsilon_{45} = 200 \cdot 10^{-6}$$

Determine the principle stresses and the principle stress directions.

Problem 7.6. A thin-walled steel pipe, closed in both ends, is a part of a larger structure. The outer diameter of the pipe is 500 mm, and the wall thickness is 20 mm. The pipe is subjected to an axial tensile force N, a torque M, and an internal pressure p. A 45°–90° strain rosette, see Problem 5.5, is fixed to the surface of the pipe. The x-direction of the rosette is parallel to axis of the pipe. For a certain load on the structure the following strains are recorded:

$$\varepsilon_x = 620 \cdot 10^{-6}, \varepsilon_y = 320 \cdot 10^{-6}, \varepsilon_{45} = 230 \cdot 10^{-6}$$

Assume homogeneous state of strain in the pipe wall. Determine N, M and p for this load.

Problem 7.7. Show how the elastic energy per unit volume may be decomposed into volumetric energy and distortion energy as shown by the formulas (7.2.30).

Problem 7.8. Determin the displacement $u_2(x,y)$ according to the alternative boundary condition 2) by adding to the displacement $u_2(x,y)$ for the alternative boundary condition 1) found in Example 7.4 a rigid-body counterclockwise rotation given by the angle:

$$\alpha = -\,u_{1,2}\big|_{x=L,y=0} = \frac{3\,(1+\nu)\,F}{E\,b\,h}$$

found from the displacement $u_2(x,y)$ for the alternative boundary condition 1).

Problem 7.9. Derive the Navier equations for plane displacements and for plane stress.

Problem 7.10. Derive the basic equations of thermoelasticity under the condition of plane stress and plane displacements. Check with Sect. 7.5.2 and Sect. 7.5.3.

Problem 7.11. Show that the state of stress given in Problem 3.3 satisfies the compatibility equation (7.3.45) for plane stress.

Problem 7.12. Show that the state of stress given in Problem 3.5 satisfies the compatibility equation (7.3.45) for plane stress.

Problem 7.13. A copper plate is restrained from expansion in the plane of the plate. $E = 118\,\mathrm{GPa}, \nu = 0.33$, and $\alpha = 17 \cdot 10^{-6}\,{}^{\circ}\mathrm{C}^{-1}$. Compute the stresses in the plate due to a temperature increase of 15 °C.

Answer: –45 MPa.

Problem 7.14. A circular plate with a concentric hole is mounted on an undeformable shaft. The radius of the plate is b. The radius of the hole and of the shaft is a. The plate has the modulus of elasticity E and the Poisson's ratio ν. The plate is subjected to an external pressure p. Assume plane stress and determine the pressure against the shaft. Answer:

$$\frac{2}{(1-\nu)\,(a/b)^2+1+\nu}p$$

Problem 7.15. A simply supported horizontal beam has the length L and a rectangular cross section with horizontal width b and vertical height h. A coordinate system has the origin O on the beam axis a distance $L/2$ from left support, the x-axis along the beam axis, and a y-axis vertically downward. The beam is subjected to uniform pressure p on the surface $y = -h/2$.

a) Show that the Airy's stress function:

$$\Psi = -\frac{p}{h^3}x^2y^3 + \frac{p}{5h^3}y^5 + \frac{3p}{4h}x^2y - \frac{p}{4}x^2 - \left[\frac{3}{2}\left(\frac{L}{h}\right)^2 - \frac{3}{5}\right]\frac{p}{6h}y^3$$

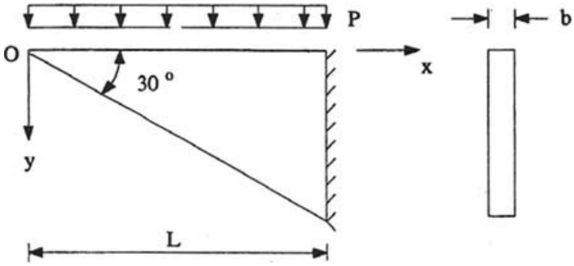
satisfies the compatibility equation (7.3.49).

b) Compute the stresses and show that they satisfy the boundary conditions on the surfaces $y = \pm h/2$. Show also the stress resultants over the cross section at $x = \pm L/2$ satisfy the support conditions.

c) Show that the stresses computed in b) satisfy the Cauchy equations for the beam.

d) Show that the stresses computed in b) gives the correct axial force, shear force and bending moment over the cross-section of the beam. Compare the stresses with those obtained from elementary (engineering) beam theory.

Fig. Problem 7.16



Problem 7.16. A triangular plate with constant thickness b is rigidly fixed along the side $x = L$. The plate is subjected to a uniform pressure p on the surface $y = 0$. The surface $y = x\tan 30°$ is free of stresses. The following state of plane stress is suggested:

$$\sigma_x = a\left[\frac{\pi}{6} - \arctan\frac{y}{x} - \frac{xy}{x^2+y^2}\right]p,\ a = \frac{6}{2\sqrt{3}-\pi}$$

$$\sigma_y = -a\left[\frac{1}{a} + \arctan\frac{y}{x} - \frac{xy}{x^2+y^2}\right]p,\ \tau_{xy} = -a\,\frac{y^2}{x^2+y^2}p$$

a) Show that the Cauchy equations are satisfied.
b) Show that the equation of compatibility (7.3.45) is satisfied.
c) Check that the boundary conditions on the surfaces $y = 0$ and $y = x\tan 30°$ are satisfied.
d) Determined the principal stresses, the principal stress directions, and the maximum shear stress at the surfaces $y = 0$ and $y = x\tan 30°$.
e) Consider the plate as a beam and compare the stresses σ_x and τ_{xy} with those obtained from the elementary (engineering) beam theory.

Problem 7.17. The rectangular plate in Fig. 7.3.12 is subjected to stresses $\sigma_x = \sigma$ on the sides $x = \pm b/2$ and $\sigma_y = -\sigma$ on the sides $y = \pm h/2$. Use the solution provided by Example 7.8 to determine the stresses σ_R, σ_θ, and $\tau_{R\theta}$. Determine the extremal principal stresses at the hole.

Problem 7.18. The rectangular plate in Fig. 7.3.12 is subjected to shear stresses $\tau_{xy} = \tau$ on the surfaces $x = \pm b/2$ and $y = \pm h/2$. Use the solution in Example 7.8, or Problem 7.17, to determine the stresses σ_R, σ_θ, and $\tau_{R\theta}$. Determine the extreme principal stresses at the hole.

Problem 7.19. A thin-walled pipe with external diameter $d = 210\,\mathrm{mm}$ and wall thickness $h = 10\,\mathrm{mm}$ is subjected to a torque $M = 24\,\mathrm{kNm}$ and an axial force $S = 280\,\mathrm{kN}$. A circular hole is cut through the wall of the pipe. Assume that the diameter of the hole is very small compared to the diameter d of the pipe. Compute the maximum normal stress in the wall. Hint: Superimpose the states of stress in Example 7.8 and the Problems 7.17 and 7.18. Answer: 221 MPa.

Problem 7.20. Consider the Prandtl stress function of torsion:

$$\Omega = k\,(x-a)\left(x - y\sqrt{3} + 2a\right)\left(x + y\sqrt{3} + 2a\right), \quad k = \text{ constant}$$

a) Show that Ω is the stress function of torsion of a cylindrical bar with an equilateral triangular cross section with side a.
b) Determine the distribution of stress on the cross section.
c) Show that relation between the torque M and the torsion angle ϕ per unit length of the bar is:

$$M = \frac{9\sqrt{3}}{5} a^4 \mu\, \phi$$

Problem 7.21. Show that the condition $\phi \geq 0$ for elastic energy per unit volume implies the conditions (7.6.20) for the elastic parameters: $E \equiv \eta, G \equiv \mu$, and κ.

Problem 7.22. Derive the differential equation (7.4.14) for the warping function ψ directly from the Navier equations (7.6.28).

Problem 7.23. Show that the compatibility equations (7.6.43) implies for the Prandl's stress function Ω that:

$$\frac{\partial}{\partial x_\alpha}\left(\nabla^2 \Omega\right) = 0 \quad \Rightarrow \quad \nabla^2 \Omega = \text{constant}$$

Compare the result with the differential equations (7.4.22).

Problem 7.24. A unidirectional lamina denoted T300/5208 consists of graphite fibers imbedded in epoxy. The lamina has the following engineering parameters with respect to the lamina axes x and y:

$$\eta_x = 181 \text{ GPa}, \; \eta_y = 10.3 \text{ GPa}, \quad \mu = 7.17 \text{ GPa}, \; \nu_x = 0.28$$

Compute the following quantities:

a) ν_y.
b) The stiffness and compliance matrices $\bar{S}$ and $\bar{K}$.
c) The stiffness and compliance matrices S and K for $\phi = (\mathbf{e}_1, \mathbf{e}_x) = 45°$.
d) The engineering parameters with respect to the laminate axes x_1 and x_2.

Problem 7.25. A laminate is made of layers of the lamina T300/5208 described in Problem 7.24. The laminate is symmetric with respect to the x_1x_2-plane. The direction of the fibers makes an angle with respect to the x_1-axis of $\phi = 90°$ for 30 laminas and of $\phi = 0°$ for 20 laminas. The thickness of each lamina is 0.125 mm.

a) Determine the plate stiffness matrix A and the plate compliance matrix B.
b) A circular cylindrical container with internal diameter $d = 1200$ mm is made of the laminate. The x_1-direction is parallel to the axis of the cylinder. The container is subjected to an internal pressure $p = 8$ MPa. Determine the stresses T_1, T_2, and T_6.

Problem 7.26. Derive the following formulas for the modulus of elasticity η_1 and the shear modulus μ_6 with respect to the laminate axes for a unidirectional lamina with fibers making a 45° angle with the laminate axes.

$$\frac{1}{\eta_1} = \frac{1}{4}\left[\frac{1-2\nu_x}{\eta_x} + \frac{1}{\eta_y} + \frac{1}{\mu}\right], \frac{1}{\eta_6} = \frac{1+2\nu_x}{\eta_x} + \frac{1}{\eta_y}$$

Check if the formulas give the correct values for an isotropic laminate.