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Waveguide propagation

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1 Introduction

In electromagnetism a transmission line transmits the fundamental TEM mode which propagates signals at arbitrarily low frequencies. This requires at least two conductors. In this note we will consider components which do not transmit the TEM mode, but nevertheless are used for transmission of signals. These components are here called waveguides. One example is a hollow metallic waveguide which transmits waves in its interior and contains only one conductor. Another example is a dielectric slab which does not contain a conductor at all. A third example is a regular transmission line operated in a mode different from the TEM mode. In a certain sense the two terms are overlapping: A transmission line is guiding waves, and a waveguide does transmit signals. Nevertheless, we shall define the transmission line as one transmitting the TEM mode, and a waveguide as propagating other modes. A consequence of this is that metallic waveguides (but not dielectric ones) are associated with a distinct frequency below which the mode of interest can not propagate. Also, in a waveguide the group velocity and phase velocity are both different from the plane wave velocity.

Applications are primarily within all uses of high frequencies where the specific properties are advantageous. A metallic waveguide has lower loss and can transmit higher power than regular transmission lines. The signal also leaks out to a much less extent. A dielectric waveguide such as an optical fiber can transmit significantly more information than any other waveguiding device or transmission line.

In the following discussion we shall consider only lossless materials. This means that all conductors and dielectrics have perfect material properties.

2 Propagation in waveguides, general relations

We will consider some relations which are valid for waveguide structures of arbitrary cross section. We assume that the waveguide structures are ***cylindrical*** which means that the cross section is ***uniform***, so that it does not vary along the propagation path.

We assume that the wave propagation is in the $+z$-direction and is harmonic with time. This means that all field components can be described by the factor $e^{j\omega t}e^{-\gamma z} = e^{\alpha z}e^{j(\omega t-\beta z)}$, where γ is the propagation constant and ω is the angular frequency. As an example, a ***physical*** or ***instantaneous*** electric vector field $\vec{E}(x,y,z,t)$ that is part of an electromagnetic wave has three cartesian components and varies in space and time. With harmonic wave propagation in z-direction this physical field can be represented by an ***electric field phasor vector*** $\hat{\vec{E}}(x,y,z)$ so that

$$\vec{E}(x,y,z,t) = \mathrm{Re}\left\{\hat{\vec{E}}(x,y,z)e^{j\omega t}\right\} = \mathrm{Re}\left\{\hat{\vec{E}}^0(x,y)e^{j\omega t-\gamma z}\right\}, \quad (1)$$

where the z-independent electric field phasor vector $\hat{\vec{E}}^0(x,y)$ has been introduced. This quantity varies with the transverse coordinates only and is convenient to introduce to suppress the z-variation in the following equations. Since we have harmonic propagation, we can do formal substitutions in the equations so that

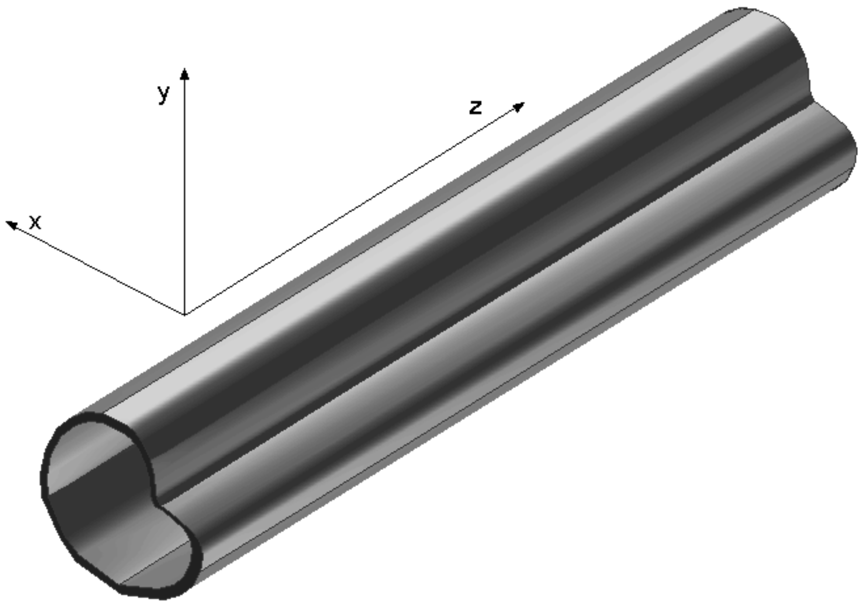


Fig. 1. Uniform waveguide with constant cross section along z-axis.

$$\frac{\partial}{\partial t} \to j\omega,$$
$$\frac{\partial}{\partial z} \to -\gamma. \tag{2}$$

We shall now discuss what this means to propagation in a waveguide as shown in Fig. 1. Previously we have developed the Helmholtz's equations for the electric and magnetic fields expressed by the phasor vectors so that we obtain (when suppressing the *z*-variation)

$$\nabla^2 \hat{\vec{E}} + k^2 \hat{\vec{E}} = 0, \tag{3}$$
$$\nabla^2 \hat{\vec{H}} + k^2 \hat{\vec{H}} = 0. \tag{4}$$

Here the wave number of an unbounded medium $k = \omega\sqrt{\mu\varepsilon}$ has been used, and $\hat{\vec{E}}$ and $\hat{\vec{H}}$ are functions of three space coordinates.

Since we know that the solution represents a wave in the *z*-direction, we want to separate the *z*-part of the Laplacian operator. As an example, we write for the electric field in Cartesian coordinates

$$\nabla^2 \hat{\vec{E}} = \left(\nabla_\perp^2 + \nabla_z^2\right)\hat{\vec{E}} = \left(\nabla_{xy}^2 + \nabla_z^2\right)\hat{\vec{E}} = \left(\nabla_{xy}^2 + \frac{\partial^2}{\partial z^2}\right)\hat{\vec{E}} = \left(\nabla_{xy}^2 + \gamma^2\right)\hat{\vec{E}}, \tag{5}$$

where we have replaced the transverse part of the Laplacian operator $\nabla_\perp^2$ by the Cartesian version $\nabla_{xy}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Substituted into Eq. (3) and multiplied by $e^{\gamma z}$ this gives

$$\nabla_{xy}^2 \hat{\vec{E}}^0 + \left(\gamma^2 + k^2\right)\hat{\vec{E}}^0 = \nabla_{xy}^2 \hat{\vec{E}}^0 + h^2 \hat{\vec{E}}^0 = 0, \tag{6}$$

so that now only variation in the transverse coordinates is involved. By the same procedure we obtain

$$\nabla_{xy}^2 \hat{\vec{H}}^0 + \left(\gamma^2 + k^2\right)\hat{\vec{H}}^0 = \nabla_{xy}^2 \hat{\vec{H}}^0 + h^2 \hat{\vec{H}}^0 = 0, \tag{7}$$

where we have introduced

$$h^2 = \gamma^2 + k^2 . \tag{8}$$

The substitution we did above for the transverse part of the Laplacian operator is an arbitrary substitution. If we had chosen cylindrical coordinates, for example, we would have written $\nabla_\perp^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}$.

The two vector equations (6) and (7) actually represent six second order differential equations, one for each spatial component of $\vec{\hat{E}}^0$ or $\vec{\hat{H}}^0$. In solving for these unknown variables, we have to specify the transverse geometry and the boundary conditions. The six vector components (each dependent on space coordinates) are dependent through Maxwell's equations. We will use the two curl equations (source-free case) for this purpose. Using the assumed variation with *z* and *t*, we obtain

$\nabla \times \vec{\hat{E}} = -j\omega\mu\vec{\hat{H}} \rightarrow$	$\nabla \times \vec{\hat{H}} = j\omega\varepsilon\vec{\hat{E}} \rightarrow$
$\left.\begin{aligned} &\frac{\partial \hat{E}_z^0}{\partial y} + \gamma \hat{E}_y^0 = -j\omega\mu \hat{H}_x^0 \quad \text{(a)} \\ &-\gamma \hat{E}_x^0 - \frac{\partial \hat{E}_z^0}{\partial x} = -j\omega\mu \hat{H}_y^0 \quad \text{(b)} \\ &\frac{\partial \hat{E}_y^0}{\partial x} - \frac{\partial \hat{E}_x^0}{\partial y} = -j\omega\mu \hat{H}_z^0 \quad \text{(c)} \end{aligned}\right\} \quad (9)$	$\left.\begin{aligned} &\frac{\partial \hat{H}_z^0}{\partial y} + \gamma \hat{H}_y^0 = j\omega\varepsilon \hat{E}_x^0 \quad \text{(a)} \\ &-\gamma \hat{H}_x^0 - \frac{\partial \hat{H}_z^0}{\partial x} = j\omega\varepsilon \hat{E}_y^0 \quad \text{(b)} \\ &\frac{\partial \hat{H}_y^0}{\partial x} - \frac{\partial \hat{H}_x^0}{\partial y} = j\omega\varepsilon \hat{E}_z^0 \quad \text{(c)} \end{aligned}\right\} \quad (10)$

Because of the discussion that follows later, it is practical to consider the four transverse field components as unknown variables, and express them as functions of the field components in *z*-direction. This can be obtained as follows. If we inspect Eqs. (9)(a) and(10)(b), these two equations contain the two "unknown" variables $\hat{E}_y^0$ and $\hat{H}_x^0$. We can eliminate $\hat{E}_y^0$ by multiplying the first equation by $j\omega\varepsilon$ while the second is multiplied by γ, followed by adding the two resulting equations. This gives an expression for $\hat{H}_x^0$. Multiplying the same two equations by other factors (find them!) eliminates $\hat{H}_x^0$ and obtains $\hat{E}_y^0$. By operating in the same way on Eqs. (9)(b) and (10)(a), we obtain results for $\hat{E}_x^0$ and $\hat{H}_y^0$. The result is

$$\hat{H}_x^0 = -\frac{1}{h^2}\left(\gamma\frac{\partial \hat{H}_z^0}{\partial x} - j\omega\varepsilon\frac{\partial \hat{E}_z^0}{\partial y}\right) \quad , \tag{11}$$

$$\hat{H}_y^0 = -\frac{1}{h^2}\left(\gamma\frac{\partial \hat{H}_z^0}{\partial y} + j\omega\varepsilon\frac{\partial \hat{E}_z^0}{\partial x}\right) \quad , \tag{12}$$

$$\hat{E}_x^0 = -\frac{1}{h^2}\left(\gamma\frac{\partial \hat{E}_z^0}{\partial x} + j\omega\mu\frac{\partial \hat{H}_z^0}{\partial y}\right) \quad , \tag{13}$$

$$\hat{E}_y^0 = -\frac{1}{h^2}\left(\gamma\frac{\partial \hat{E}_z^0}{\partial y} - j\omega\mu\frac{\partial \hat{H}_z^0}{\partial x}\right) \quad , \tag{14}$$

where, as before, $h^2 = \gamma^2 + k^2$, $\gamma = \alpha + j\beta$ and $k = \omega\sqrt{\mu\varepsilon}$. We therefore solve the complete field problem in ***two basic steps:***

1. We first solve Eqs. (6) and (7) for the longitudinal wave field components.

2. Then we find the transverse components from Eqs. (11) - (14).

It turns out that the electromagnetic waves we will discuss, can be conveniently divided into three groups according to the z-components of the electric and magnetic fields.

1. ***Transverse electromagnetic (TEM) waves***. In this case both E_z and H_z are zero. An example of this is a plane electromagnetic wave which has both electric and magnetic field perpendicular to the propagation direction.

2. ***Transverse electric (TE) waves*** contain no electric field component in the z-direction. However, the magnetic field contains a z-component.

3. ***Transverse magnetic (TM) waves*** contain no magnetic field component in the z-direction. However, the electric field contains a z-component.

These wave groups will be discussed in the following.

2.1 TEM waves

In this case both E_z and H_z are zero. From Eqs. (11) - (14) we see that this implies the trivial condition that all transverse fields are also zero unless $h^2 = 0$. Therefore, TEM waves exist only when

$$h_{TEM}{}^2 = \gamma_{TEM}{}^2 + k^2 = 0. \tag{15}$$

This further implies that

$$\boxed{\gamma_{TEM} = jk = j\omega\sqrt{\mu\varepsilon}}, \tag{16}$$

which is the same relation we know from a uniform plane wave in an infinite medium, and also from a TEM wave on a lossless transmission line. In all three cases we therefore obtain the same phase velocity given by

$$\boxed{u_{p(TEM)} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}}}. \tag{17}$$

We obtain the ratio between the x-component of the electric field and the y-component of the magnetic field by using Eqs. (9)(b) and (16). This gives the ***wave impedance***

$$Z_{TEM} @ \frac{\hat{E}_x^0}{\hat{H}_y^0} = \frac{j\omega\mu}{\gamma_{TEM}} = \frac{j\omega\mu}{j\omega\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}} = \eta. \tag{18}$$

This shows that the phase velocity u_p and Z_{TEM} are

- equal to the ***plane wave velocity*** u and the ***intrinsic impedance*** η, respectively, for plane waves in the medium,
- both independent of frequency.

We can also find (from Eqs. (9)(a) and (16)) that the ratio of the two other transverse field components can be written

$$\frac{\hat{E}_y^0}{\hat{H}_x^0} = -\frac{j\omega\mu}{\gamma_{TEM}} = -Z_{TEM} = -\sqrt{\frac{\mu}{\varepsilon}}\,. \tag{19}$$

The results we have obtained for the transverse fields in Eqs. (18) and (19) can be written as the vector equation

$$\boxed{\vec{H} = \frac{1}{Z_{TEM}}\vec{a}_z \times \vec{E}}, \tag{20}$$

after we have multiplied each wave component by $e^{j\omega t}e^{-\gamma z}$ and taken the real part. This form is equal to the relation we know from propagation of a uniform plane wave in an unbounded medium.

We will now show that waveguides of a closed cross section with only one conductor can not support TEM waves. In magnetostatics we have learned that magnetic flux lines always close upon themselves. This is also true in time-varying cases. Therefore a possible TEM wave in a waveguide would contain B and H fields forming closed loops in the transverse plane. By using Ampere's law around one such loop, we would find that there would be a current inside the loop in the longitudinal direction. This current would be the sum of conduction and displacement current. The absence of a center conductor would give zero conduction current. Also, the displacement current ($\partial\vec{D}/\partial t$) would have no longitudinal component since for TEM waves $E_z = 0$ by definition. Since no total longitudinal current occurs, the solution does not exist if only one conductor is present.

We conclude that a hollow waveguide consisting of a single conductor can not support TEM waves. If we have a hollow waveguide with a center conductor such as in a coaxial cable, however, TEM waves are supported. Other two-conductor waveguides such as striplines and two-wire lines also support these waves.

2.2 TE waves

Transverse electric (TE) waves have no electric field component in the z-direction. They can be analyzed by solving for the z-component of Eq. (7) to give:

$$\nabla_{xy}^2 \hat{H}_z^0 + h^2 \hat{H}_z^0 = 0. \tag{21}$$

As discussed above, the field problem can be solved by first solving this second-order partial differential equation with appropriate boundary conditions at the walls. Thereafter, we can find all transverse field components by setting E_z equal to zero in Eqs. (11) to (14). This gives the equations

$$\hat{H}_x^0 = -\frac{\gamma}{h^2}\frac{\partial \hat{H}_z^0}{\partial x}, \tag{22}$$

$$\hat{H}_y^0 = -\frac{\gamma}{h^2}\frac{\partial \hat{H}_z^0}{\partial y}, \tag{23}$$

$$\hat{E}_x^0 = -\frac{j\omega\mu}{h^2}\frac{\partial \hat{H}_z^0}{\partial y}, \tag{24}$$

$$\hat{E}_y^0 = \frac{j\omega\mu}{h^2}\frac{\partial \hat{H}_z^0}{\partial x}. \tag{25}$$

The ***wave impedance*** is the ratio between two perpendicular field components. We find

$$\boxed{Z_{TE} \triangleq \frac{\hat{E}_x^0}{\hat{H}_y^0} = -\frac{\hat{E}_y^0}{\hat{H}_x^0} = \frac{j\omega\mu}{\gamma}}. \tag{26}$$

It should be noted that although the field components are functions of the transverse coordinates, the wave impedance does not vary over the cross section. Eq. (26) also implies the following vector equation:

$$\boxed{\vec{E} = -Z_{TE}\left(\vec{a}_z \times \vec{H}\right)}, \tag{27}$$

As stated above, the waveguide problem can be solved by starting with the Helmholtz' equation for the longitudinal field H_z (Eq. (21)). It turns out that solutions are possible only for ***discrete values of h***. There is an infinite number of these values, but not all values are possible. These values are called ***characteristic values*** or ***eigenvalues*** of this specific problem. Each of these eigenvalues is associated with a particular ***mode*** which is characterized by a unique combination of its own

- velocity and
- field distribution.

We shall now show that the eigenvalues are all real numbers. Some important conclusions that are valid in the general case can be found from this fact. We know from Eq. (8) that

$$\gamma = \sqrt{h^2 - k^2} = \sqrt{h^2 - \omega^2 \mu\varepsilon} . \tag{28}$$

Two distinct ranges for the propagation constant can be found, depending on whether the argument under the square root is positive or negative. Let us start with the value zero for this argument. The value $\gamma = 0$ gives the condition

$$\omega_c^2 \mu\varepsilon = h^2 . \tag{29}$$

From this equation we conclude that h is real. We find the ***cutoff frequency***

$$\boxed{f_c = \frac{\omega_c}{2\pi} = \frac{h}{2\pi\sqrt{\mu\varepsilon}}}, \tag{30}$$

which depends on the eigenvalue of the particular eigenmode. We can then rewrite Eq. (28) in terms of f_c into

$$\gamma = h\sqrt{1 - \frac{\omega^2 \mu\varepsilon}{h^2}} = h\sqrt{1 - \left(\frac{f}{f_c}\right)^2} . \tag{31}$$

We will find the wavelength for a plane wave in an unbounded medium given by μ and ε corresponding to f_c. In this medium $u = 1/\sqrt{\mu\varepsilon}$ is the velocity of light. Therefore,

$$\boxed{\lambda_c = \frac{u}{f_c} = \frac{1}{f_c\sqrt{\mu\varepsilon}} = \frac{2\pi}{h}}. \tag{32}$$

Thus ***the eigenvalue can be interpreted as the wavenumber for plane waves at the cutoff frequency***.

We shall consider two main cases; frequencies above and under the cutoff frequency.

Case 1. $f > f_c$ or $\left(f/f_c\right)^2 > 1$.

Frequency is higher than the cutoff frequency. In this range γ is imaginary. Its value is

$$\gamma = j\beta = j\sqrt{k^2 - h^2} = jk\sqrt{1-\left(\frac{h}{k}\right)^2} = jk\sqrt{1-\left(\frac{\omega_c/u}{\omega/u}\right)^2} = jk\sqrt{1-\left(\frac{f_c}{f}\right)^2}. \tag{33}$$

Thus we have a propagating mode with a phase constant

$$\boxed{\beta = k\sqrt{1-\left(\frac{f_c}{f}\right)^2}}. \tag{34}$$

This phase constant is characteristic of the propagation in the z-direction which is along the waveguide axis. Thus we can define a periodicity along the waveguide, the ***guide wavelength***

$$\lambda_g = \frac{2\pi}{\beta} = \frac{2\pi}{k}\frac{k}{\beta} = \frac{\lambda}{\sqrt{1-\left(f_c/f\right)^2}} > \lambda, \tag{35}$$

where

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\omega\sqrt{\mu\varepsilon}} = \frac{u}{f} \tag{36}$$

is the wavelength of a plane wave in an unbounded medium given by μ and ε.

The relation between γ, k and h in Eq. (28) can be transformed into a relation between the wavelengths we have introduced above. From Eq. (28) this gives

$$k^2 = \beta^2 + h^2 \tag{37}$$

or

$$\frac{1}{\lambda^2} = \frac{1}{\lambda_g^2} + \frac{1}{\lambda_c^2}. \tag{38}$$

The ***phase velocity*** of the wave in the guide is

$$u_p = \frac{\omega}{\beta} = \frac{\omega}{k}\frac{k}{\beta} = \frac{u}{\sqrt{1-\left(f_c/f\right)^2}} = \frac{\lambda_g}{\lambda}u > u\,. \tag{39}$$

This equation shows that the phase velocity for a propagating mode in a waveguide is always higher than the plane wave velocity in an unbounded material of the same properties, and that it is frequency dependent. Thus the mode is ***dispersive*** in a single-conductor waveguide, although a plane wave in an unbounded medium of the same properties is nondispersive.

The ***group velocity*** is

$$u_g = \frac{1}{d\beta/d\omega}, \tag{40}$$

where from Eq. (34) we can write

$$\beta = k\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2} = \frac{k}{\omega}\sqrt{\omega^2-\omega_c^2} = \frac{1}{u}\sqrt{\omega^2-\omega_c^2}\,. \tag{41}$$

This yields

$$u_g = u\sqrt{1-\left(\frac{f_c}{f}\right)^2} = \frac{\lambda}{\lambda_g}u < u. \tag{42}$$

By multiplying Eqs. (39) and (42) we find

$$u_p u_g = u^2. \tag{43}$$

An illustration of these relations will be found later.

From Eqs. (26) and (33) we find the wave impedance to be

$$Z_{TE} = \frac{j\omega\mu}{jk\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{j\omega\mu}{j\omega\sqrt{\mu\varepsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{\eta}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}. \tag{44}$$

This shows that the wave impedance of a propagating TE mode in a waveguide consisting of a lossless dielectric is purely resistive.

Case 2. $f < f_c$ or $\left(\frac{f}{f_c}\right)^2 < 1$.

Frequency is lower than the cutoff frequency and γ is real. From Eq. (31) we see:

$$\gamma = \alpha = h\sqrt{1-\left(\frac{f}{f_c}\right)^2}. \tag{45}$$

This means that the phase constant is zero, so the mode is ***non-propagating*** or ***evanescent***. The ***attenuation constant*** α causes the amplitude to be attenuated according to $e^{-\alpha z}$.

We have seen that the waveguide acts as a ***high-pass filter*** such that only frequencies $f > f_c$ for the relevant mode can propagate in the waveguide. Lower frequencies will not be transmitted through the waveguide.

2.3 TM waves

Transverse magnetic (TM) waves have no magnetic field component in the z-direction. They can be analyzed by solving Eq. (6) for $\hat{E}_z^0$:

$$\nabla_{xy}^2 \hat{E}_z^0 + h^2 \hat{E}_z^0 = 0 . \qquad (46)$$

In the same way as for TE waves, this equation can be solved by applying the appropriate boundary conditions at the walls. The detailed solutions have some distinct differences from the TE case. On the other hand, the methods and solutions for the two mode classes have many similarities. We will therefore not pursue this theme further here.

We shall next look into wave propagation of TE waves in specific geometries.

3 TE modes in metallic waveguides

We shall now discuss metallic waveguides. This means that propagation of the electric and magnetic fields is along one or several electric conductors.

3.1 TE modes in a parallel-plate waveguide

The ***parallel-plate waveguide*** consists of two parallel metal sheets as shown in Fig. 2. This configuration is also known as a ***planar transmission line***. From earlier we know that this configuration can transmit signals of frequencies all the way down to zero. Operated in this way the signal is transmitted as a TEM mode. We shall see that this is not the only mode that can be transmitted.

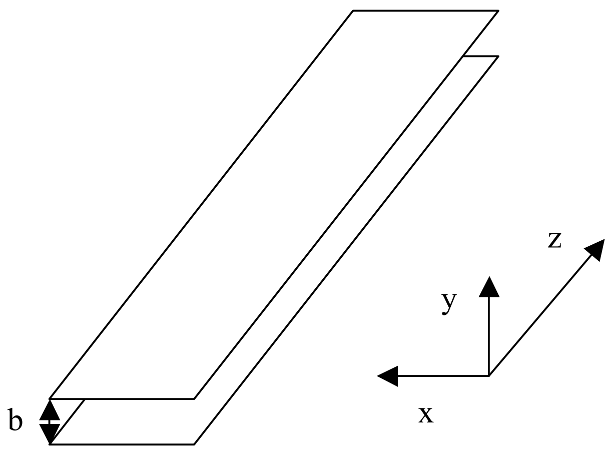


Fig. 2. An infinite parallel-plate waveguide.

In Fig. 2 two parallel metal plates with a dielectric material in between constitute the transmission line with propagation in *z*-direction. The dimension in this direction is assumed to be infinite. In this analysis we assume the plate distance *b* to be much smaller than the plate width in the *x*-direction. We therefore neglect fringe fields at the plate edges, and assume uniform fields along *x*-direction as if the plates have an infinite width. This means that we can do a two-dimensional analysis where any variation in the *x*-direction is zero. After the mathematical analysis we shall discuss a physical interpretation of this case and also illustrate some more general concepts related to waveguide propagation.

3.1.1 Mathematical analysis

Since there is no *x*-variation, Eq. (21) can be written

$$\frac{d^2 \hat{H}_z^0(y)}{dy^2} + h^2 \hat{H}_z^0(y) = 0. \tag{47}$$

The general solution of this problem can be written

$$\hat{H}_z^0(y) = A\sin(hy) + B\cos(hy). \tag{48}$$

The boundary condition we need to determine the constants can be obtained from Eq. (24). The tangential electric field E_x must vanish at the boundaries, therefore

$$\frac{d\hat{H}_z^0(y)}{dy} = 0 \quad \text{at } y = 0 \text{ and } y = b. \tag{49}$$

The first condition gives $A = 0$ while the second condition gives

$$h = \frac{n\pi}{b}. \tag{50}$$

where n is an integer. Therefore,

$$\boxed{\hat{H}_z^0(y) = B_n \cos\left(\frac{n\pi y}{b}\right)}, \tag{51}$$

and the only two other fields are

$$\boxed{\hat{H}_y^0(y) = \frac{\gamma}{h} B_n \sin\left(\frac{n\pi y}{b}\right)}, \tag{52}$$

$$\boxed{\hat{E}_x^0(y) = \frac{j\omega\mu}{h} B_n \sin\left(\frac{n\pi y}{b}\right)}. \tag{53}$$

We note that ***the integer n gives the number of half periods*** of the variation of the fields over the cross section. Therefore these modes are denoted ***TE_n modes***. The value $n = 0$ must be excluded since this case corresponds to fields having no variations over the cross section, and we have already the condition that the electric field is zero at the boundaries. In the following discussion we shall limit ourselves to the case $n = 1$.

3.1.2 Physical interpretation

We now want to discuss more physical aspects of the TE_1 mode obtained from these results. In accordance with the previous section, let us start by assuming a two-dimensional problem with the electric field parallel to the x-axis. This is illustrated in Fig. 3.

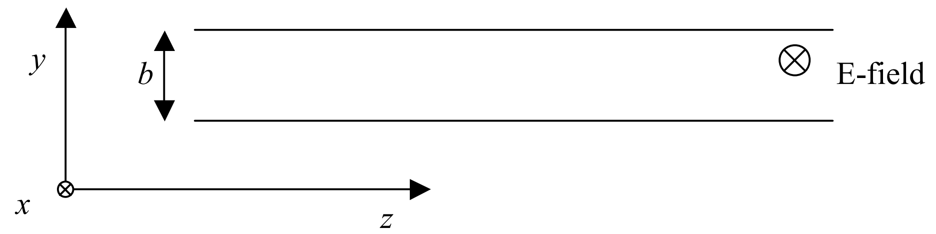


Fig. 3. Two-dimensional TE problem.

Let us attempt to transmit a single plane uniform TEM wave of this polarization through this structure. We see that any such single wave propagating within the structure will have tangential electric fields different from zero at the metallic boundaries. This violates the boundary conditions for perfect conductors. Therefore this solution does not exist.

It is, however, possible to satisfy the boundary conditions if we introduce **two** plane waves of this polarization which propagate in different directions within the structure.

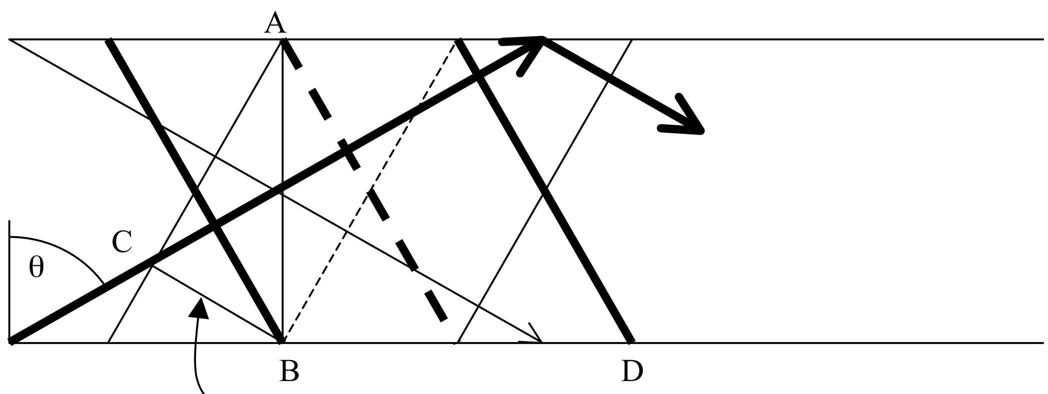


Fig. 4. TE mode as a sum of two TEM waves.

Assume a plane wave traveling along the inner part of the structure at an angle θ with the surface normal as indicated by **thick** lines in Fig. 4. The propagation direction is shown by an arrow and the phase planes by full lines (+ phase) and dashed lines (- phase), respectively. The phase planes are perpendicular to the propagation direction. This upward directed wave will be reflected by the metallic surface into a downward propagating wave (thin lines) at the same angle θ. Upon reflection from the upper plane, $y = b$, the boundary conditions for the tangential field, $E_{tang} = 0$ must be satisfied. This means that the upward propagating wave will have a positive extremum value at the same location as the downward propagating wave has a negative extremum. A similar reflection will occur at the lower boundary, $y = 0$. For the two waves to give zero total field also here, the two waves must have the same amplitude and be at opposite phase here. From the triangle ABC in Fig. 4 we find that

$$\cos\theta = \frac{\lambda}{2b}. \tag{54}$$

Since we must have $\cos\theta \le 1$ we require $\lambda \le 2b = \lambda_c$ where λ_c is ***the cutoff wavelength***. If we write this condition

$$\frac{\lambda}{u} \le \frac{\lambda_c}{u}, \tag{55}$$

we obtain the following frequency condition,

$$\boxed{f \ge f_c = \frac{u}{\lambda_c}}, \tag{56}$$

where f_c is the ***cutoff frequency***. The meaning of this is that the mode can be propagated only at frequencies above this cutoff frequency as was also found in Chapter 2.

Returning to Eq. (54), we see that it can be written

$$\cos\theta = \frac{\lambda}{\lambda_c} = \frac{f_c}{f}, \qquad (57)$$

which means that the angle θ varies with frequency.

We can define the periodicity along the waveguide by the guide wavelength (distance BD)

$$\lambda_g \equiv \frac{2\pi}{\beta} = \frac{\lambda}{\sin\theta} = \frac{\lambda}{\sqrt{1-\cos^2\theta}} = \frac{\lambda}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} \geq \lambda. \qquad (58)$$

Thus we have shown on a field basis that this propagating mode can be interpreted as the interference of two plane waves which are coupled together by boundary conditions satisfied by reflections at these boundaries.

3.1.3Velocities

We can define three different velocities in the planar waveguide. With reference to Fig. 5, we consider two phase planes at a distance δr, corresponding to the distance the plane wave has propagated during the time δt.

This defines the ***plane wave velocity*** $u = \frac{\delta r}{\delta t} = \frac{1}{\sqrt{\mu\varepsilon}}$. If we consider the velocity at which a phase plane intersection with the boundary propagates in z-direction, we find that this ***phase velocity*** is

$$u_p = \frac{u}{\sin\theta} = \frac{u}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}. \qquad (59)$$

If we send a short pulse of energy along the waveguide, it bounces up and down between the two boundaries by reflection. The velocity of the energy propagation in the direction of the waveguide axis is the component of u along the guide axis. It is called the ***energy*** or ***group velocity*** and it is given by

$$u_g = u\sin\theta = u\sqrt{1-\left(\frac{f_c}{f}\right)^2}\,, \tag{60}$$

where θ is found from Eq. (57). The variation of these three velocities with frequency is shown in Fig. 6. The figure demonstrates the dispersion of u_p and u_g.

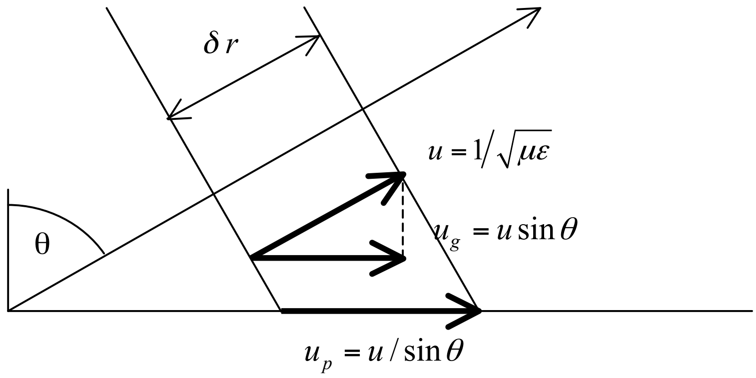


Fig. 5. Demonstration of three different velocities.

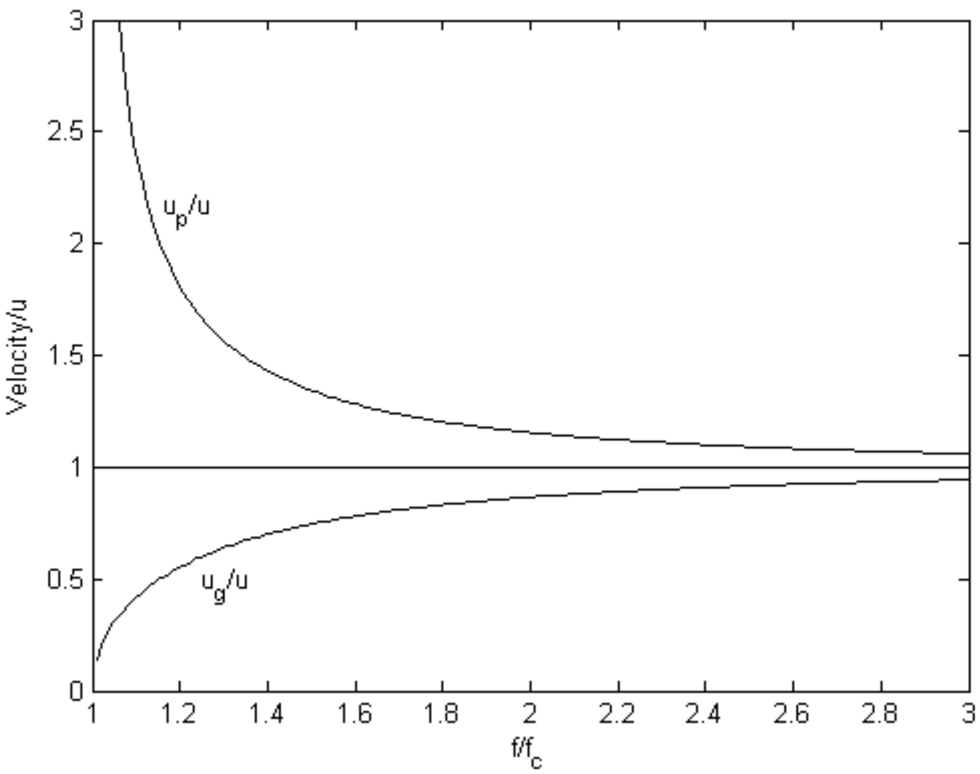


Fig. 6. Phase velocity and group velocity as functions of frequency in relative magnitudes. Straight line corresponds to plane wave velocity.

As is apparent from these two equations, the velocities satisfy the following relations:

$$u_g \le u \le u_p, \tag{61}$$

$$u_g u_p = u^2. \tag{62}$$

This agrees with our previous general discussion of the different velocities. In particular, it is seen that when $f \to f_c$, $u_g \to 0$ and $u_p \to \infty$.

3.1.4Fields

The electric field has one component which is directed in the *x*-direction. This field is zero at the boundaries and varies sinusoidally between the boundaries by one half period.

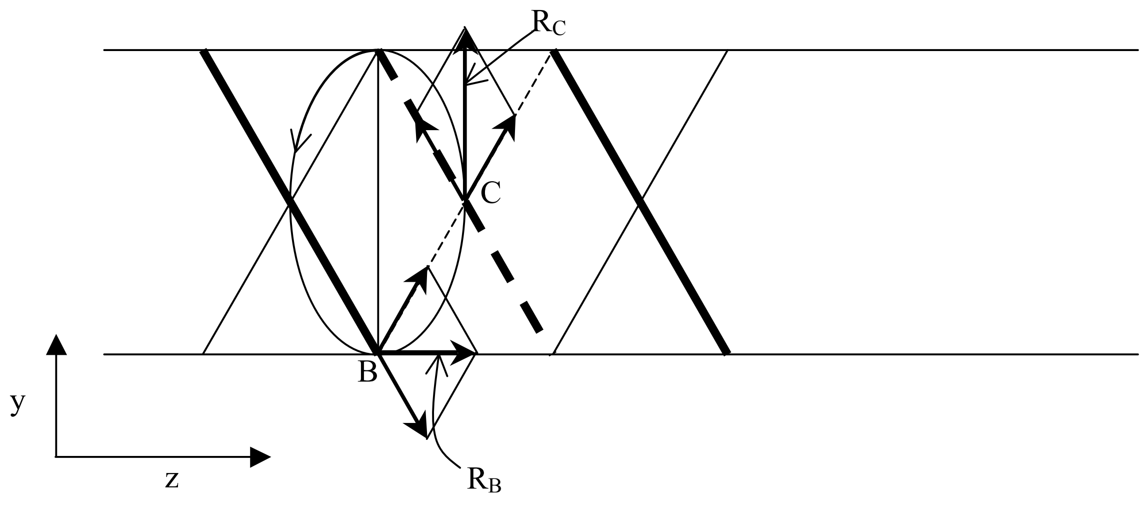


Fig. 7. Visualization of the magnetic field. Phase planes of the two interfering plane waves between the metal boundaries are shown as straight lines.

The magnetic field has both *y*- and *z*- components. We can find some important properties of this field by considering the concept of two interfering plane waves. We first consider point B on the metal boundary in Fig. 7 where both waves have a maximum amplitude, and where they are of opposite phase. At this point the magnetic field of one plane wave will be directed to the right of and perpendicular to the propagation direction while the other contribution will be directed to the left. This gives a resulting magnetic field vector R_B which is directed in the *z*-direction. If on the other hand we consider points along the center line of the waveguide such as point C, the phases of the two plane waves are the same. The two magnetic field contributions will therefore be directed to the same side compared to the respective propagation direction of each plane wave. As a function of *z*, point C represents maximum magnitude of both fields, and the resulting field R_C will therefore be a maximum and be directed in the *z*-direction.

We further note that the two points B and C are displaced by ¼ period ($\lambda_g/4$) along the waveguide. By combining this information with the discussion above and also with information from similar points, we conclude that ***it is possible for the H-field to describe closed loops in the y-z-plane***. This agrees with our fundamental knowledge of the magnetic field. One such closed loop is indicated in Fig. 7.

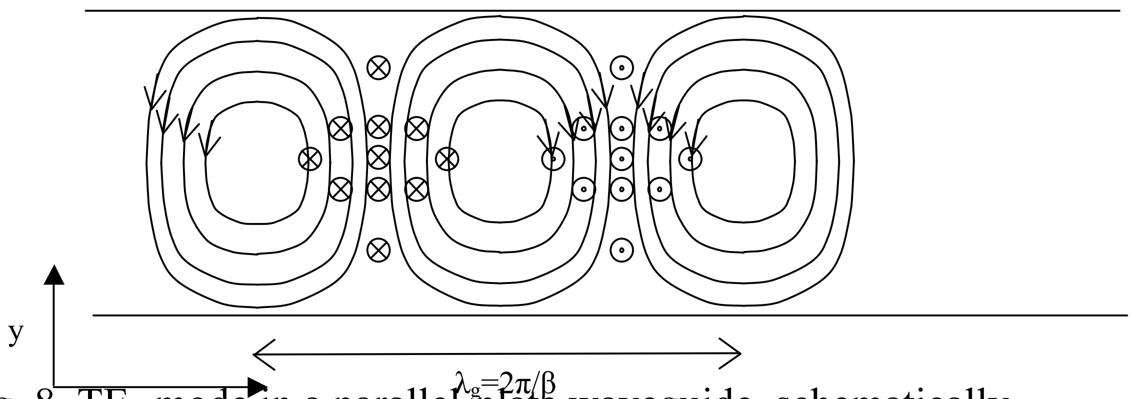


Fig. 8. TE$_1$ mode in a parallel-plate waveguide, schematically.

→ Magnetic field

⊙, ⊗ Electric field

To sum up, the instantaneous fields are as shown roughly in Fig. 8. The electric field normal to the paper plane is shown as arrow signs in both directions. The density of arrows indicates the electric field strength. The closed loops indicate the magnetic fields. Both fields vary periodically along the propagation direction. We note that the tangential electric field and the normal magnetic field are both zero at the boundaries. Thus the boundary conditions are satisfied.

3.2 TE modes in rectangular waveguides

In the previous sections we discussed the planar transmission line geometry. We will now discuss a waveguide consisting of only ***one single hollow conductor*** surrounding a region of a dielectric where the electric and magnetic fields propagate. We know that such a conductor does not transmit direct current; however, at higher frequencies this configuration can transmit signals.

The shape of the cross section can take many forms. For technical reasons the preferred shape is rectangular. This type of waveguide is used in many critical high frequency applications where low loss and high power handling are desired properties. Of these, high power radars may be best known.

We shall first give a description of a possible mode that can be transmitted in this structure, followed by a more general mathematical treatment of the subject.

3.2.1 Transition to a TE mode in a rectangular waveguide

The fields of the TE_1 mode we discussed in the previous section, were situated between two metal plates extending infinitely in the transverse direction. We will show the similarity between this mode and a mode in a waveguide with rectangular cross section. This cross section can be obtained by introducing two new metallic boundaries to

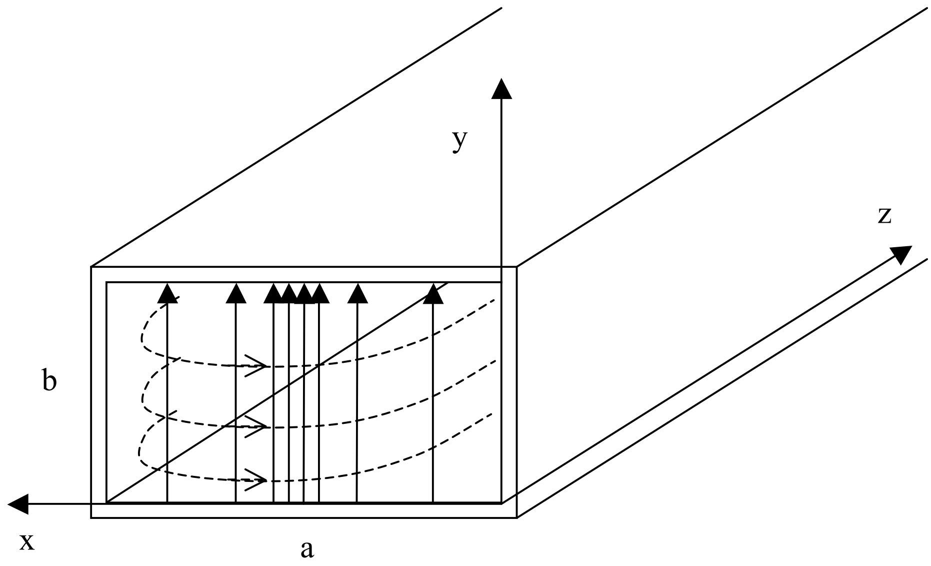


Fig. 9. TE_{10} mode in a rectangular waveguide. Full lines: Electric field. Dashed lines: Magnetic field.

the planar transmission line geometry, thus limiting the extent of the fields as indicated in Fig. 9. The dimensions of the rectangle are a and b in x and y directions, respectively. The same fields we calculated for the TE_1 mode in the planar transmission line have been sketched in the volume limited by these four boundaries. We note that the orientation of the fields has been changed so that the electric field now is along the y-axis. We realize that the boundary conditions are satisfied automatically at the new boundaries (y = 0 and b) since the electric field is normal and the magnetic field is tangential to the surface. This new mode is uniform in y-direction. The mode is called a TE_{10} mode and is one of several modes which can propagate in a rectangular waveguide. We shall look into this more quantitatively in the next section.

3.2.2TE modes in rectangular waveguides; general case

To obtain the solution for TE modes in rectangular waveguides, we follow the prescription according to Eqs. (21) to (25). We first solve the differential equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\hat{H}_z^0+h^2\hat{H}_z^0=0, \tag{63}$$

where the relation between the two quantities with and without z-dependence is given by $\hat{H}_z(x,y,z)=\hat{H}_z^0(x,y)e^{-\gamma z}$.

To solve the problem we apply the method of separation of variables, writing

$$\hat{H}_z^0(x,y)=X(x)Y(y). \tag{64}$$

Substituting this expression into Eq. (63) and dividing by $X(x)Y(y)$ gives a relation that can be written

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2}+\frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2}+h^2=0. \tag{65}$$

In this equation, we note that the first term depends on x only, just as the second term depends on y only. The last term does not depend on any coordinate. From this observation we can conclude that all three terms must be constants. The two first terms we set equal to $-k_x^2$ and $-k_y^2$, respectively. This gives a new set of ordinary differential equations:

$$\frac{d^2X(x)}{dx^2}+k_x^2X(x)=0, \tag{66}$$

$$\frac{d^2Y(y)}{dy^2}+k_y^2Y(y)=0, \tag{67}$$

with

$$k_x^2+k_y^2=h^2. \tag{68}$$

The general solution to the two differential equations can be written

$$X(x) = C_1 \sin k_x x + C_2 \cos k_x x, \tag{69}$$

$$Y(y) = C_3 \sin k_y y + C_4 \cos k_y y. \tag{70}$$

We use the boundary conditions to determine relations between the four constants. We apply the fact that the electric field has no tangential components at the metal boundaries. By using Eqs. (24) and (25) we therefore get the following conditions:

$$\frac{\partial \hat{H}_z^0}{\partial x} = 0 \ \left(\hat{E}_y^0 = 0\right) \quad \text{at} \quad x = 0, \tag{71}$$

$$\frac{\partial \hat{H}_z^0}{\partial x} = 0 \ \left(\hat{E}_y^0 = 0\right) \quad \text{at} \quad x = a, \tag{72}$$

$$\frac{\partial \hat{H}_z^0}{\partial y} = 0 \ \left(\hat{E}_x^0 = 0\right) \quad \text{at} \quad y = 0, \tag{73}$$

$$\frac{\partial \hat{H}_z^0}{\partial y} = 0 \ \left(\hat{E}_x^0 = 0\right) \quad \text{at} \quad y = b. \tag{74}$$

The first two conditions imply that $\partial X(x)/\partial x$ is equal to zero at the boundaries $x = 0$ and a. Therefore C_1 must be zero and

$$\boxed{k_x = \frac{m\pi}{a}}, \tag{75}$$

where m is an integer. Likewise, the two last conditions imply that $\partial Y(y)/\partial y$ is equal to zero at the boundaries $y = 0$ and b. Therefore C_3 must be zero and

$$\boxed{k_y = \frac{n\pi}{b}}, \tag{76}$$

where n is an integer.

From Eq. (64) we can write the solution of the form

$$\boxed{\hat{H}_z^0(x, y) = \hat{H}_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)}, \tag{77}$$

where we have introduced the constant $\hat{H}_0 = C_2 C_4$.

The eigenvalue can be found from Eq. (68) combined with (75) and (76):

$$\boxed{h^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}. \tag{78}$$

Knowing $\hat{H}_z^0$, we can apply Eqs. (22) - (25) to obtain the transverse fields. The result is:

$$\hat{H}_x^0 = \frac{\gamma}{h^2}\left(\frac{m\pi}{a}\right)\hat{H}_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right), \tag{79}$$

$$\hat{H}_y^0 = \frac{\gamma}{h^2}\left(\frac{n\pi}{b}\right)\hat{H}_0 \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right), \tag{80}$$

$$\hat{E}_x^0 = \frac{j\omega\mu}{h^2}\left(\frac{n\pi}{b}\right)\hat{H}_0 \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right), \tag{81}$$

$$\hat{E}_y^0 = -\frac{j\omega\mu}{h^2}\left(\frac{m\pi}{a}\right)\hat{H}_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right), \tag{82}$$

where the value of γ is found from Eqs. (33), (68), (75) and (76) to be

$$\gamma = j\beta = j\sqrt{k^2 - h^2} = j\sqrt{\omega^2\mu\varepsilon - k_x^2 - k_y^2} = j\sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}. \tag{83}$$

Every combination of *m* and *n* defines a mode, and this is called a ***TE$_{mn}$* *mode***. The first subscript denotes the number of half periods of the field variation in the *x*-direction, and the second subscript gives the number of half periods of the field variation in the *y*-direction. A few examples of different TE modes are shown in Fig. 10.

All modes have a ***cutoff frequency*** which can be found from this relation by putting $\gamma = 0$. This gives

$$f_{c,mn} = \frac{\omega_{c,mn}}{2\pi} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}. \tag{84}$$

which is consistent with Eq. (30). The value of the corresponding ***cutoff wavelength*** is

$$\lambda_{c,mn} = \frac{u}{f_{c,mn}} = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}} \qquad (85)$$

From the transverse field expressions we see that either m or n, but not both, can be zero. If $a > b$, the mode with the lowest cutoff frequency has $m = 1$ and $n = 0$. For this TE_{10} mode we have

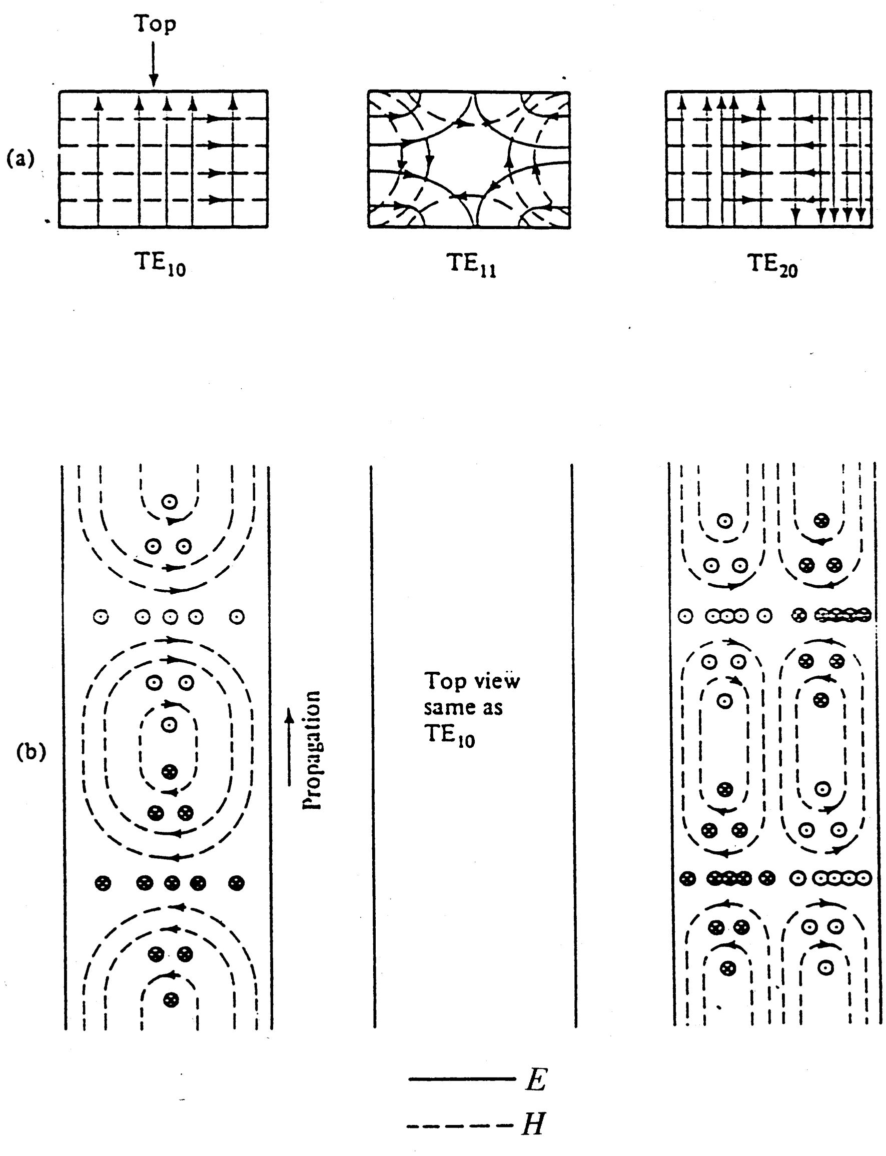


Fig. 10. Some TE modes in a rectangular metallic waveguide. a) end views; b) top views.

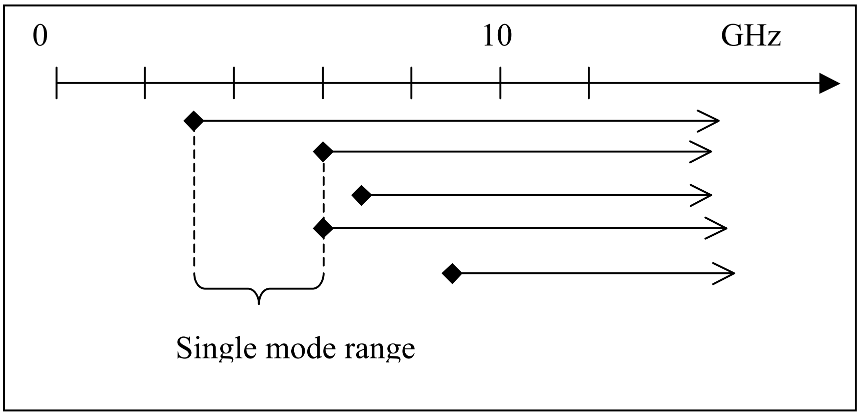
$$f_{c,10} = \frac{1}{2a\sqrt{\mu\varepsilon}} = \frac{u}{2a} \tag{86}$$

Below this frequency there is no propagating mode, and all other modes have higher cutoff frequencies. Thus there is a certain frequency range where ***only one mode propagates***. This is an advantage for transmitting signals with low distortion. For this reason the TE_{10} mode is denoted ***the dominant mode.***

Example

We want to discuss the TE modes occurring in a metallic waveguide of rectangular cross section where a = 5 cm and b = 2.5 cm, in particular the operating frequencies and at what frequencies a single mode is transmitted. Let us find the cutoff frequencies for some lower modes. Using Eq. (84) and a value $u = 3\cdot10^8$ m/s, we find the values shown in the table below. The result is also indicated graphically to the right of the table. It is seen that in this example the lowest, ***dominant mode*** is the only propagating mode in the frequency range 3 to 6 GHz. Therefore this frequency range is the best for signal transmission in this case.

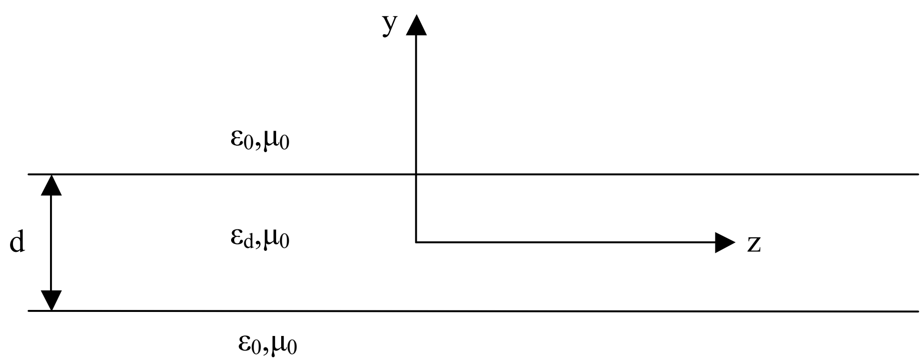
Mode	$f_{c,mn}$
(m,n)	(GHz)
1,0	3
0,1	6
1,1	6.7
2,0	6
2,1	8.5



4 TE modes in a dielectric slab waveguide

4.1 Discussion

Metallic waveguides are used in high frequency techniques up to the microwave range. When the frequency increases, the losses are becoming more severe, and in the optical range dielectric waveguides are predominant. In ***integrated optics*** thin dielectric films are deposited on the surface of a substrate to serve as waveguides, and circular dielectric waveguides are used in ***fiber optics***. Based on what was developed in the first parts of this note, we shall find some key



properties of a ***dielectric slab waveguide*** which is depicted in Fig. 11. This will illustrate the properties of dielectric films in integrated optics, although we discuss a simplification of the real situation where the waveguide is surrounded by two different dielectrics. Our model has vacuum on both sides of the guiding structure.

Fig. 11. Dielectric slab waveguide.

The thickness of the slab is d, and it is symmetrically located around $y = 0$. Its permittivity is ε_d and the permeability is assumed to be the same as for the surrounding vacuum, μ_0. Vacuum has permittivity ε_0. The dielectric material is assumed to be lossless. The propagation is in the z-direction while we assume no variation in x-direction. We shall analyze the TE modes of this configuration.

Since the fields exist in all regions, we have to consider them in both the slab region as well as in the surrounding medium. For TE waves, $\hat{E}_z^0 = 0$, and from Eq. (21) we have the following differential equation that must be satisfied in both regions (though with different parameters),

$$\frac{d^2\hat{H}_z^0(y)}{dy^2} + h^2\hat{H}_z^0(y) = 0, \tag{87}$$

where

$$h^2 = \gamma^2 + k^2 = \begin{cases} \gamma^2 + \omega^2\varepsilon_d\mu_0 = h_d^2 & \text{in the dielectric} \\ \gamma^2 + \omega^2\varepsilon_0\mu_0 = h_0^2 & \text{in vacuum} \end{cases}, \tag{88}$$

and $\gamma = j\beta$ expresses the propagation in the z-direction.

Solutions of Eq. (87) for the different regions must be matched at the boundaries. In the slab region we seek solutions which vary sinusoidally over the cross section. This requires the quantity

$$\boxed{h_d^2 = \omega^2\varepsilon_d\mu_0 - \beta^2 = k_y^2} \tag{89}$$

to be positive, with general solution

$$\hat{H}_z^0(y) = \hat{H}_o \sin k_y y + \hat{H}_e \cos k_y y, \ |y| \le \frac{d}{2} \tag{90}$$

where k_y is real, and $\hat{H}_o$ corresponds to asymmetric (odd) functions while $\hat{H}_e$ corresponds to symmetric (even) functions around $y = 0$, respectively.

In the vacuum region we require that the waves do not radiate from the waveguide structure. This means that solutions to Eq. (87) must have exponential decay in the outward y-direction, which is to say that

$$\boxed{h_0^2 = \omega^2\varepsilon_0\mu_0 - \beta^2 = -\alpha^2} \tag{91}$$

where α is real and positive. This gives the following solutions:

$$\hat{H}_z^0(y) = \begin{cases} C_1 e^{-\alpha(y-d/2)}, \quad y \geq \dfrac{d}{2} & \text{(a)} \\ C_2 e^{\alpha(y+d/2)}, \quad y \leq -\dfrac{d}{2} & \text{(b)} \end{cases} \tag{92}$$

Equations (89) and (91) are known as **dispersion** relations since they show the (nonlinear) dependence of the phase constant β on ω. We should note that we have still not determined the constants α or k_y in these equations. We also need to determine the relations between the coefficients $\hat{H}_o$, $\hat{H}_e$, C_1 and C_2. This is done in the following where we consider the odd and even TE modes separately. However, we first realize that $\hat{H}_z^0$ has no x-variation. From Eqs. (22) - (25) this means that the electric and magnetic fields have only one transverse component each, in addition to the longitudinal magnetic field $\hat{H}_z^0$.

From Eq. (89) we conclude that

$$\beta < \omega\sqrt{\varepsilon_d \mu_0} = k_d, \tag{93}$$

where k_d is the wave number of a plane wave in the dielectric, and from Eq. (91)

$$\beta > \omega\sqrt{\varepsilon_0 \mu_0} = k_0 \tag{94}$$

where k_0 is the wave number of a plane wave in vacuum. Combining these two equations, we conclude that

$$u_d < u_p < u_0 \tag{95}$$

where u_d and u_0 are the plane wave velocity of the two media, and u_p is the phase velocity in this structure. Thus for a mode to propagate we require the plane wave velocity of the dielectric to be smaller than the surrounding medium. The propagation velocity of this mode is always between these two plane wave velocities.

a) Odd TE modes. From Eqs. (23), (25) and (90) we obtain

1) In the dielectric slab region, $|y| \le d/2$:

$$\hat{H}_z^0(y) = \hat{H}_o \sin k_y y \tag{96}$$

$$\hat{H}_y^0(y) = -\frac{\gamma}{h_d^2}\frac{\partial \hat{H}_z^0(y)}{\partial y} = -\frac{j\beta}{k_y}\hat{H}_o \cos k_y y \tag{97}$$

$$\hat{E}_x^0(y) = -\frac{j\omega\mu_0}{h_d^2}\frac{\partial \hat{H}_z^0(y)}{\partial y} = -\frac{j\omega\mu_0}{k_y}\hat{H}_o \cos k_y y \tag{98}$$

2) In the upper vacuum region, $y \ge d/2$:
Continuity of the tangential magnetic field $\hat{H}_z^0$ across the boundary combined with Eq. (92) determines the constant C_1 so that

$$\hat{H}_z^0(y) = \left(\hat{H}_o \sin k_y d/2\right)e^{-\alpha(y-d/2)} \tag{99}$$

thus

$$\hat{H}_y^0(y) = -\frac{\gamma}{h_0^2}\frac{\partial \hat{H}_z^0(y)}{\partial y} = -\frac{j\beta}{\alpha}\left(\hat{H}_o \sin k_y d/2\right)e^{-\alpha(y-d/2)} \tag{100}$$

$$\hat{E}_x^0(y) = -\frac{j\omega\mu_0}{h_0^2}\frac{\partial \hat{H}_z^0(y)}{\partial y} = -\frac{j\omega\mu_0}{\alpha}\left(\hat{H}_o \sin k_y d/2\right)e^{-\alpha(y-d/2)} \tag{101}$$

3) In the lower vacuum region, $y \le -d/2$:
The boundary condition requiring continuity of the longitudinal magnetic field at the boundary $y = -d/2$ determines the constant C_2 so that

$$\hat{H}_z^0(y) = -\left(\hat{H}_o \sin k_y d/2\right)e^{\alpha(y+d/2)} \tag{102}$$

and further

$$\hat{H}_y^0(y) = -\frac{\gamma}{h_0^2}\frac{\partial \hat{H}_z^0(y)}{\partial y} = -\frac{j\beta}{\alpha}\left(\hat{H}_o \sin k_y d/2\right)e^{\alpha(y+d/2)} \tag{103}$$

$$\hat{E}_x^0(y) = -\frac{j\omega\mu_0}{h_0^2}\frac{\partial \hat{H}_z^0(y)}{\partial y} = -\frac{j\omega\mu_0}{\alpha}\left(\hat{H}_o \sin k_y d/2\right)e^{\alpha(y+d/2)} \tag{104}$$

So far we have determined the constants C_1 and C_2 by using the boundary conditions for $\hat{H}_z^0$ at the interface between the dielectric and vacuum. We also need to determine α, k_y and β. We start by requiring $\hat{E}_x$ to be continuous at $y = d/2$, which gives (from Eqs. (98) and (101))

$$\frac{j\omega\mu_0}{k_y}\hat{H}_o \cos k_y d/2 = \frac{j\omega\mu_0}{\alpha}\hat{H}_o \sin k_y d/2 \tag{105}$$

or

$$\boxed{\alpha = k_y \tan\frac{k_y d}{2}} \tag{106}$$

From the two dispersion relations (89) and (91) we can eliminate β. This gives

$$\boxed{\alpha = \left[\omega^2\mu_0\left(\varepsilon_d - \varepsilon_0\right) - k_y^2\right]^{1/2}} \tag{107}$$

We eliminate α by comparing the last two equations, giving

$$\boxed{\left[\omega^2\mu_0\left(\varepsilon_d - \varepsilon_0\right) - k_y^2\right]^{1/2} = k_y \tan\frac{k_y d}{2}} \tag{108}$$

This is a transcendental equation to determine the transverse wave number k_y in the dielectric. The equation can not in general be solved analytically; it has to be solved numerically or graphically. When $\omega, d, \varepsilon_d, \varepsilon_0$ and μ_0 are known, the two sides of this equation can be plotted as functions of k_y. The intersections of the two curves determine the value of k_y corresponding to different odd TE modes. Subsequently the remaining unknowns α and β can be found from two of the above equations, for example, Eqs. (107) and (89).

b) Even TE modes. We now use the second term of Eq. (90), i.e.,

$$\hat{H}_z^0\left(y\right) = \hat{H}_e \cos k_y y, \quad |y| \le \frac{d}{2}\ , \tag{109}$$

From this starting point we obtain the remaining field components inside and outside of the dielectric slab by following the same procedure as for the odd TE case. We then essentially do the following substitutions: cos → sin, sin → -cos. This means that Eq. (106) is replaced by

$$\boxed{\alpha = -k_y \cot\frac{k_y d}{2}} \tag{110}$$

To obtain solutions, this equation must be solved together with the same equation (107) as in the previous case. This gives the equation

$$\boxed{\left[\omega^2\mu_0\left(\varepsilon_d-\varepsilon_0\right)-k_y^2\right]^{1/2} = -k_y \cot\frac{k_y d}{2}} \tag{111}$$

4.2 Example

To illustrate these results we shall find possible propagating modes for a dielectric slab applied in the optical range. The following data is used: $\varepsilon_d = 2.5\varepsilon_0$ (corresponding to an index of refraction $n=\sqrt{\varepsilon_d/\varepsilon_0}=\sqrt{2.5}=1.58$), d=1 μm, f = 300 THz corresponding to an optical wavelength of 1 μm in vacuum.

In Fig. 12 are plotted three curves to solve this problem. The thick full curve and the thin dashed curve represent the right hand side of Eqs. (108) and (111), respectively, while the circular curve represents the left hand side of both equations. There are three intersections between this circle and the two other curves, marked by small circles. They are denoted TE_1, TE_2 and TE_3, respectively. Thus there are two odd mode solutions and one even solution. The values of k_y and α for these solutions can be read on the horizontal and vertical axis, respectively. Only positive values of α are acceptable since the field must be finite. After k_y and α have been determined, β can be found as described above.

We note that the only frequency dependent terms in the expressions above occur in the left hand parts of the equations. These parts correspond to a circle with its radius proportional to frequency. We then realize that more modes are included if the frequency is increased. On the other hand, if the frequency is reduced, there is a certain frequency below which only one TE mode propagates. This odd mode propagates down to zero frequency.

The transverse distribution of the two magnetic field components for the two odd modes TE_1 and TE_3 is plotted in Fig. 13. The electric field E_x has not been plotted, but the curve shape is the same as for H_y. It is seen that

- Odd modes have antisymmetric distribution of longitudinal field and symmetric distribution of transverse field components (also the electric field, not shown). The converse would be the case for even modes.
- The transverse field component can be dominating, at least for the lowest mode.
- Higher modes vary more quickly over the cross section.

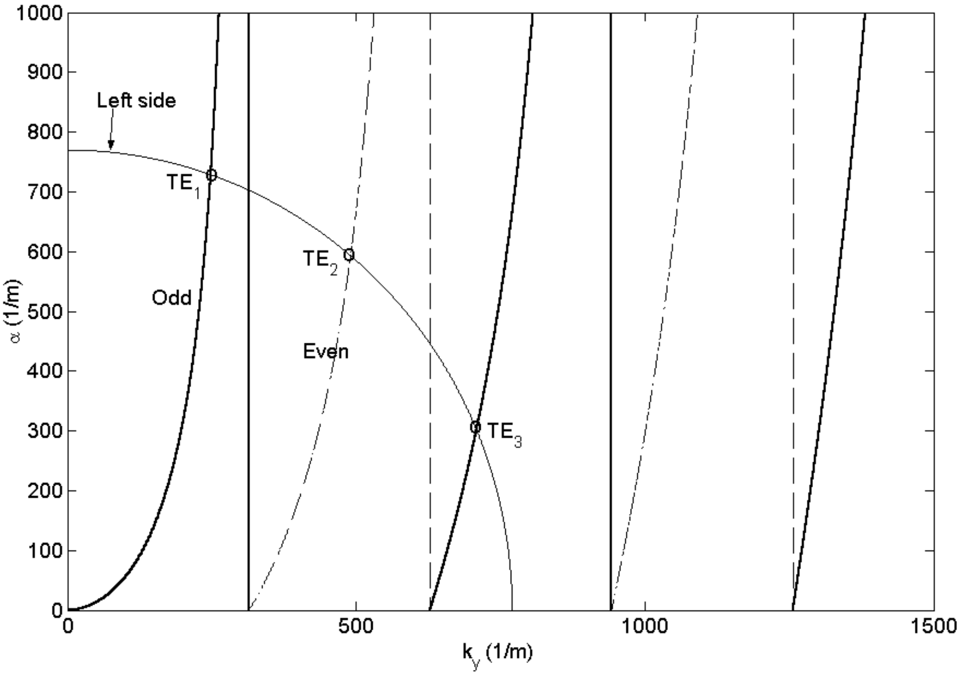


Fig. 12. Graphic solution of optical waveguide problem.

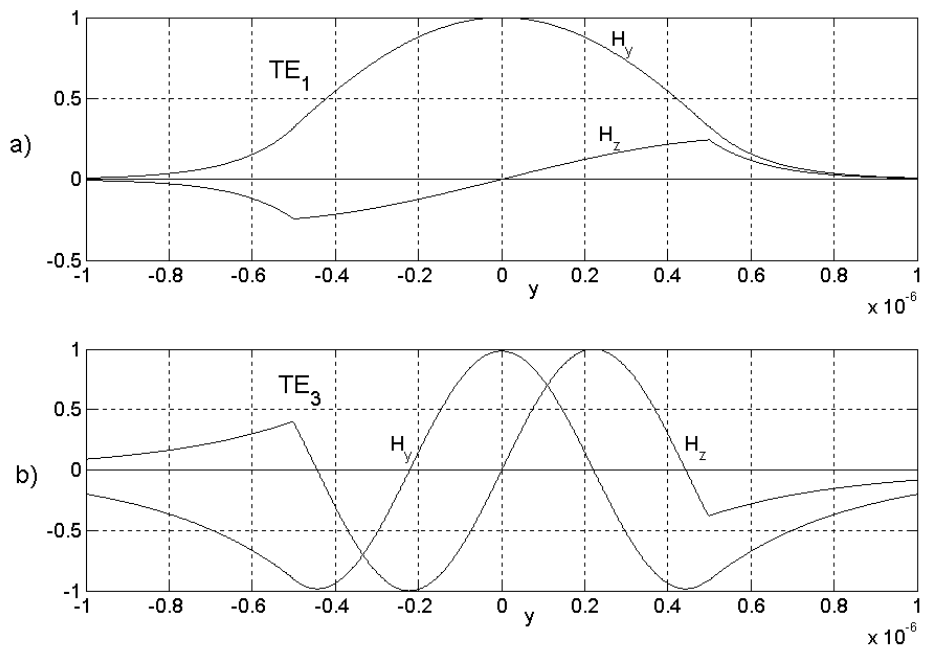


Fig. 13. Transverse distribution of the two components of the magnetic field. a) Lowest odd mode, b) Next odd mode. The two components are 90° out of phase.

Finally we reiterate that there is another group of propagating waves, the ***TM modes***, also in this configuration. They have many similar properties to the TE modes, but we shall not discuss them here.

5 Optical fiber

5.1 Introduction

Today optical fibers are predominant within telecommunications. In particular, they are well suited for transmitting large amounts of data. A discussion of this type of dielectric waveguide is somewhat more involved than for rectangular cross sections. This is because of the curved geometry, the two transverse field components are coupled together through the boundary conditions. There are, however, simplified theories which can be used. In the following such a simplified theory will be outlined, based on material that has been discussed previously in this note.

5.2 Definition of the problem

The geometry of an optical step index fiber can be described according to Fig. 14. The fiber has a ***core*** of radius a with an index of refraction n_1, while the surrounding ***cladding*** has an index of refraction n_2. We should bear in mind that the index of refraction n is related to the permittivity ε through $n = \sqrt{\varepsilon / \varepsilon_0}$ where ε_0 is the permittivity of vacuum. In the discussion here we assume the cladding extends to infinity.

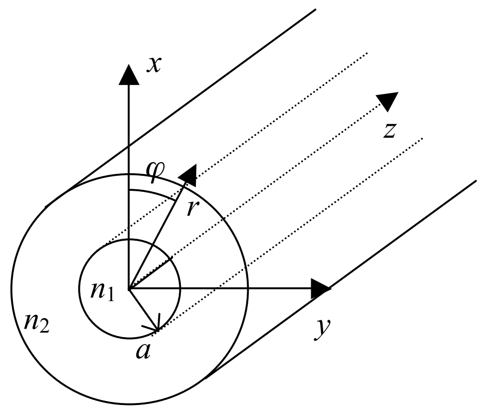


Fig. 14. Geometry for optical fiber.

$$n_1 - n_2 << 1 \tag{112}$$

As indicated above, this geometry is more involved than geometries which have been considered earlier. On the other hand most fibers used in telecommunications are designed such that the index of refraction of the core and the cladding differ very little, i.e.

This means that the the solutions for different polarizations (previously called TE and TM waves) vary very little. In many cases this difference is disregarded, as we shall do here. Under this condition it turns out that we end up with a scalar Helmholtz's equation involving a quantiy ψ to represent both E_x and E_y. In cylindrical coordinates this equation can be written

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 \psi}{\partial \varphi^2}-\beta^2\psi=-k_0^2 n^2 \psi \quad . \tag{113}$$

where k_0 is the wavenumber of vacuum. The hat symbol denoting phasors has been deleted from the equation.

5.3 Solutions

We base the solution on separation of variables in the two transversal coordinates r and φ, i.e.,

$$\psi(r,\varphi) = R(r)\Phi(\varphi) \ . \tag{114}$$

We also assume that Φ is a periodic function of φ so

$$\Phi(\varphi) = A\cos l\varphi + B\sin l\varphi \quad . \tag{115}$$

where l is an integer equal to zero or larger to make the solution unique. Substitution into Eq. (113) gives

$$\Phi\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)-\frac{l^2}{r^2}R\Phi-\beta^2 R\Phi=-k_0^2 n^2 R\Phi \quad . \tag{116}$$

Multiplication of this equation by r^2/Φ on both sides obtains a general equation

$$r^2 \frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r} + \left(\left(k_0^2 n^2 - \beta^2\right) r^2 - l^2\right) R = 0 \quad . \tag{117}$$

The two materials have different properties, giving two differential equations. They are:

$$r^2 \frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r} + \left(\beta_{i1}^2 r^2 - l^2\right) R = 0, \quad r<a \tag{118}$$

$$r^2 \frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r} + \left(-\beta_{i2}^2 r^2 - l^2\right) R = 0, \quad r > a \tag{119}$$

where $\beta_{i1}^2 = k_0^2 n_1^2 - \beta^2$ and $\beta_{i2}^2 = -\left(k_0^2 n_2^2 - \beta^2\right)$ and the suffixes 1 and 2 stand for core and cladding, respectively. Both equations are of the form of ***Bessel's differential equation***. Solutions of this equation are called ***Bessel functions***. The solution to our problem is found by introducing boundary conditions similar to what was done for the dielectric slab case discussed above. In this case continuity of both $R(r)$ and its derivative is required. Discarding solutions which diverge for $r = 0$ and large values of r, respectively, the solutions can be written

$$R(r) = \frac{C_0}{J_l\left(\beta_{i1} a\right)} J_l\left(\beta_{i1} r\right) , \quad r < a \tag{120}$$

$$R(r) = \frac{C_0}{K_l\left(\beta_{i2} a\right)} K_l\left(\beta_{i2} r\right) , \quad r > a . \tag{121}$$

Here $J_l(x)$ is a ***Bessel function of the first kind and order l*** and $K_l(x)$ is a ***modified Bessel function of the second kind and order l***. Some of these functions are shown in the Appendix. In the same way as before, the boundary conditions result in transcendental equations which can be solved graphically or numerically. Without going in more detail on this procedure, we sum up the result in this way: For a given circumferential index l, we obtain a set of radial functions each with a particular value of β_{i1} and β_{i2} to be applied in the last two equations. Each solution therefore gives a mode characterized by a double index (l,i). Here l determines the number of periods of the field for one period in φ- direction and i is the number of intensity maxima in r-direction.

As mentioned above, the index of refraction of the two main regions of a practical fiber differs only slightly. A consequence of this is that the fields are almost transversal, with only small longitudinal components. Real fields can therefore be considered as combinations of two linearly polarized mode fields. For this reason the modes are denoted ***LP modes***. The approximate boundary conditions we have applied give inaccuracies of higher order which can usually be disregarded.

As was found for the dielectric slab, there is a frequency range where single mode transmission is possible. This is usually stated through a ***normalized frequency V*** as

$$V = k_0 a\sqrt{n_1^2 - n_2^2} < 2.4048 \tag{122}$$

We shall not go in more detail on this topic. To illustrate some results of the discussion, a transverse distribution of the fields of some lower LP modes are sketched in Fig. 15. The intensity (which is what we actually detect by our eyes) is obtained by squaring this amplitude. Its distribution is shown in Fig. 16.

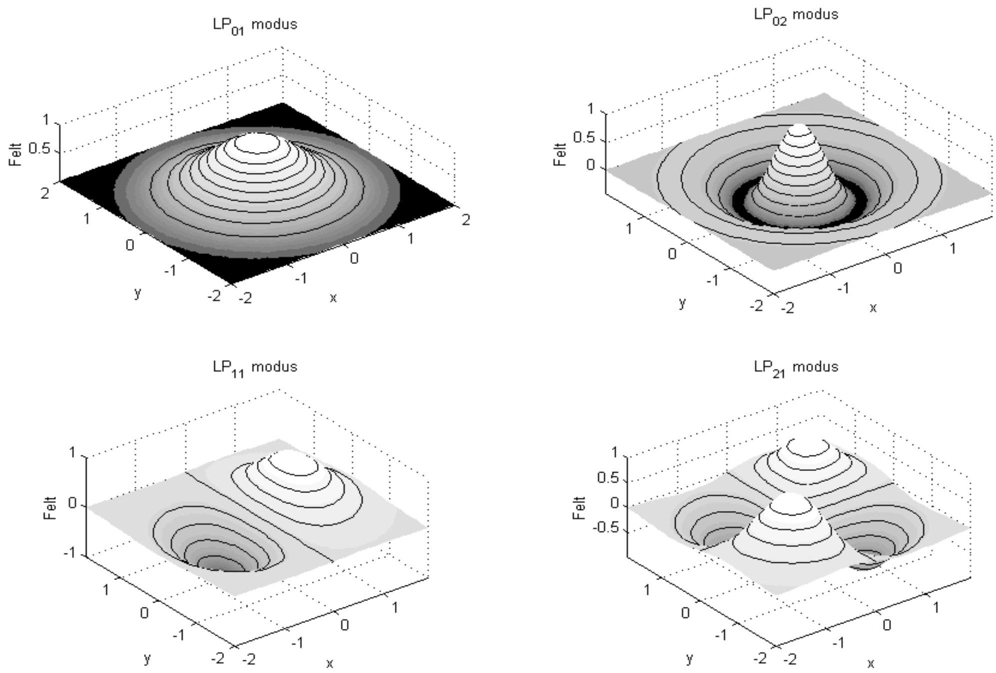


Fig. 15. Field distribution for some LP modes.

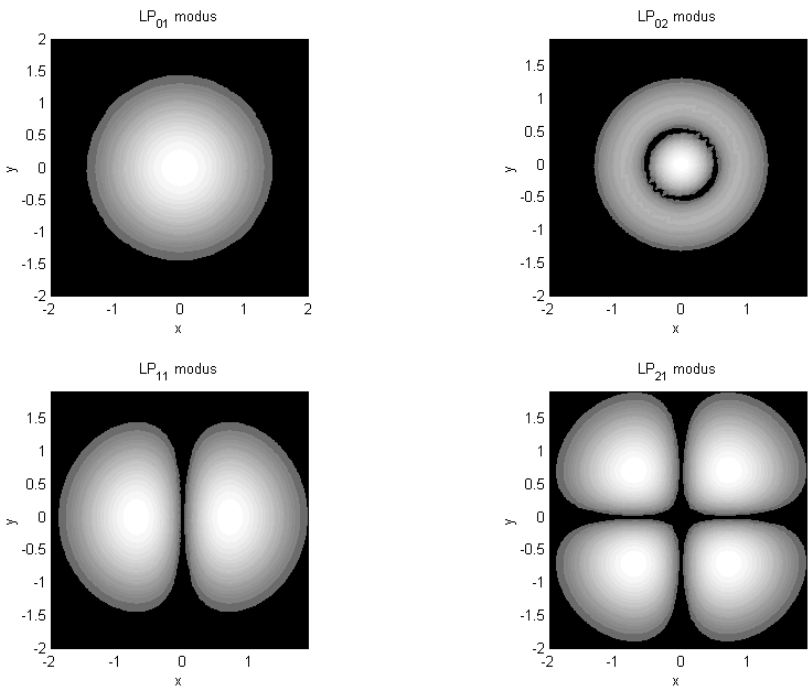


Fig. 16. Intensity distribution for the same modes as shown in the previous figure.

6 Appendix. Bessel functions

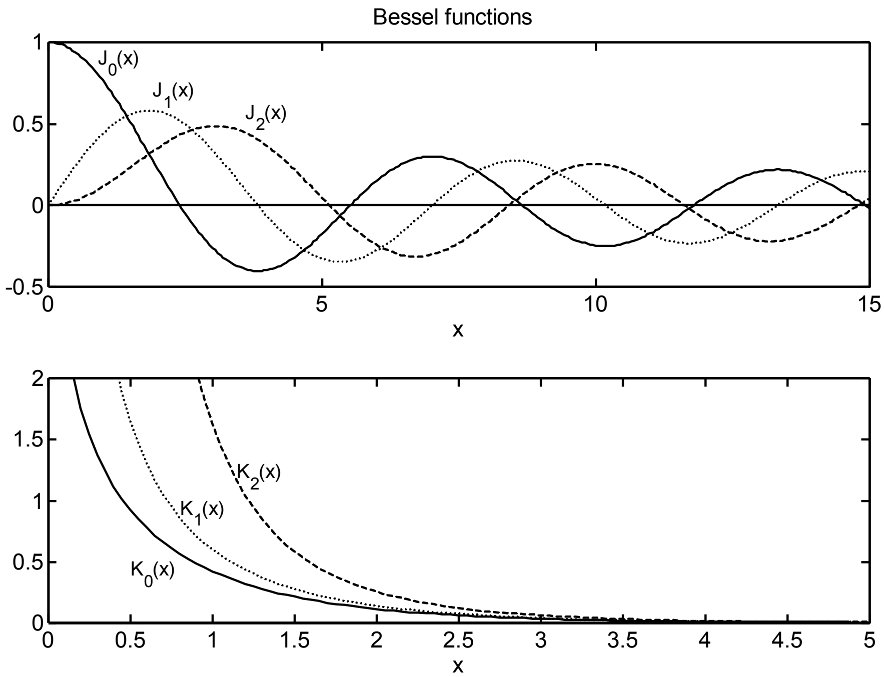


Fig. A1. Some Bessel functions.