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Piecewise Deterministic Markov Process (PDMP)

As a modelling framework in RAMS

RAMS seminar NTNU
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Outline

- General introduction to PDMP
- PDMP in dynamic reliability
- PDMP for CBM – unit level
- Summary

PDMP - Intro

- M.H.A. Davis, 1984, “Piecewise-Deterministic Markov Processes: A General Class of Non-Diffusion Stochastic Models”
  - “Almost all **continuous-time** stochastic process models of applied probability consist of some combination of the following”:
    a) Diffusion
    b) Deterministic motion
    c) Random jumps

Techniques

a) Diffusion ⇎ b) Deterministic motion, c) Random jumps

a) Diffusion:
- Unified, highly developed theory.
- Ito Calculus, Stochastic differential equations

b) Deterministic motion, c) Random jumps:
- Heterogeneous
- Special models and methodologies appropriate to specific problems

PDMP - Intro

M.H.A. Davis, 1984

“It can be argued that”:

1. The class of “piecewise-deterministic” Markov process, provides a general family of stochastic models covering **virtually all non-diffusion** applications
2. These process can be analyzed by methods which are directly analogous to those of diffusion theory.

Objective. To place non-diffusion models on the same footing as diffusion theory, with availability of:

- A “canonical model” including a variety of applications as special cases,
- General methods based on stochastic calculus for analyzing the canonical model

It is not implied that methods for studying non-diffusion models are obsolete. In many fields, **efficient techniques** for calculations have been built up, making use of the **special structure** of specific models.

PDMP - formalism

- Markov process consisting of a mixture of **deterministic motion** and **random jumps**
- Hybrid stochastic process $\{I_t, X_t\}, t \geq 0$ with values in a discrete-continuous space $E \times R$

$$\{I_t, X_t\}$$

Discrete component — Continuous component

Law

Determined by three local characteristics:

- Jump rate $\mathrm{z}(i, x)$
- Flow $\phi(i, x)$
- Transition measure Q

PDMP - motion

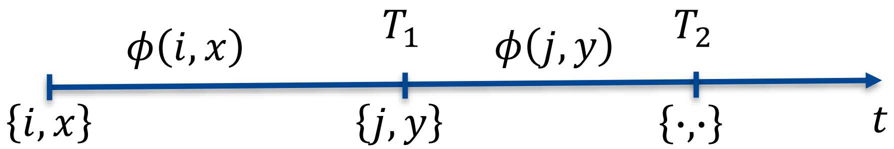
The process starts at (i, x):

1. Follows the flow $\phi(i, x)$ until a first jump occurs at T_1
   A jump can occur:
   - Randomly, with rate $z(i, x)$
   - When the flow hits a boundary in the continuous-state space R

2. The post jump location is selected from the transition measure $Q[(i, x), (j, y)]$

3. The motion restarts from this point.

The flow is deterministic and generally described by differential equation



Dynamic reliability

- **Reliability**

  "the ability of an item to perform a required function, **under given** environmental and operational conditions and for a stated period of time" (Rausand M.)

- **Dynamic** reliability

  Extension of traditional reliability models and methods, with ones that are capable of **capturing the dynamics** of the operational and environmental conditions in which systems evolve.

Example

Case-study from

Safety, Reliability and Risk Analysis: Beyond the Horizon – Steenbergen et al. (Eds)
© 2014 Taylor & Francis Group, London, ISBN 978-1-138-00123-7

Dynamic reliability analysis of three nonlinear aging components with different failure modes characteristics

G. Manno
Det Norske Veritas, Research and Innovation, Høvik, Norway

A. Zymaris & N.M.P. Kakalis
Det Norske Veritas, Research and Innovation, Piraeus, Greece

F. Chiacchio, F.E. Cipollone, L. Compagno, D. D'Urso & N. Trapani
University of Catania, Department of Industrial and Mechanical Engineering, Catania, Italy

In the paper:

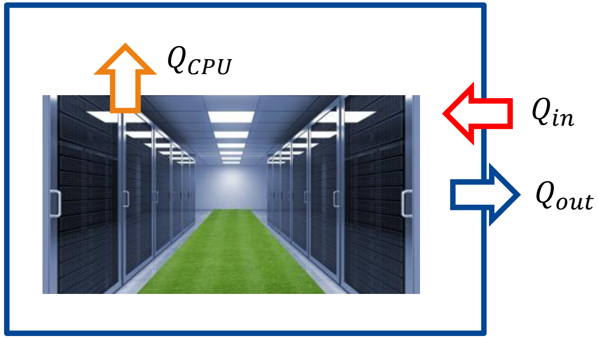
PDMP mentioned as framework for dynamic reliability (deterministic motion + random jumps)

However, the authors propose a modeling formalism based on Stochastic activity networks with flowsheet models, and a Discrete Event simulation algorithm

Dynamic reliability

- **Problem**

Server room



Air cooling system

- The air cooling system has an on-off control
- Works at a fixed working point T_{cool}

Change in room temperature

$$\frac{\partial T_{room}}{\partial t} = (Q_{in}) - (Q_{out} \cdot 1_{on}) + (Q_{CPU})$$

$$\frac{\partial T_{room}}{\partial t} = \frac{K}{J}(T_{ext} - T_{room}) - \frac{P}{J}(T_{room} - T_{cool})1_{on} + Q_{CPU}$$

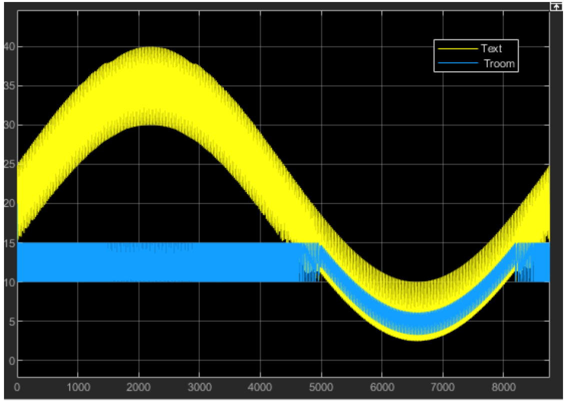
- T_{ext} – External temp
- T_{room} – Room temp
- T_{cool} – Cooling temp (fixed working point)
- 1_{on} – indicator function of the operative state of the cooling system (1–on, 0–off(standby))
- K, P, J – Physical coefficients (heat transfer coefficients, heat capacity)

Dynamic reliability

$$\frac{\partial T_{room}}{\partial t} = \frac{K}{J}(T_{ext} - T_{room}) - \frac{P}{J}(T_{room} - T_{cool})1_{on} + Q_{CPU}$$

$T_{ext}(t) = 20 + 15 \cdot \sin(7.17 \times 10^{-4}) + 5 \cdot \sin(0.2618 \cdot t)$
$T_{room}(0) = 20$ c
$T_{cool} = 5$ c
$Q_{cpu} = 0$
$\mathrm{K} = 0.1$ W/C
$\mathrm{J} = 1$ W/C
$\mathrm{P} = 0.5$ W/C

Threshold_off = 10 c
Threshold_on = 15 c



"Failure free"

Dynamic reliability – random jumps

- **Weibull distributed lifetime**

  Failure rate, scale α, shape β

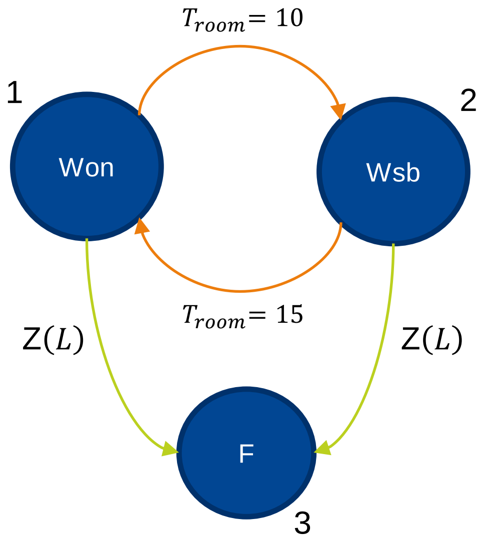
  – $\mathrm{z}(t) = \frac{\beta}{\alpha} \cdot \left(\frac{t}{\alpha}\right)^{\beta-1}$

- **Non-linear aging**

  – $\mathrm{z}(l) = \frac{\beta}{\alpha} \cdot \left(\frac{l}{\alpha}\right)^{\beta-1}$

  – L denotes actual time in operation (air cooling system in “on” position, governed by deterministic dynamics)

Dynamic reliability - PDMP



$$\{I_t, X_t\}$$

$$i = \{1,2,3\}$$

$$\boldsymbol{x} = (t_{room}, l)$$

Model state:

$$\{(i), (t_{room}, l)\}$$

Jump rate

$$z(1, (t_{room}, l)) = \frac{\beta}{\alpha} \cdot \left(\frac{l}{\alpha}\right)^{\beta-1}$$

$$z(2, (t_{room}, l)) = \frac{\beta}{\alpha} \cdot \left(\frac{l}{\alpha}\right)^{\beta-1}$$

Flow

$$\phi(1, (t_{room}, l)) = \left(\frac{K}{J}(T_{ext} - T_{room}) - \frac{P}{J}(T_{room} - T_{cool}), 1\right)$$

$$\phi(2, (t_{room}, l)) = \left(\frac{K}{J}(T_{ext} - T_{room}), 0\right)$$

Transition measure

- **Random jumps (not boundary hit)**

  $Q[(1, x), (3, x)] = 1$

  $Q[(2, x), (3, x)] = 1$

- **Boundary hit**

  $q\big[(1, (10, l)), (2, (10, l))\big] = 1$

  $q\big[(2, (15, l)), (1, (15, l))\big] = 1$

Dynamic reliability - PDMP

- Quantification
  - Simulations
    - Random jumps + deterministic motion (flow)
      (Matlab + Simulink)
  - Numerical approach
    - Based on **discretization** of the continuous state space and time
    - Probability mass balance

PDMP - numerical approach

Renny Arismendi, Anne Barros, Antoine Grall,
Piecewise deterministic Markov process for condition-based maintenance models — Application to critical infrastructures with discrete-state deterioration,
Reliability Engineering & System Safety, Volume 212, 2021, 107540, ISSN 0951-8320,

Numerical approach: (Prob. mass balance: Markov property + total probability)

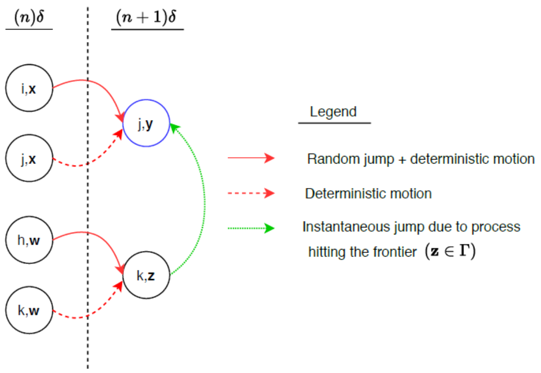


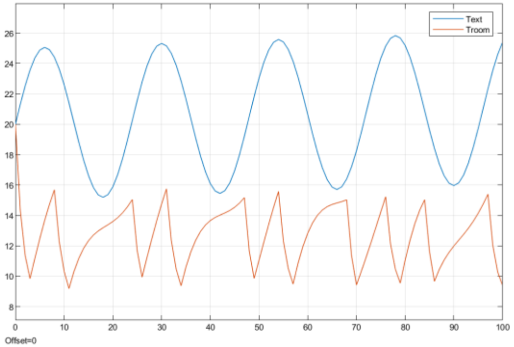
Figure 2: Transitions into state $(j, \mathbf{y})$ in $(n\delta, (n+1)\delta]$

$$
\begin{aligned}
\underline{\pi}_{(n+1)\delta}(j,\mathbf{y}) \approx & \sum_{\substack{i\neq j\\ \mathbf{y}=\mathbf{x}+\mathbf{v}\delta}}^{N-1} \underline{\pi}_{n\delta}(i,\mathbf{x})[\lambda(i,\mathbf{x})Q(i,\mathbf{x},j)\delta] \qquad (6)\\
& + \mathbb{1}_{\{\mathbf{y}=\mathbf{x}+\mathbf{v}\delta\}}\, \underline{\pi}_{n\delta}(j,\mathbf{x})[1-\lambda(j,\mathbf{x})\delta]\\
& + \sum_{\substack{k=1\\ \mathbf{y}=m_\Gamma(k,\mathbf{z},j)}}^{N} \sum_{\substack{h=1\\ h\neq k\\ \mathbf{z}=\mathbf{w}+\mathbf{v}\delta\\ \mathbf{z}\in\Gamma}}^{N-1} \underline{\pi}_{n\delta}(h,\mathbf{w})[\lambda(h,\mathbf{w})Q(h,\mathbf{w},k)\delta][q(k,\mathbf{z},j)]\\
& + \sum_{\substack{k=1\\ \mathbf{z}=\mathbf{w}+\mathbf{v}\delta\\ \mathbf{z}\in\Gamma\\ \mathbf{y}=m_\Gamma(k,\mathbf{z},j)}}^{N} \underline{\pi}_{n\delta}(k,\mathbf{w})[1-\lambda(k,\mathbf{w})\delta][q(k,\mathbf{z},j)]
\end{aligned}
$$

Approximation of the Chapman-Kolmogorov equation (backward)

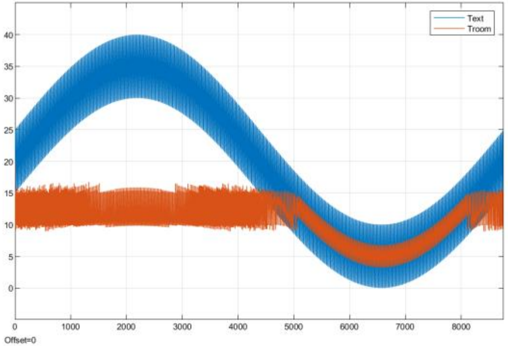
Interesting remarks

- Simulations
  - Computational cost, dynamics

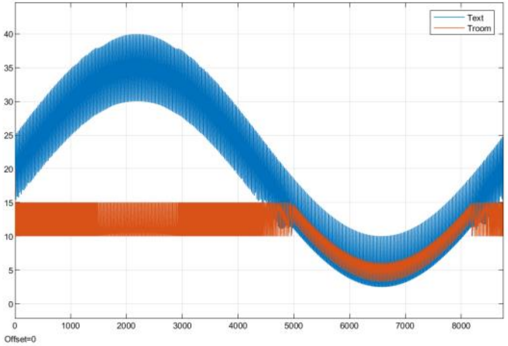


100 hours

- Numerical approach
  - Discretization: fixed vs variable-step



fixed

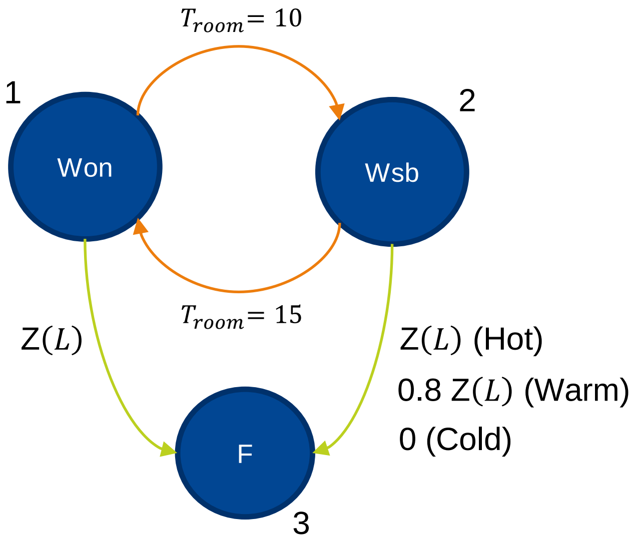


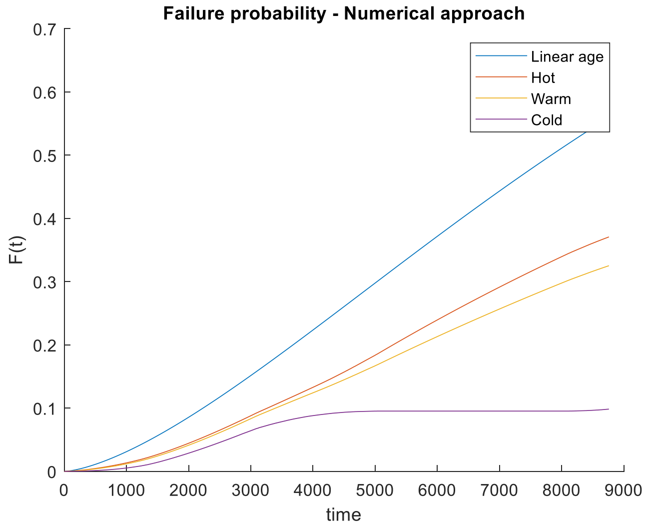
variable (auto-selected Simulink)

Dynamic reliability - PDMP

3 cases

- **‘Hot’ model**
  - Same failure rate when on or standby
- **‘Warm’ model**
  - The failure rate when in standby is 80% of the failure rate when on
- **‘Cold’ model**
  - No failure while on standby





PDMP for CBM

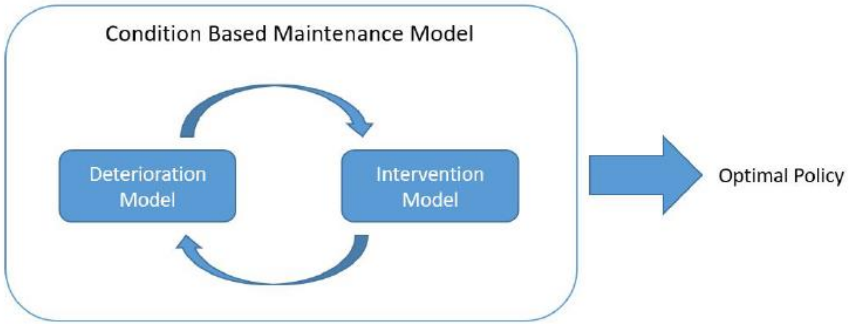


Figure 1: Condition-based maintenance model

In our case: not modelling a physical law with a differential equation, then why PDMP?

Interventions:

- Deterministic durations
- Non-continuous monitoring (the state of the item is only revealed by the operator at inspections)
- Delay before maintenance task

Deterioration:

- Time-dependent transition rates

PDMP for CBM

$\{I_t, X_t\}$

- Discrete-state deterioration

- Continuous component
  - not related to any physical phenomena, but used **as an artifact** to track time, either for:
    - Not constant random jump rate.
    - Introducing jumps at specified times.

PDMP for CBM – bridge management

Periodic Inspections:

Laws & rules

Maintenance scheduling

Based on the condition assigned at inspection, laws and rules dictate when to maintain.

Condition is assigned according to the damage degree / severity:

1. Small
2. Medium
3. Large
4. Critical

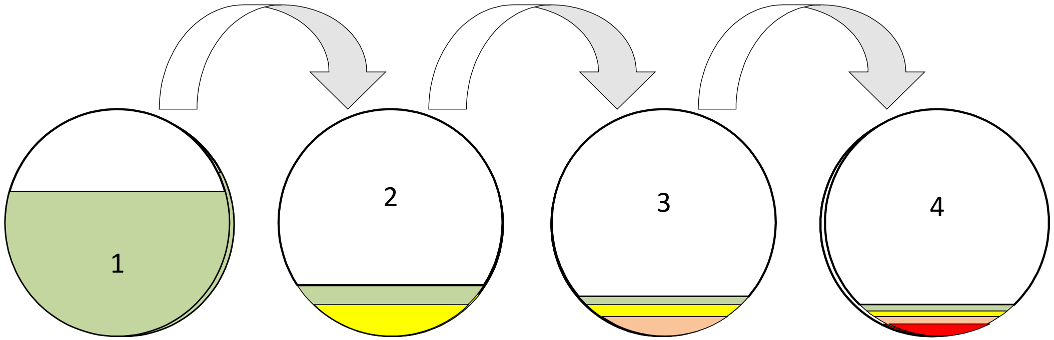
Maintenance action:

1. No action required
2. Action required between 4 to 10 years
3. Action required between 1 to 3 years
4. Action required before 6 months

PDMP for CBM

Assumptions:

- Constant jump rate
- Perfect periodic inspections
- Delay (fixed duration) before maintenance
- Good-as-new repairs



Process starts

Inspection

- No maintenance
- Maintenance in 8 years
- Maintenance in 3 years
- Maintenance in ½ year

PDMP for CBM

Model state $\{\boldsymbol{I_t}, \boldsymbol{X_t}\} = \{(\boldsymbol{i_A}, \boldsymbol{i_B}), (\boldsymbol{x_A}, \boldsymbol{x_B}, \boldsymbol{t})\}$

- i_A - **deterioration state** of the structure
- i_B - **type of maintenance** scheduled
- x_A - date of **next inspection**
- x_B - date of **next maintenance action**
- t - **time**

Jumps at random times

Used to model the deterioration process of the structure

- The **deterioration state** $\boldsymbol{i_A}$, jumps to a more deteriorated state
- The **type of maintenance** $\boldsymbol{i_B}$ does not change

Jumps at deterministic times

Used to model the **inspection** and **maintenance** of the structure

Inspection, $\boldsymbol{t = x_A}$

Used to model the deterioration process of the structure

- The **type of maintenance** $\boldsymbol{i_B}$, is updated
- The **date of next inspection** $\boldsymbol{x_A}$, is updated
- A **maintenance action** $\boldsymbol{x_B}$ is scheduled or re-scheduled

Maintenance, $\boldsymbol{t = x_B}$

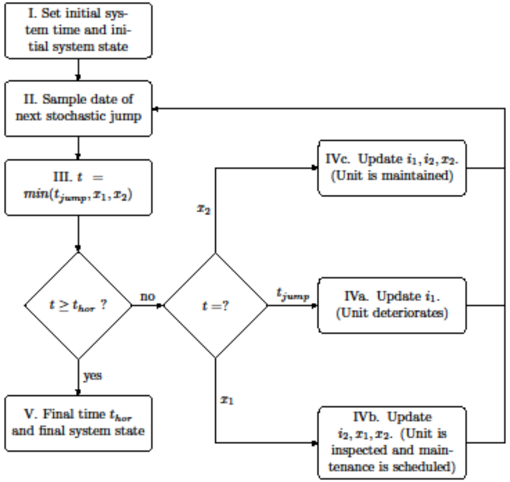
Used to model the deterioration process of the structure

- The **discrete component** $(\boldsymbol{i_A}, \boldsymbol{i_B})$ jumps to (1,1)
- The **date of next inspection** $\boldsymbol{x_A}$, does not change
- The **date of next maintenance action** $\boldsymbol{x_B}$ is set to infinite

PDMP for CBM

- **Simulations**

Model state $\{I_t, X_t\} = \{(i_A, i_B), (x_A, x_B, t)\}$



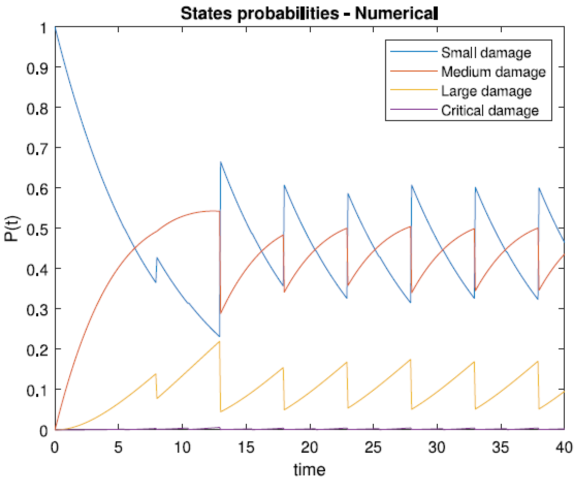


Fig. 5. Deterioration state probabilities.

PDMP for CBM

- Expected cost per unit of time

$$E[C] = E[N_{in}]C_{in} + E[N_{mr}]C_{mr} + E[N_{lr}]C_{lr} + E[N_{cr}]C_{cr} \quad (13)$$

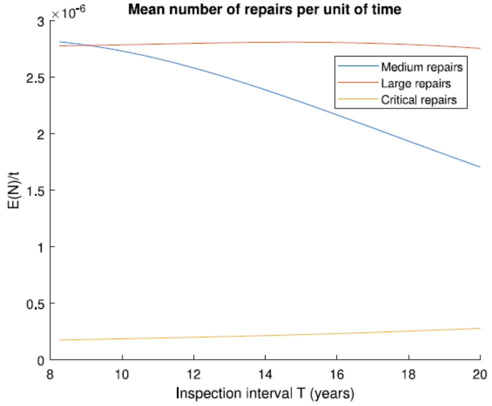


Fig. 8. Mean number of repairs per unit of time.

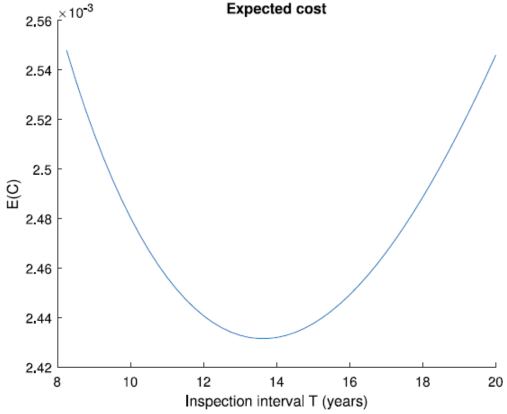


Fig. 9. Expected cost per unit of time.

Illustration with symbolic values for cost

PDMP (time-dependent rates)

- Example:
  - Two states
  - Weibull dist. Lifetime
  - Fixed repair duration (corrective)

$$\{\boldsymbol{I_t}, \boldsymbol{X_t}\}$$

$i = \{1,0\}$ where: 1-working, 0-failed

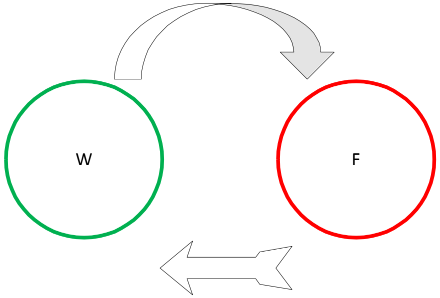
x: amount of time spent in the current discrete state i a time t

Flow

$$\phi(i, x) = 1$$

Jump rate

The failure rate from $i = 1$ to $i = 0$ is: $z(1, x) = \frac{\alpha}{\mu}\left(\frac{x}{\mu}\right)^{(\alpha-1)}$



Continuous variable
Bounded when i=0 due to repair duration, (intervention jump to i=1, x=0)

If age-based PM, then x can be bounded when i=1, resetting x to zero.

The challenge:

Numerical approach implementation

Summary

PDMP

- Markov process consisting of a mixture of **deterministic motion** and **random jumps**

- Hybrid stochastic process $\{I_t, X_t\}, t \geq 0$ with values in a discrete-continuous space $E \times R$. Can be considered a canonical model including a variety of applications as special cases.

- A **general class of non-diffusion** stochastic models that provides a framework for studying optimization problems

- It is not implied that methods for studying non-diffusion models are obsolete. In many fields, **efficient techniques** for calculations have been built up, making use of the **special structure** of specific models.

- Two approaches are commonly used in the quantification of such models : Montecarlo simulation and Finite-volume scheme