COMPLEX NETWORK DYNAMICS

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Baruch Barzel

RESILIENCE of COMPLEX NETWORKS

Universal resilience patterns in complex networks.
Nature **530**, 307 (2016).

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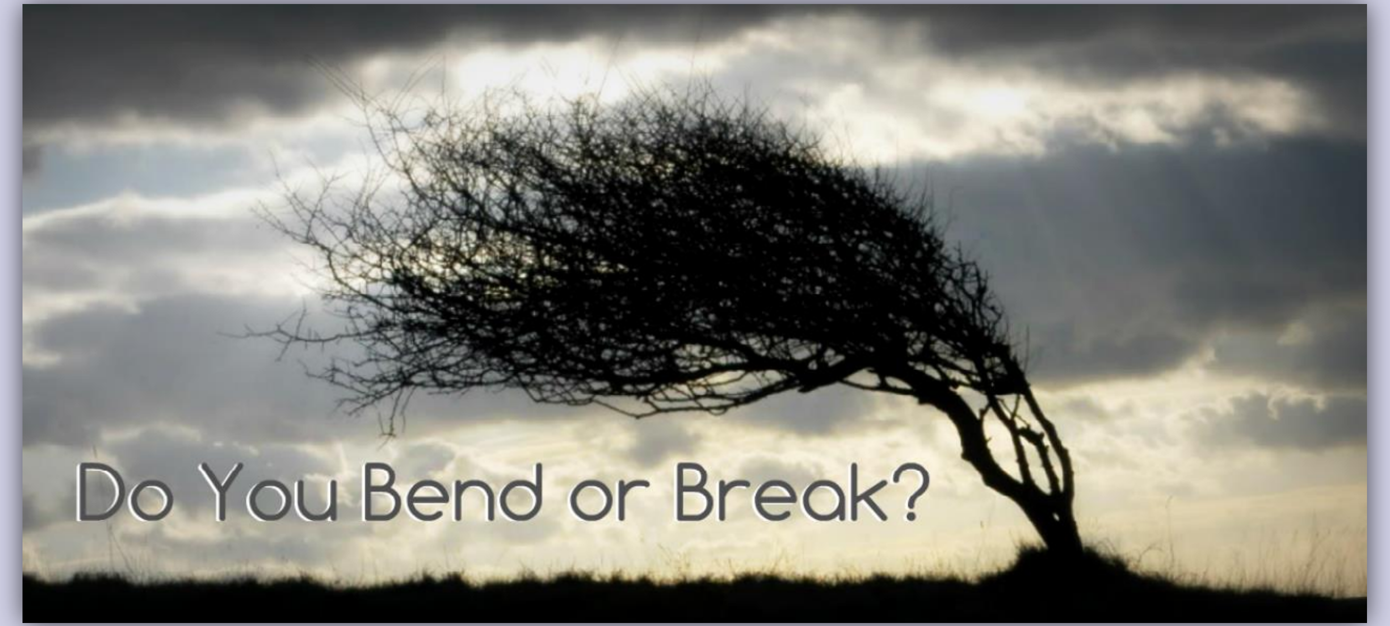
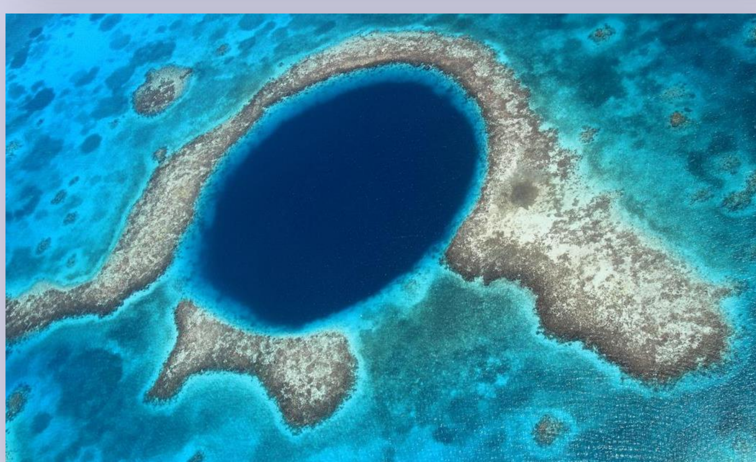


RESILIENCE

Resilience loss



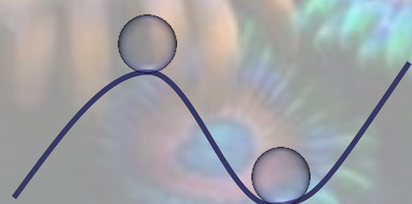
RESILIENCE

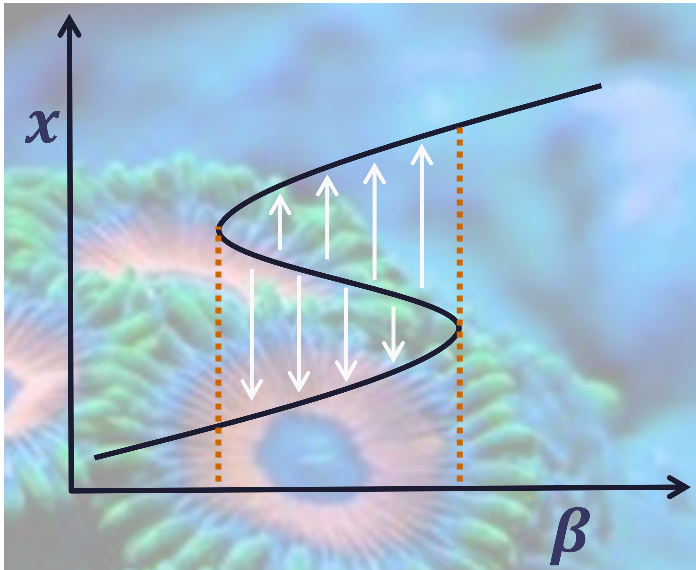
Resilience function

$$\frac{dx}{dt} = f(\beta, x(t))$$

$$f(\beta, x) = 0 \longrightarrow x(\beta)$$

$$\frac{df}{dx} < 0$$





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$$f(\beta, x) = 0$$

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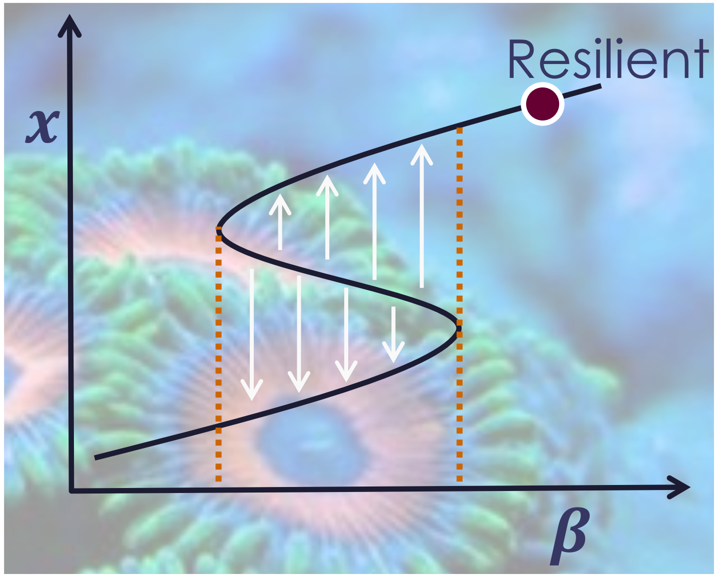

RESILIENCE

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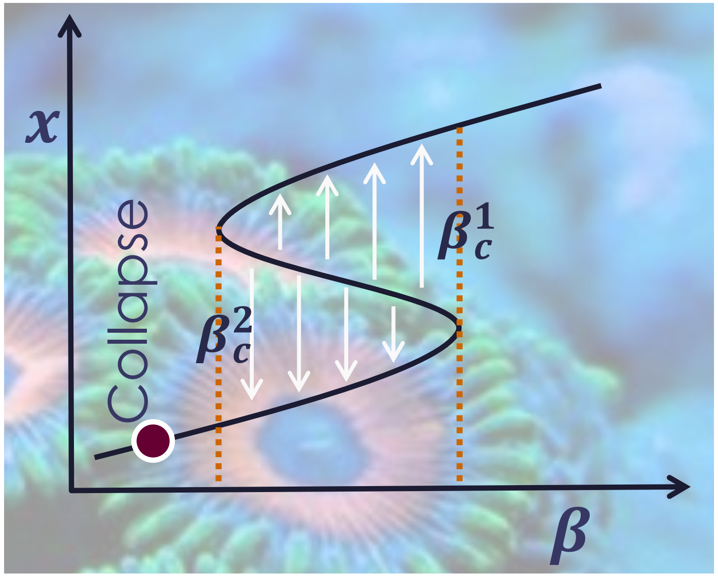

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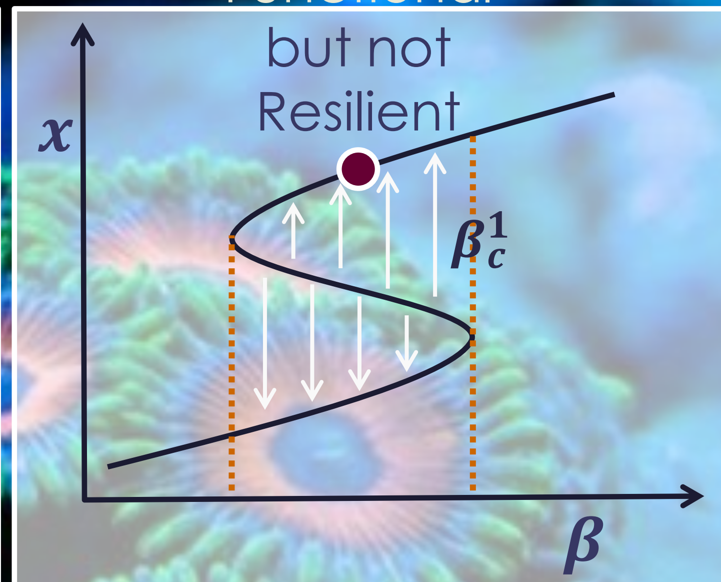

Resilience function

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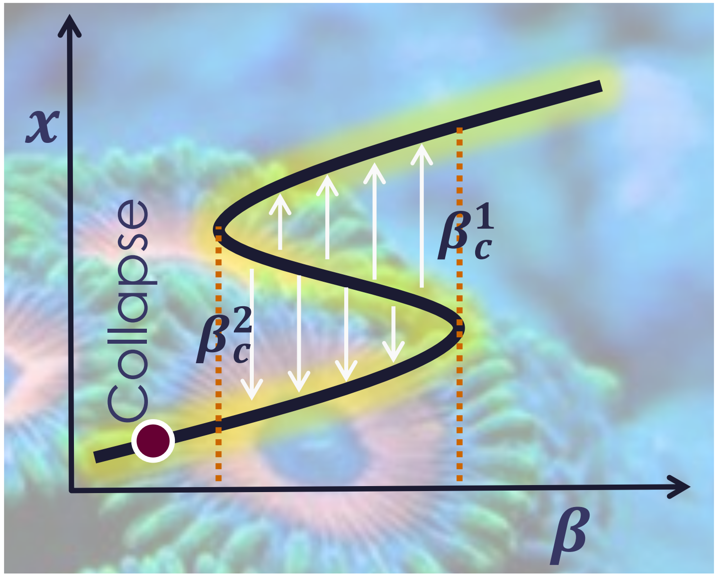

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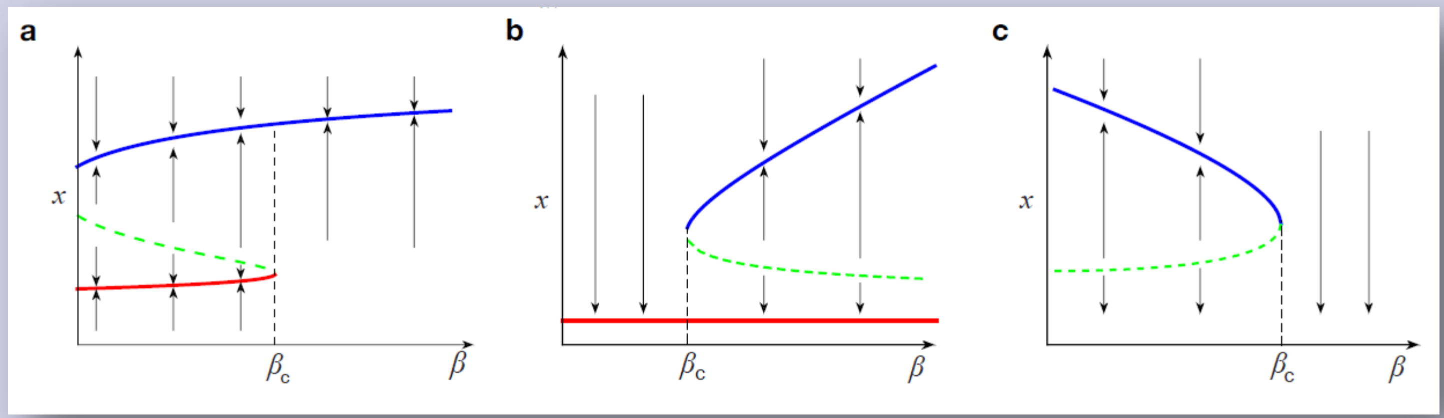
Resilience functions

$$\frac{dx}{dt} = f(\beta, x(t))$$



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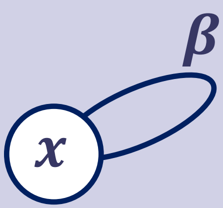
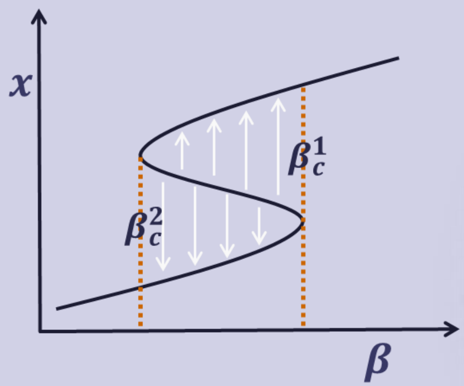
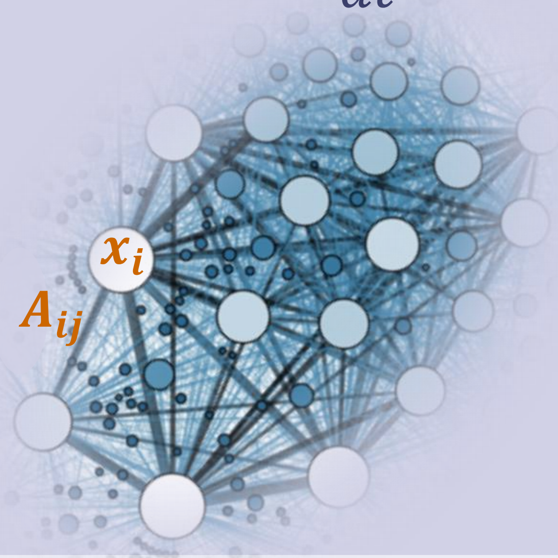
Multidimensional systems

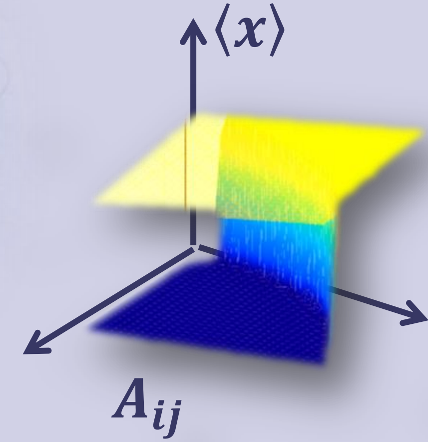
$$\frac{dx}{dt} = f\big(\beta, x(t)\big)$$

$$\frac{dx_i}{dt} = f\big(A_{ij}, \vec{x}(t)\big)$$









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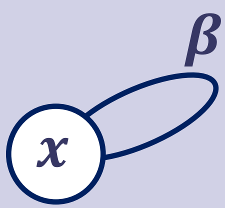
Multidimensional systems

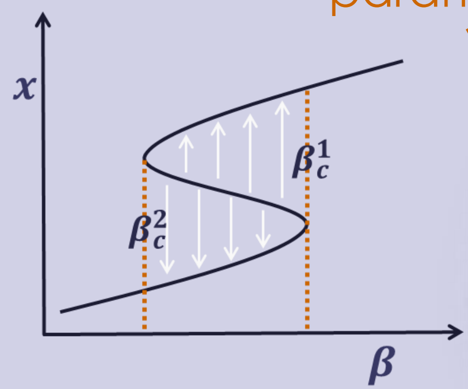
$$\frac{dx}{dt} = f(\beta, x(t))$$

N dimensional

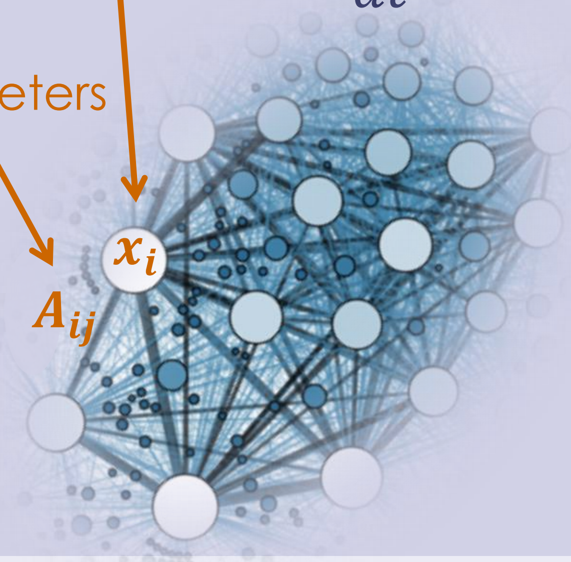
$$\frac{dx_i}{dt} = f(A_{ij}, \vec{x}(t))$$

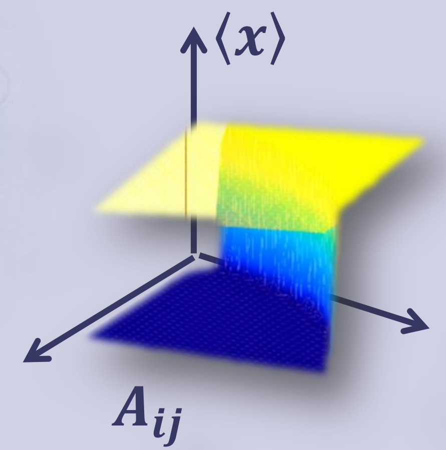
N^2 parameters





A_{ij}





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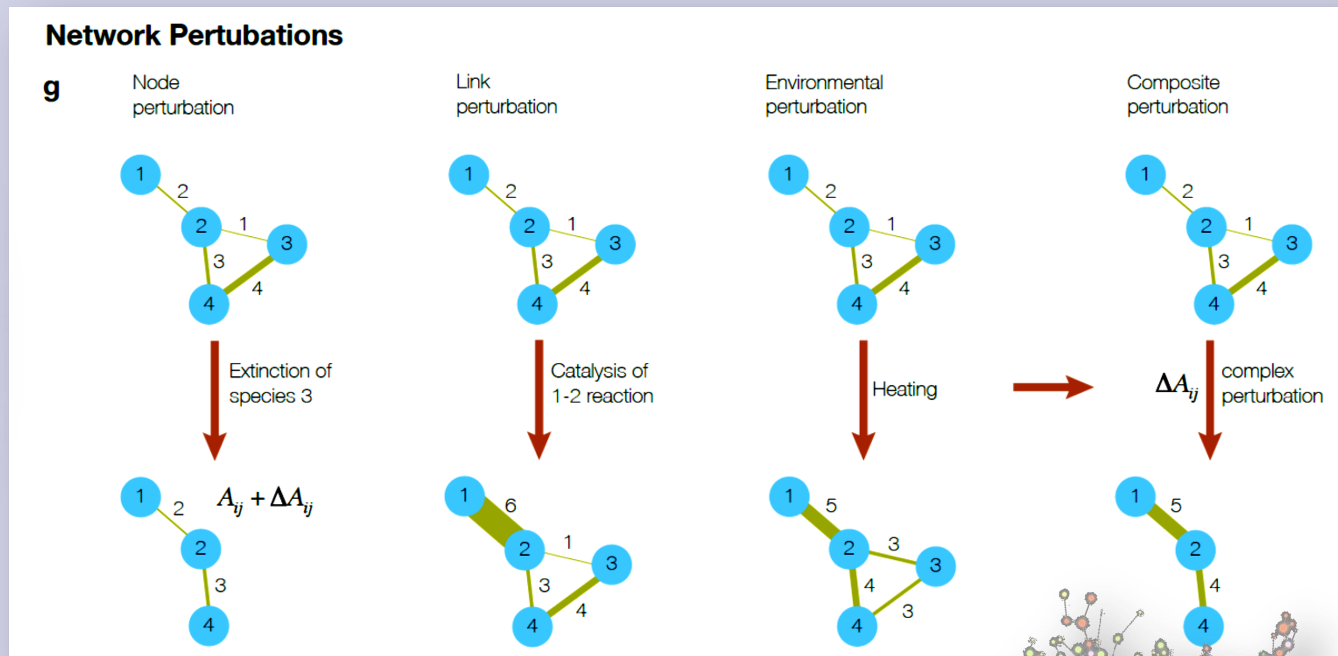
Multidimensional systems

$$\beta \to \beta + \Delta\beta \longrightarrow A_{ij} \to A_{ij} + \Delta A_{ij}$$



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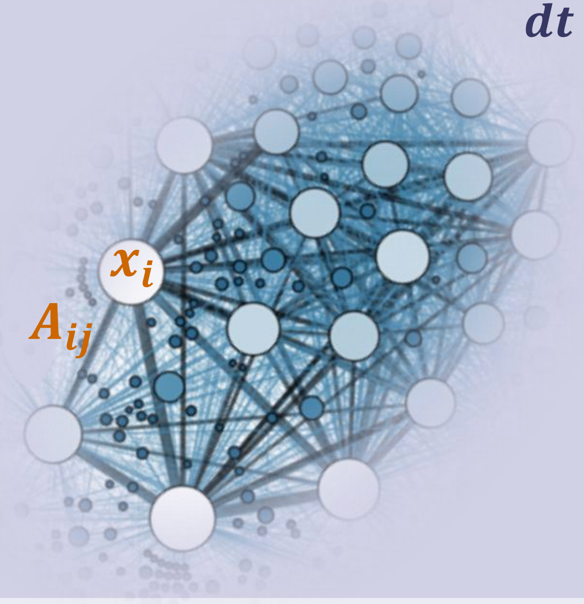
Multidimensional systems

$$\frac{dx_i}{dt} = F(x_i(t)) + \sum_{j=1}^{N} A_{ij} Q(x_i(t), x_j(t))$$



Self dynamics

Parameters – $N \times N$ weighted, directed (positive) interaction network

Interaction Mechanism

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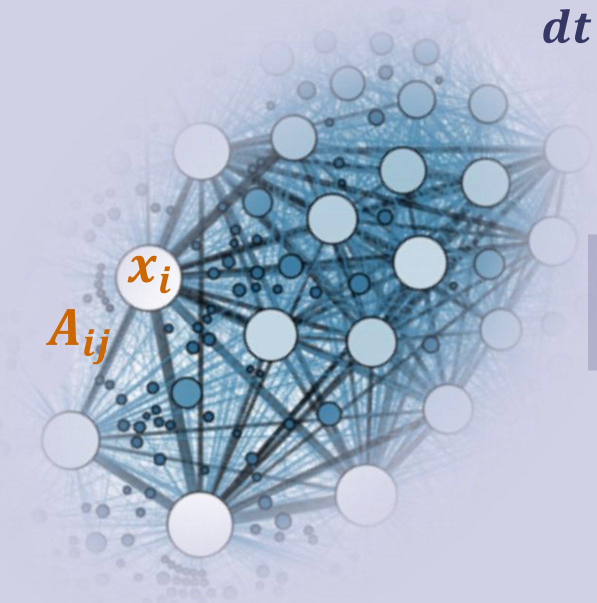
Multidimensional systems

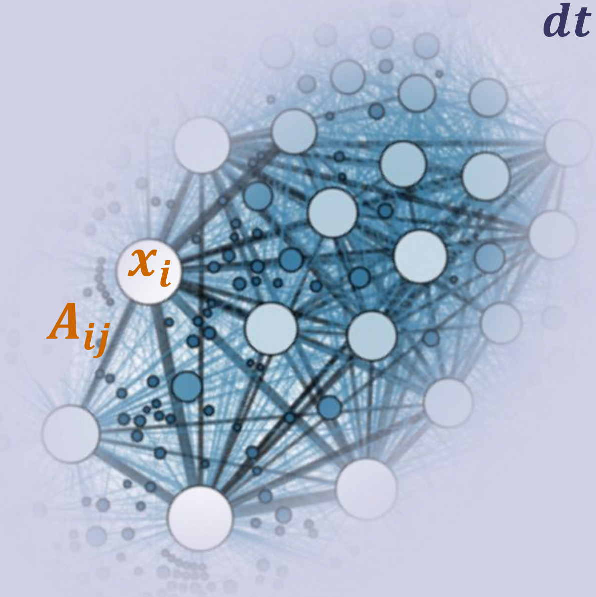
$$\frac{dx_i}{dt} = F(x_i(t)) + \sum_{j=1}^{N} A_{ij} Q\big(x_i(t), x_j(t)\big)$$

- $F(x_i(t))$ → Self dynamics
- A_{ij} → Parameters – $N \times N$ weighted, directed interaction network
- $Q\big(x_i(t), x_j(t)\big)$ → Interaction Mechanism



Multidimensional systems

$$\frac{dx_i}{dt} = F(x_i(t)) + \sum_{j=1}^{N} A_{ij} Q\big(x_i(t), x_j(t)\big)$$



Epidemic dynamics: $$\frac{dx_i}{dt} = -Bx_i + \sum_{j=1}^{N} A_{ij}(1 - x_i)x_j$$

Gene regulation: $$\frac{dx_i}{dt} = -Bx_i^a + \sum_{j=1}^{N} A_{ij} \frac{x_j^h}{1 + x_j^h}$$



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Ecological Resilience

$$\frac{dx_i}{dt} = B_i + x_i\left(1 - \frac{x_i}{\kappa_i}\right)\left(\frac{x_i}{\xi_i} - 1\right) + \sum_{j=1}^{N} A_{ij}\frac{x_i x_j}{1 + \alpha x_i + \gamma x_j}$$

Self dynamics
(Logistic growth
+ Allee effect)
$F(x_i)$

Interaction
mechanism
(Symbiosis)
$Q(x_i, x_j)$

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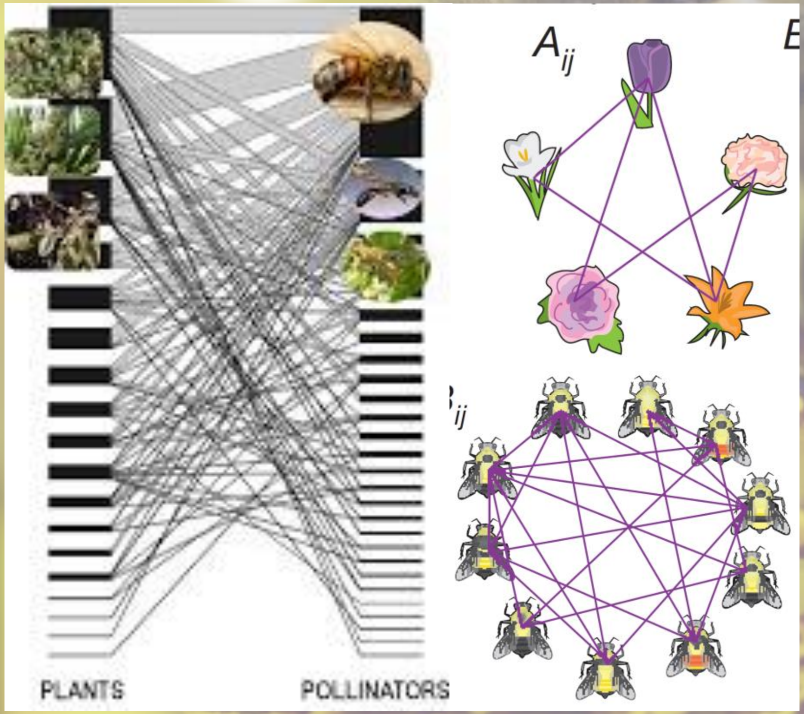
Ecological resilience

$$\frac{dx_i}{dt} = B_i + x_i\left(1-\frac{x_i}{\kappa_i}\right)\left(\frac{x_i}{\xi_i}-1\right) + \sum_{j=1}^{N} A_{ij}\frac{x_i x_j}{1+\alpha x_i+\gamma x_j}$$



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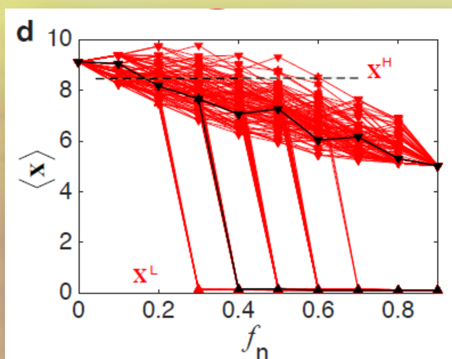
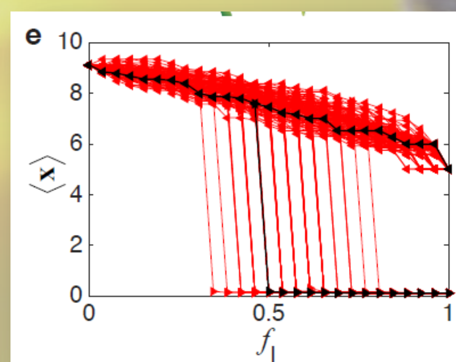
Ecological Resilience

$$\frac{dx_i}{dt} = B_i + x_i\left(1 - \frac{x_i}{\kappa_i}\right)\left(\frac{x_i}{\xi_i} - 1\right) + \sum_{j=1}^{N} A_{ij}\frac{x_i x_j}{1 + \alpha x_i + \gamma x_j}$$

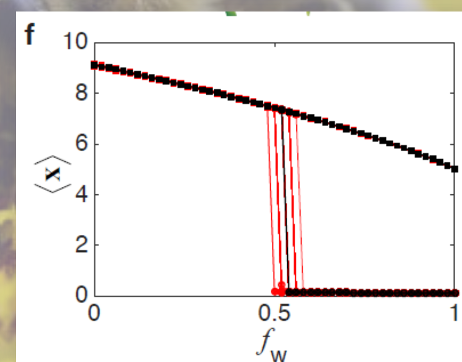
Plant Extinction



Pollinator Extinction



Environmental Change



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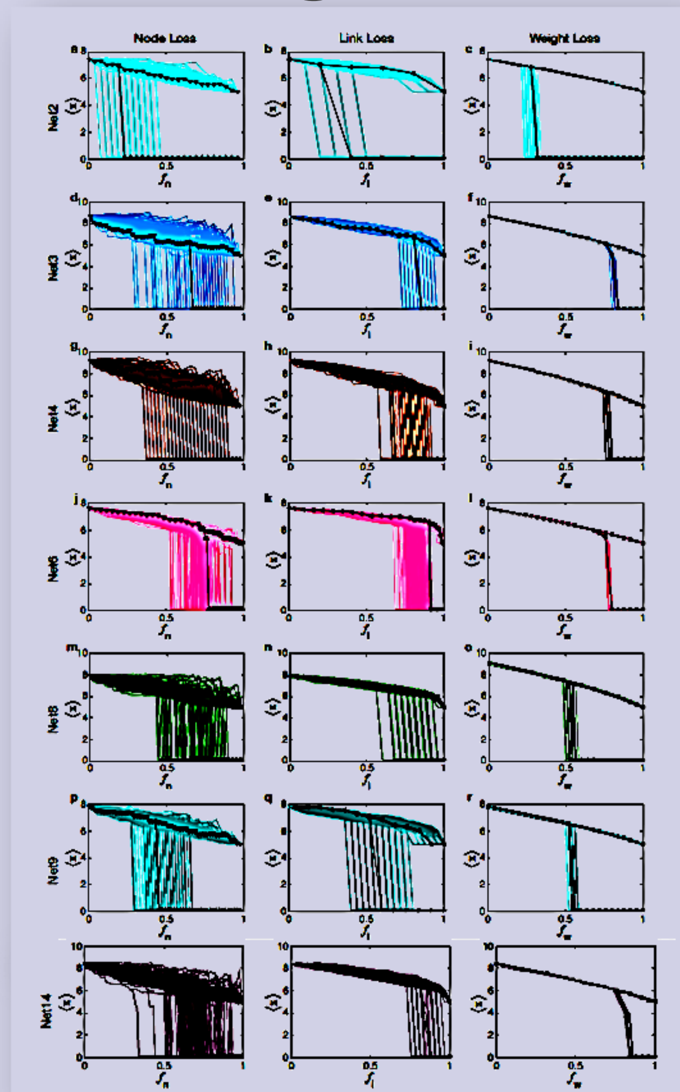
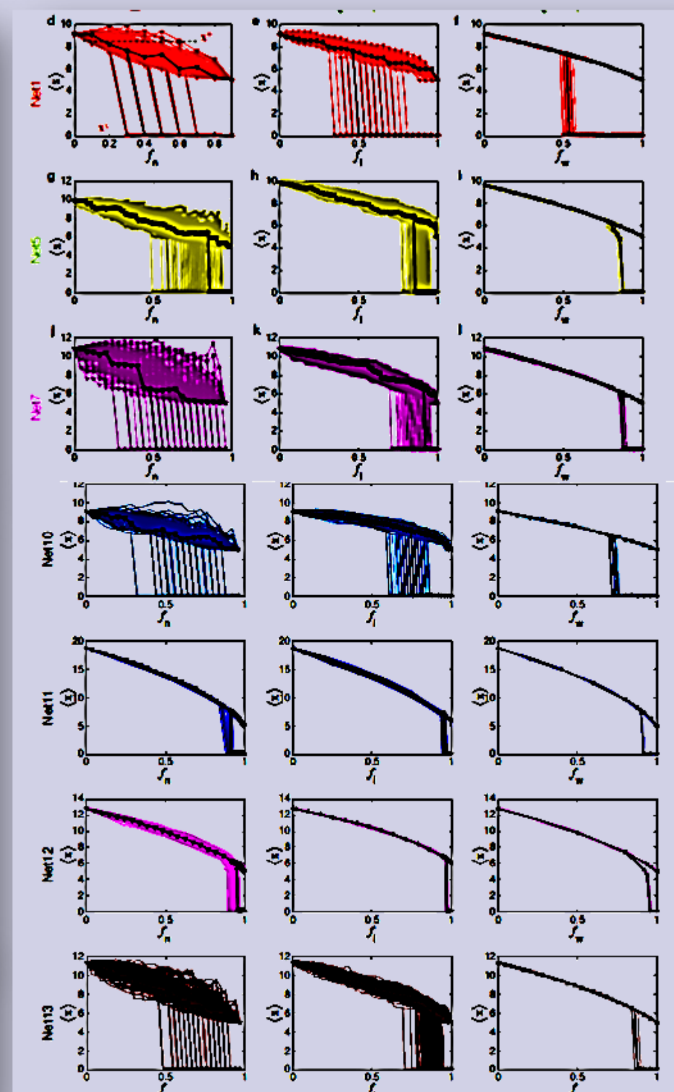
Ecological resilience





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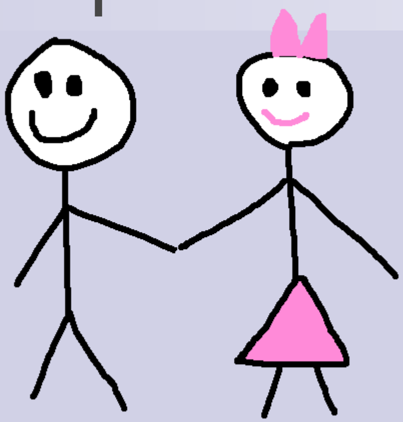
Simple – Complex - Complexer



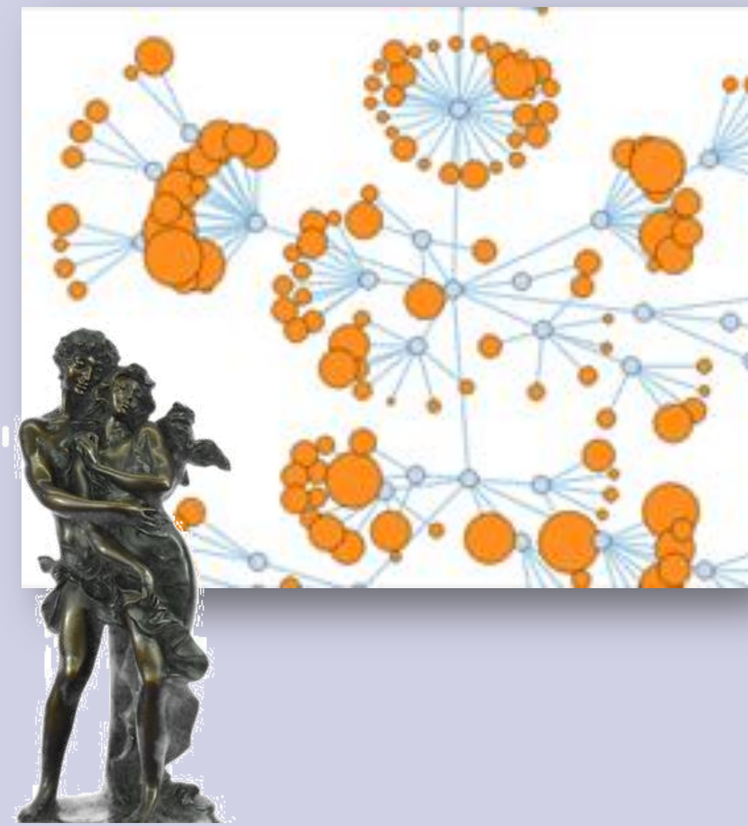
$$\frac{dx}{dt} = f(\beta, x)$$

$$\frac{dx_i}{dt} = F(x_i(t)) + \sum_{j=1}^{N} A_{ij} Q\left(x_i(t), x_j(t)\right)$$

?

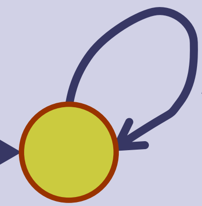
Dimension reduction

$$\frac{dx_i}{dt} = F(x_i(t)) + \sum_{j=1}^{N} A_{ij} Q\big(x_i(t), x_j(t)\big) \longrightarrow \frac{dx}{dt} = f(\beta, x)$$



Naïvely

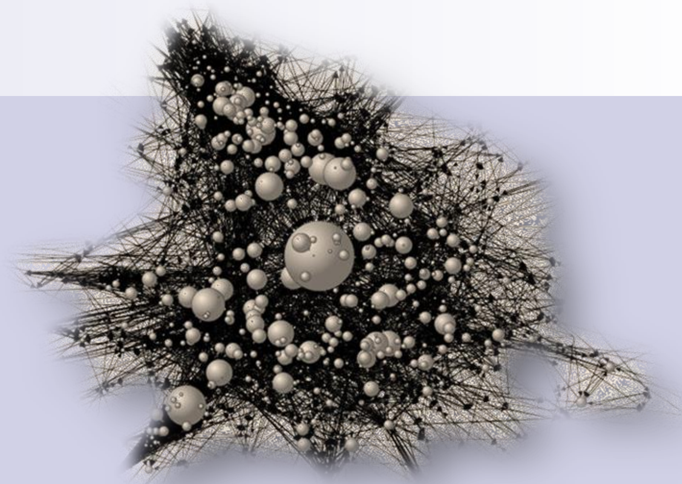
$$\beta = \langle k \rangle$$
$$x = \langle x \rangle$$

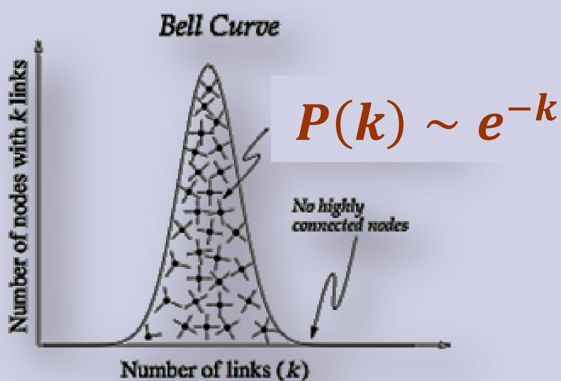


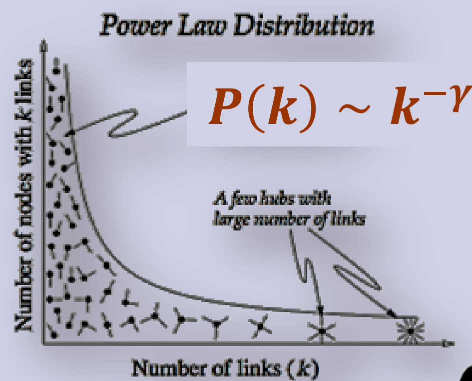
Averages are irrelevant

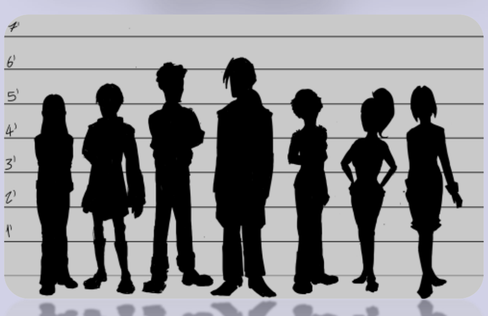
Degree heterogeneity $P(k)$

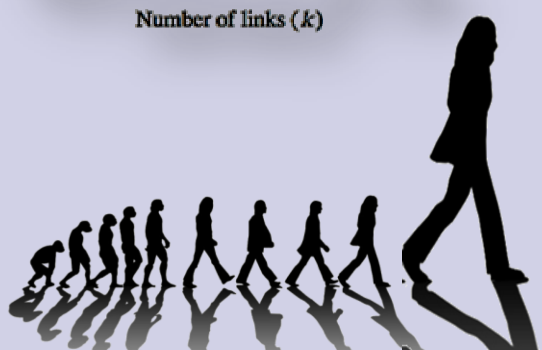
Weighted links





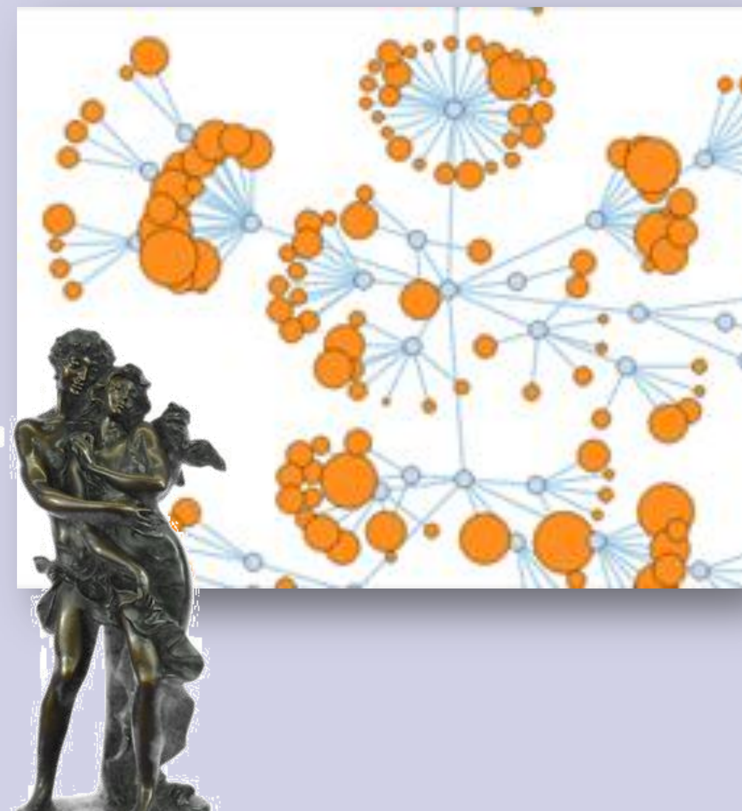






Dimension reduction

$$\frac{dx_i}{dt} = F(x_i(t)) + \sum_{j=1}^{N} A_{ij} Q\big(x_i(t), x_j(t)\big) \longrightarrow \frac{dx}{dt} = f(\beta, x)$$



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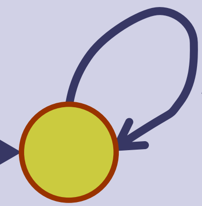
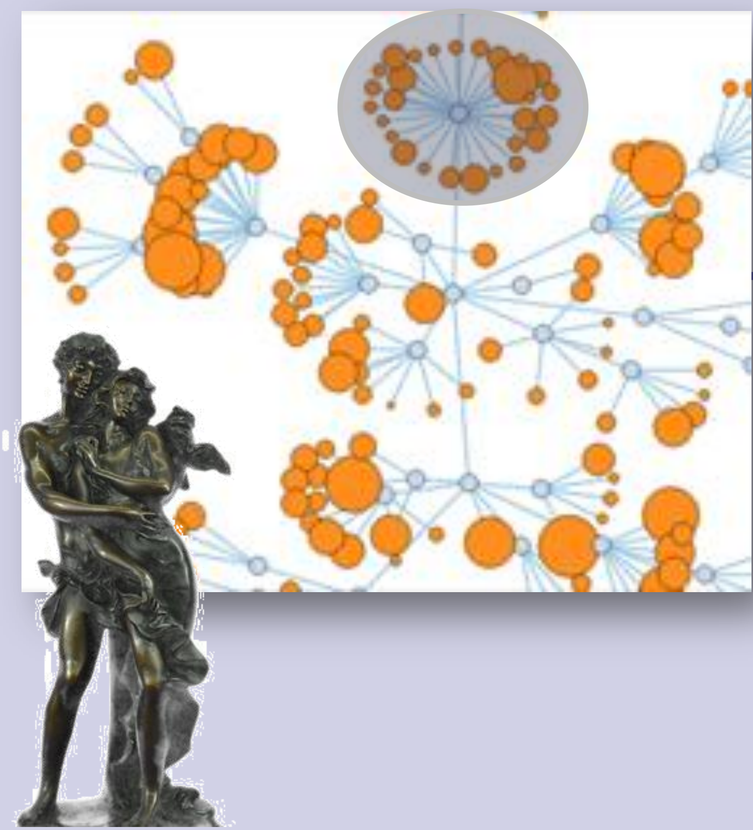


$f(\beta, x)$

Dimension reduction

$$\frac{dx_i}{dt} = F(x_i(t)) + \sum_{j=1}^{N} A_{ij} Q\left(x_i(t), x_j(t)\right)$$

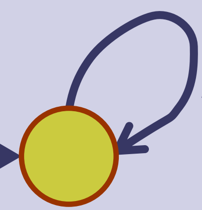


Correctly

$$\beta = \frac{\mathbf{1}^{\mathrm{T}} A \overrightarrow{s^{\mathrm{in}}}}{\mathbf{1}^{\mathrm{T}} A \mathbf{1}}$$

$$x = \frac{\mathbf{1}^{\mathrm{T}} A \vec{x}}{\mathbf{1}^{\mathrm{T}} A \mathbf{1}}$$

Weighted average over nearest neighbor nodes



$f(\beta, x)$

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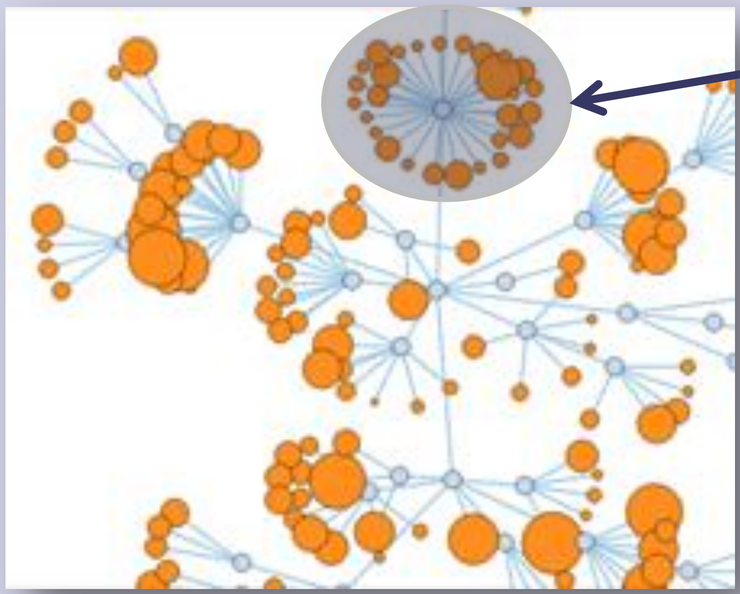
Dimension reduction

$$\frac{dx_i}{dt} = F(x_i(t)) + \sum_{j=1}^{N} A_{ij} Q\left(x_i(t), x_j(t)\right)$$



$$\mathcal{L}(\vec{v}) = \frac{\mathbf{1}^T A \vec{v}}{\mathbf{1}^T A \mathbf{1}}$$

Weighted average over nearest neighbor nodes

Configuration model:
Nodes are unique.
Neighborhoods are all alike.

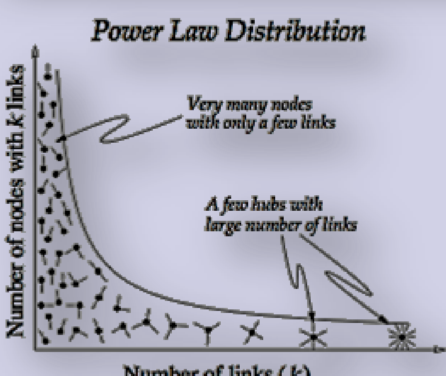
Your friends are more popular than you

Ex. 1 – Pick a node i, measure k_i

$$P(k), \qquad \langle k \rangle = \sum_k kP(k)$$

Ex. 2 – Pick a node i, measure k_j of its neighbor j

$$P_{nn}(k) = \frac{1}{\langle k \rangle} kP(k), \langle k_{nn} \rangle = \frac{\langle k^2 \rangle}{\langle k \rangle} = \langle k \rangle + \frac{\sigma^2}{\langle k \rangle}$$



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Dimension reduction

$$\frac{dx_i}{dt} = F(x_i(t)) + \sum_{j=1}^{N} A_{ij} Q\big(x_i(t), x_j(t)\big)$$

$$\mathcal{L}(\vec{v}) = \frac{\mathbf{1}^{\mathrm{T}} A \vec{v}}{\mathbf{1}^{\mathrm{T}} A \mathbf{1}}$$

Weighted average over nearest neighbor nodes

$$\frac{dx_i}{dt} \approx F(x_i(t)) + s_i^{\mathrm{in}} Q\big(x_i(t), \mathcal{L}(\vec{x}(t))\big)$$

$$s_i^{\mathrm{in}} = \sum_{j=1}^{N} A_{ij}$$

Weighted in-degree

Dimension reduction

$$\frac{dx_i}{dt} = F(x_i(t)) + \sum_{j=1}^{N} A_{ij} Q\big(x_i(t), x_j(t)\big)$$

$$\mathcal{L}(\vec{v}) = \frac{\mathbf{1}^{\mathrm{T}} A \vec{v}}{\mathbf{1}^{\mathrm{T}} A \mathbf{1}}$$

Weighted average over nearest neighbor nodes

$$\frac{dx_i}{dt} \approx F(x_i(t)) + s_i^{\mathrm{in}} Q\big(x_i(t), \mathcal{L}\big(\vec{x}(t)\big)\big)$$

$$s_i^{\mathrm{in}} = \sum_{j=1}^{N} A_{ij}$$

Weighted in-degree

$$\frac{d\mathcal{L}(\vec{x})}{dt} \approx F\big(\mathcal{L}(\vec{x})\big) + \mathcal{L}(\overrightarrow{s^{\mathrm{in}}}) Q(\mathcal{L}(\vec{x}), \mathcal{L}(\vec{x}))$$

$$\mathcal{L}\big(F(\vec{x})\big) \approx F\big(\mathcal{L}(\vec{x})\big)$$

Dimension reduction

$$\frac{dx_i}{dt} = F(x_i(t)) + \sum_{j=1}^{N} A_{ij} Q\big(x_i(t), x_j(t)\big)$$

$$\mathcal{L}(\vec{v}) = \frac{\mathbf{1}^{\mathrm{T}} A \vec{v}}{\mathbf{1}^{\mathrm{T}} A \mathbf{1}}$$

Weighted average over nearest neighbor nodes

$$\frac{dx_i}{dt} \approx F(x_i(t)) + s_i^{\mathrm{in}} Q\big(x_i(t), \mathcal{L}(\vec{x}(t))\big)$$

$$\frac{d\mathcal{L}(\vec{x})}{dt} \approx F\big(\mathcal{L}(\vec{x})\big) + \mathcal{L}(\overrightarrow{s^{\mathrm{in}}}) Q(\mathcal{L}(\vec{x}), \mathcal{L}(\vec{x}))$$

$$\frac{dx_{\mathrm{eff}}}{dt} \approx F(x_{\mathrm{eff}}) + \beta_{\mathrm{eff}} Q(x_{\mathrm{eff}}, x_{\mathrm{eff}}) = f(\beta, x)$$

$$x_{\mathrm{eff}} = \mathcal{L}(\vec{x}) = \frac{\mathbf{1}^{\mathrm{T}} A \vec{x}}{\mathbf{1}^{\mathrm{T}} A \mathbf{1}}$$

$$\beta_{\mathrm{eff}} = \mathcal{L}\left(\overrightarrow{s^{\mathrm{in}}}\right) = \frac{\mathbf{1}^{\mathrm{T}} A \overrightarrow{s^{\mathrm{in}}}}{\mathbf{1}^{\mathrm{T}} A \mathbf{1}}$$

Mapping to β-space reduces the multidimensional system into an effective one dimensional resilience function

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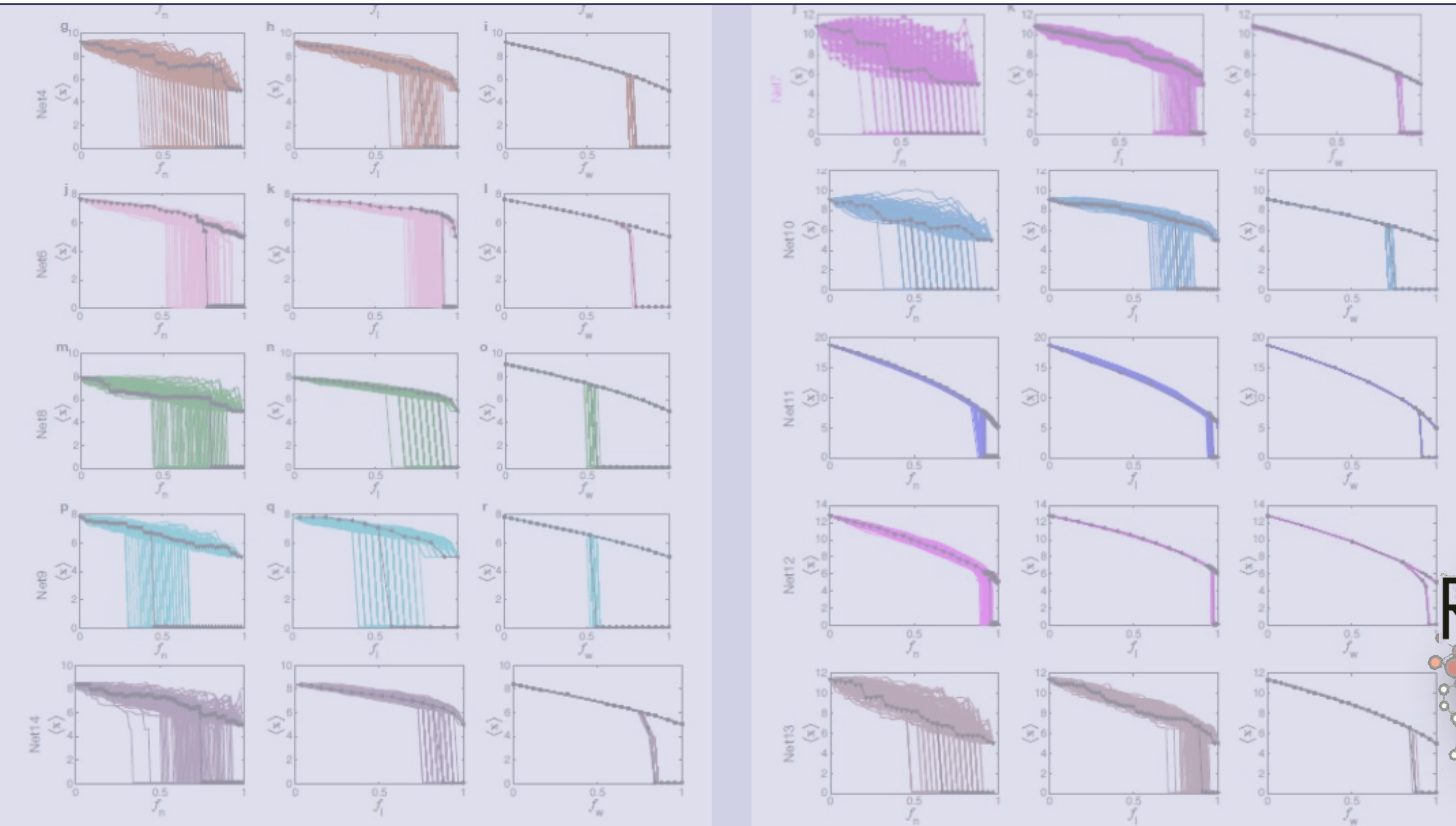
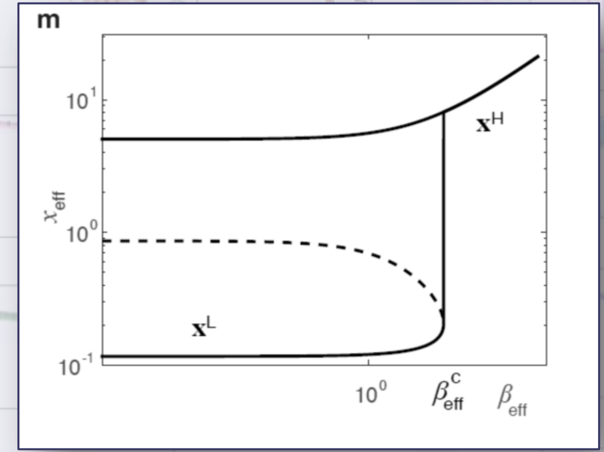
Ecological Resilience

$$\frac{dx_{\text{eff}}}{dt} = B + x_{\text{eff}}\left(1-\frac{x_{\text{eff}}}{\kappa}\right)\left(\frac{x_{\text{eff}}}{\xi}-1\right)+\beta_{\text{eff}}\frac{x_{\text{eff}}^2}{1+(\alpha+\gamma)x_{\text{eff}}}$$



Ecological resilience

$$\frac{dx_{\mathrm{eff}}}{dt} = B + x_{\mathrm{eff}}\left(1 - \frac{x_{\mathrm{eff}}}{\kappa}\right)\left(\frac{x_{\mathrm{eff}}}{\xi} - 1\right) + \beta_{\mathrm{eff}}\frac{x_{\mathrm{eff}}^2}{1 + (\alpha + \gamma)x_{\mathrm{eff}}}$$



$A_{ij} \rightarrow \beta_{\mathrm{eff}}$

$\Delta A_{ij} \rightarrow \Delta\beta_{\mathrm{eff}}$

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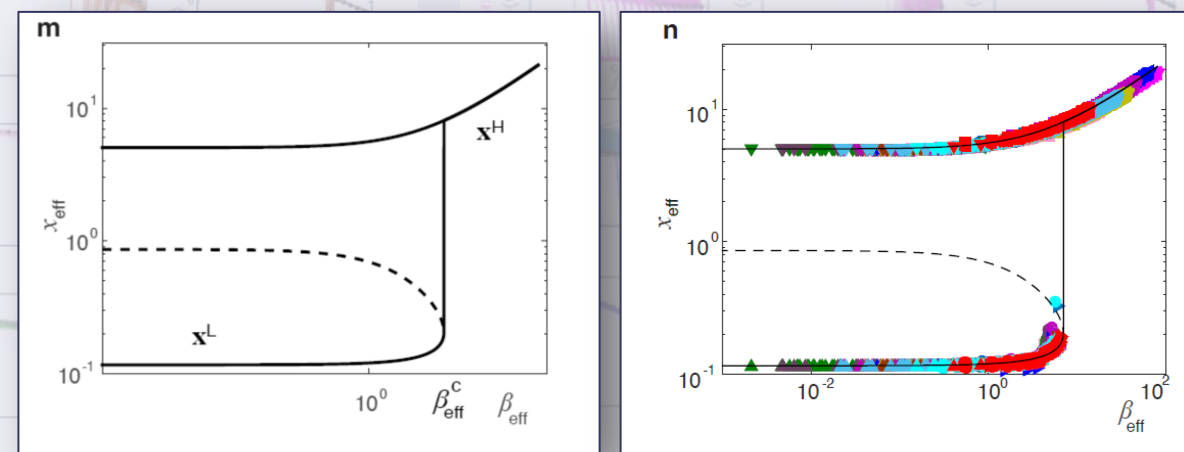
Ecological resilience

$$\frac{dx_{\rm eff}}{dt} = B + x_{\rm eff}\left(1 - \frac{x_{\rm eff}}{\kappa}\right)\left(\frac{x_{\rm eff}}{\xi} - 1\right) + \beta_{\rm eff}\frac{x_{\rm eff}^2}{1 + (\alpha + \gamma)x_{\rm eff}}$$



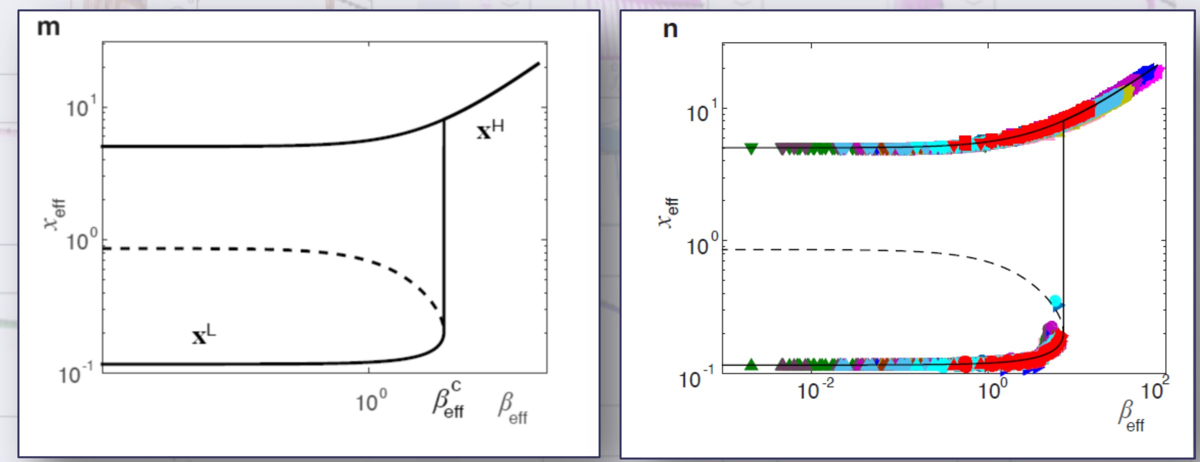
$$A_{ij} \rightarrow \beta_{\rm eff}$$

$$\Delta A_{ij} \rightarrow \Delta\beta_{\rm eff}$$

RESILIENCE

Ecological resilience

$$\frac{dx_{\mathrm{eff}}}{dt} = B + x_{\mathrm{eff}}\left(1 - \frac{x_{\mathrm{eff}}}{\kappa}\right)\left(\frac{x_{\mathrm{eff}}}{\xi} - 1\right) + \beta_{\mathrm{eff}}\frac{x_{\mathrm{eff}}^2}{1 + (\alpha + \gamma)x_{\mathrm{eff}}}$$



$A_{ij} \rightarrow \beta_{\mathrm{eff}}$

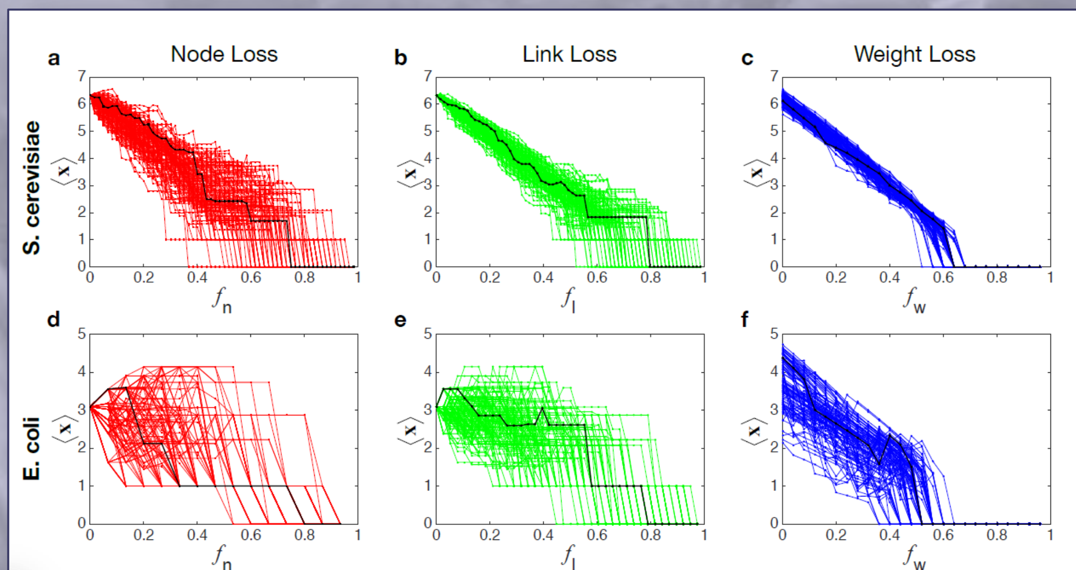
$\Delta A_{ij} \rightarrow \Delta\beta_{\mathrm{eff}}$

β-space exposes the hidden universality of the system's resilience function

RESILIENCE

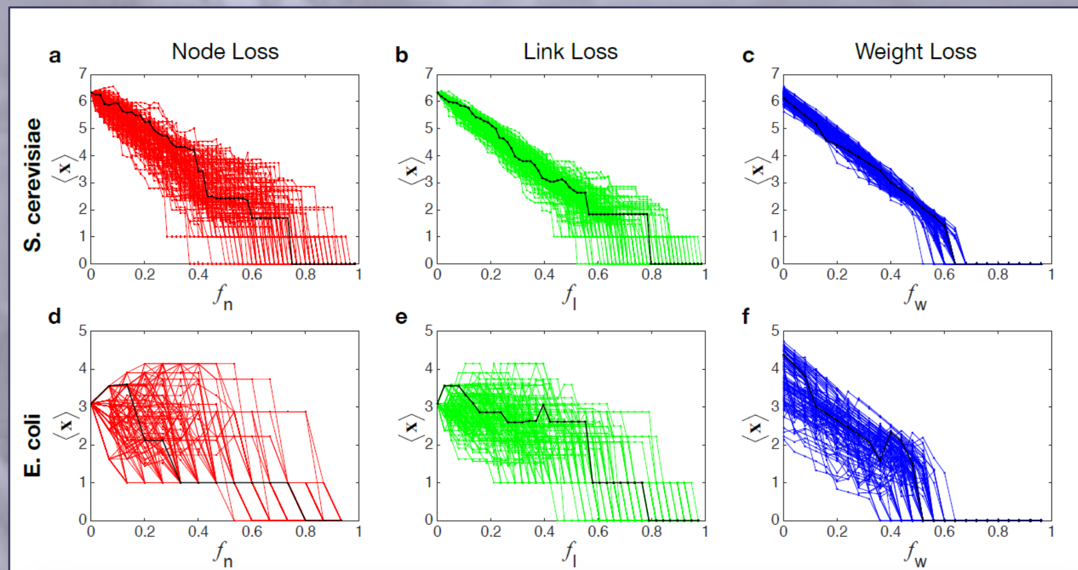
Biological Resilience

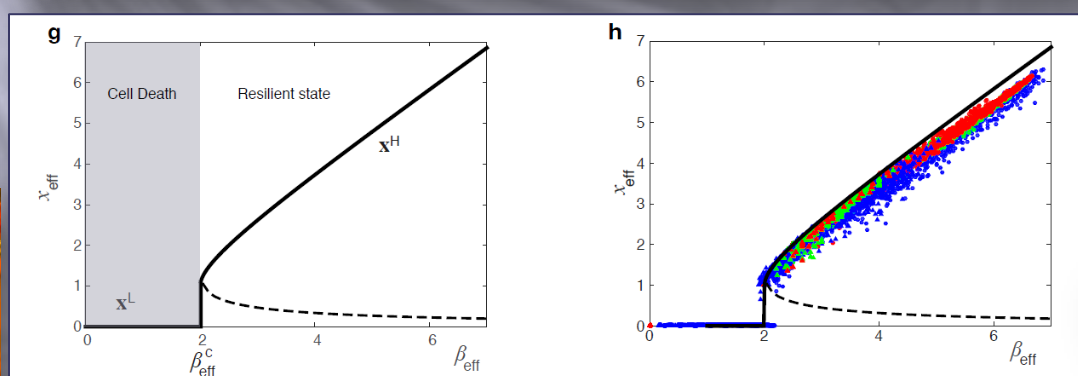
$$\frac{dx_i}{dt} = -Bx_i^{\alpha} + \sum_{j=1}^{N} A_{ij} \frac{x_j^h}{1 + x_j^h}$$

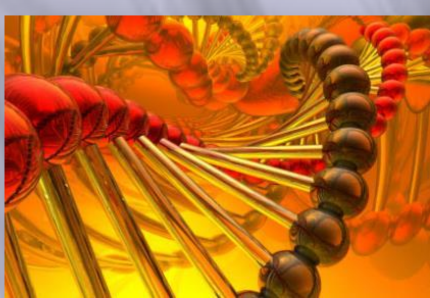


RESILIENCE

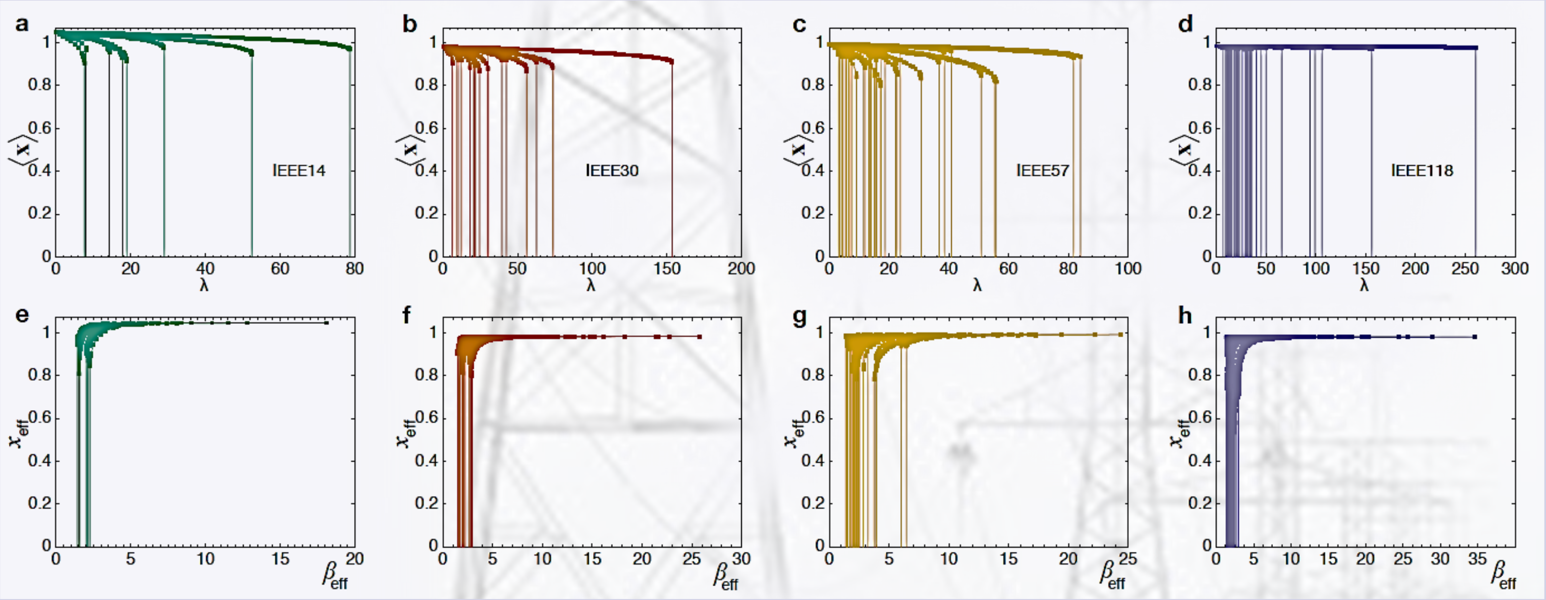
Biological Resilience



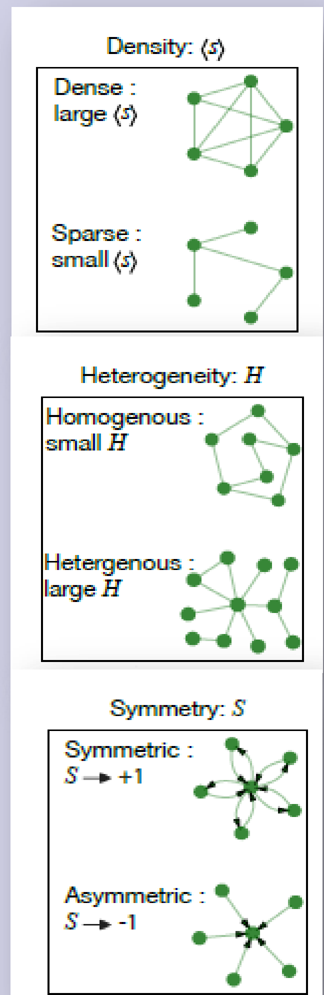




Power supply



The parameters of resilience



$$\beta_{\text{eff}} = \mathcal{L}\left(\overrightarrow{s^{\text{in}}}\right) = \langle s \rangle + \mathcal{H}\mathcal{S}$$

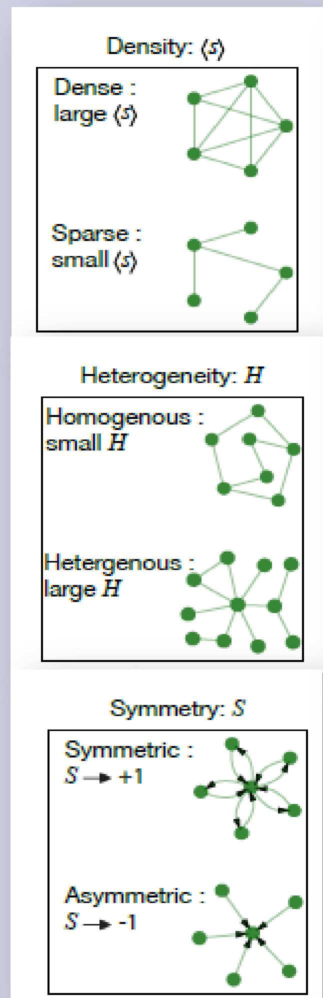
$$\langle s \rangle = \sum_{i=1}^{N} s_i = \langle s^{\text{in}} \rangle = \langle s^{\text{out}} \rangle$$

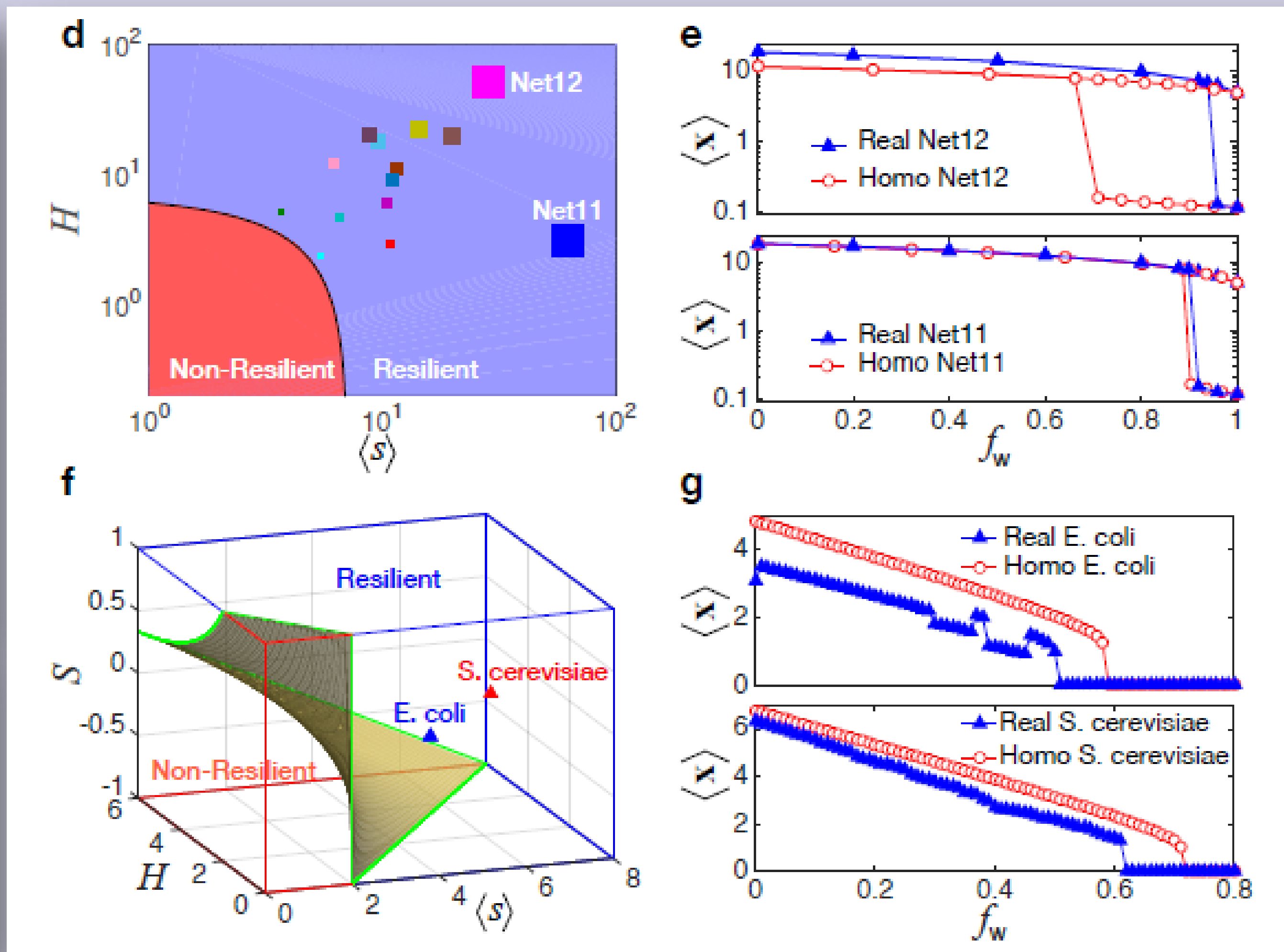
$$\mathcal{H} = \frac{\sigma^{\text{in}}\sigma^{\text{out}}}{\langle s \rangle}$$

$$\mathcal{S} = \frac{\langle s^{\text{in}} s^{\text{out}} \rangle - \langle s^{\text{in}} \rangle \langle s^{\text{out}} \rangle}{\sigma^{\text{in}}\sigma^{\text{out}}}$$

Sparse Heterogeneous Antisymmetric	→ β_{eff} →	Dense Heterogeneous Symmetric

The parameters of resilience





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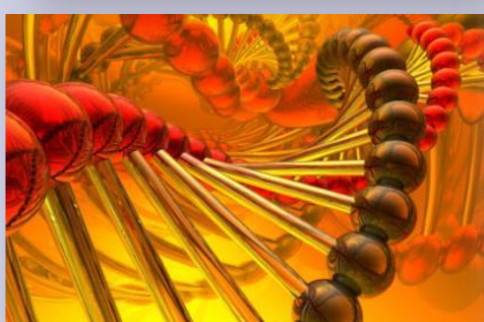

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RESILIENCE