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Monte Carlo (MC) simulation for system reliability

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Lecture plan

1. **Motivation: MC simulation is a necessary method for complexe systems reliability analysis.**
2. **MC simulation principle**
3. **Random number generation (RNG)**
4. **Probability distributions simulation**
5. **Simulation realization (History)**
6. **Output analysis and data accuracy**
7. **Example**
8. **MC simulation and uncertainty propagation**

1. Motivation: MC simulation is a necessity

Complex systems

- High system states number
- Non constant failure (component aging) and repair rates
- Components dependency: standby, ...
- Complex maintenance strategies: spare parts, priority, limited resources,...
- Multi-phase systems (MPS)
- Multi-state systems (MSS)
- Reconfigurable systems
- …

Analytical approaches become impractical:

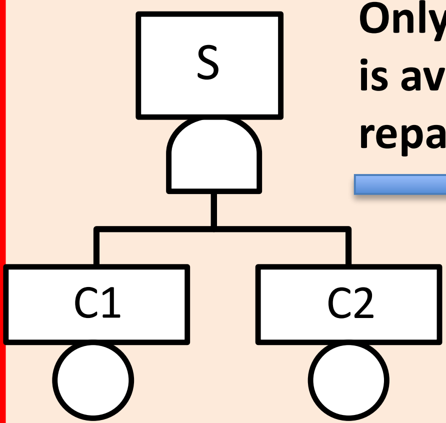
- ***Approximated representation*** that does not fit to the real system
- ***Extensive time*** for the development of the analytical model
- ***Infeasible***

Simulation is required: a ***common paradigm*** and a ***powerful tool*** for analyzing ***complex systems***, due to its capability of achieving a ***closer adherence*** to reality. It provides a simplified representation of the system under study.

1. Motivation: example

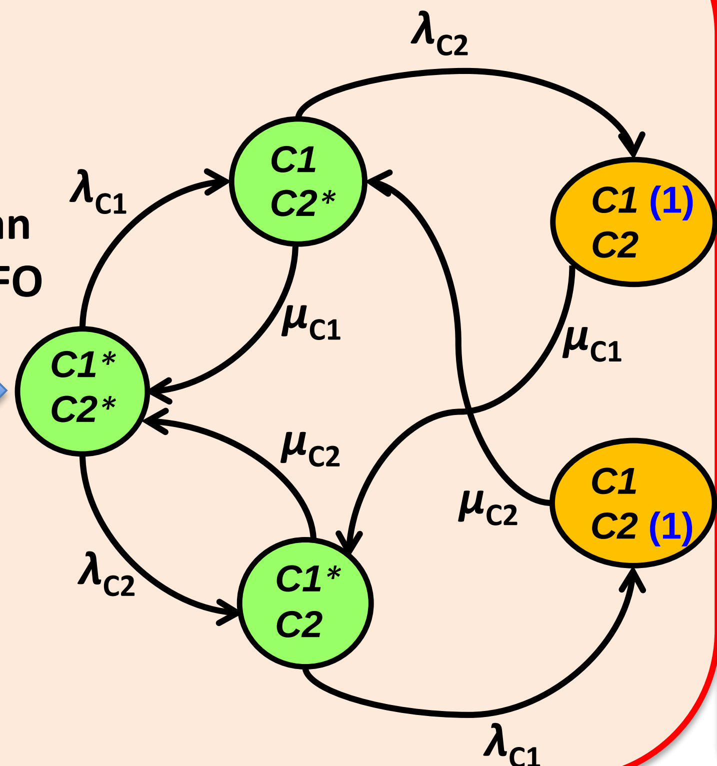
Analytical approaches

Fault tree



Only one repair man is available with FIFO repair strategy

Markov chains



Non constant repair rates (e.g. lognormal)

- Markov chains become *impractical*.
- *Strong complications* for other analytical approaches

MC simulation

2. MC simulation principle : definitions

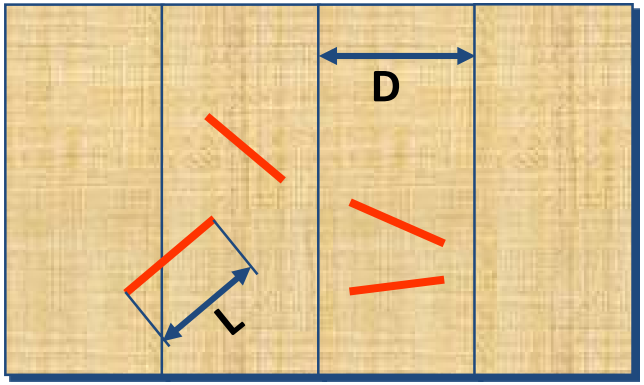
- MC simulation may be defined as a process for ***obtaining estimates (numeric values)*** for a given system (guided by a prescribed set of ***goals***) by means of ***random numbers***.

- Monte Carlo is one of Monaco's four quarters. The name "Monte Carlo" was given by **Nicholas Metropolis** after the **Monte Carlo Casinos**.

- Random numbers: a set of numbers that have nothing to do with the other numbers in the sequence. Thus, for any *n* random numbers, ***each appears*** with ***equal probability*** (independent, identically distributed).

2. MC simulation principle : history

- The needle experiment of Comte de Buffon (***Buffon's needles,*** **1733)** (french biologist, 1707-1788) ***:***



L ≤ D

What is the probability p, that a needle (length L), which randomly falls on a wooden floor, crosses one of the lines (Lathes) (distance D)?

N: *number of needles*
n: *number of needles that are crossing lines*

$$p = \frac{2\,L}{D\,\Pi} \Longrightarrow \Pi = \frac{2\,L}{D\,p} \approx \frac{2\,L}{D} \cdot \frac{N}{n}$$

- The real development of the Monte Carlo methods is performed, under the leadership of ***Nicholas Metropolis***, ***Stanislaw Marcin Ulam***, ***Enrico Fermi*** and ***John von Neumann*** during World War II and research towards the development of nuclear weapons (Manhattan Project). In particular, they used these probabilistic methods for solving certain types of differential equation.

2. MC simulation principle : Steps

1. Problem analysis and information collection:
   - Objective of the study (system oriented problem): reliability, availability, production, design options,…
   - System structure: its elements
   - Input parameters: failure and repair rates, processing time, …
   - Relationships between input parameters and performance measures
   - Rules governing the operation of systems components

2. Model construction. It consists in constructing the model (conceptual) and implementing it as a computer program:
   - General-purpose language: C^{++}, FORTRAN,…
   - Special-purpose language: simulation tools.

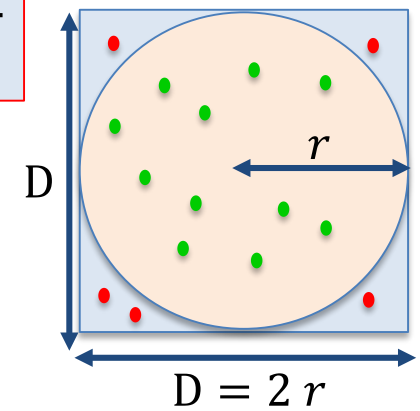
3. Model verification and validation: make sure that the model conforms to its specifications and does what it is supposed to do.

4. Simulation experiments. This step is achieved thanks to ***random numbers***. The evolution of the simulated system within a ***large number of Histories*** allows to ***statistically assess*** the established performance measure (average, variance, confidence interval, ...).

2. MC simulation principle : determining Π (classic example)

- **Π** is a ratio of a circle's circumference to its diameter: $\Pi = \dfrac{C}{D}$
  - The area of circle = $\boldsymbol{\Pi\, r^2}$
  - The area of square = $\boldsymbol{4\, r^2}$

$$\boldsymbol{\Pi} = 4 \cdot \frac{\text{Area of circle}}{\text{Area of square}}$$

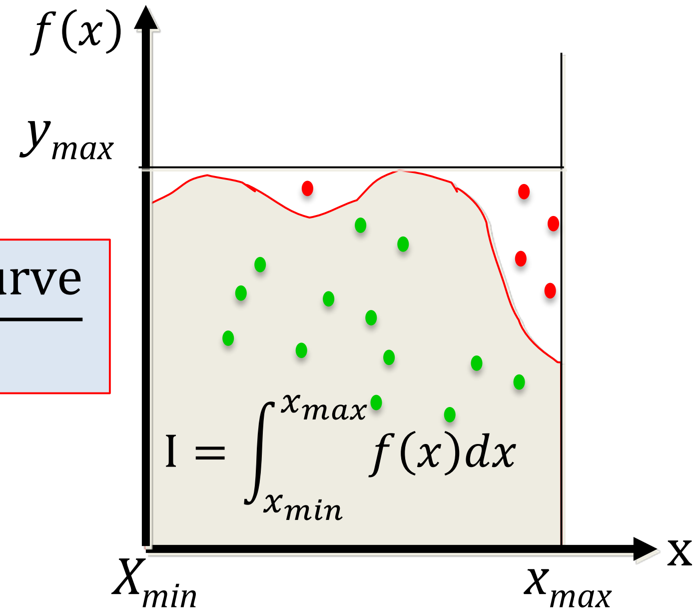


$$\boldsymbol{\Pi} \approx 4 \cdot \frac{\text{Number of dots inside the circle}}{Total\ \text{number of dots}}$$

- Using the ***hit and miss method (with*** $r = 1$):
  - Generate two sequences of random numbers: $\boldsymbol{R_i}$ and $\boldsymbol{R_j}$ (*i, j= 0,…, N*)
  - $x_i = -1 + 2\, R_i$ (distribution range for x)
  - $y_j = -1 + 2\, R_j$(distribution range for y)
  - If $x_i^2 + y_j^2 \leq 1$ $\quad then\ \ S = S + 1$ (dot inside) $\quad else \quad S = S$ (dot outside)
  - $\boldsymbol{\Pi} \approx 4 \cdot \dfrac{\text{S}}{N}$

Solution with Excel

2. MC simulation principle : integral calculus



$$\frac{I}{\text{Area of the rectangle}} \approx \frac{\text{Number of dots under the curve}}{\text{Total number of dots}}$$

- Using the ***hit and miss method (with*** $0 \le f(x) \le Y_{max}$):
  - Generate two sequences of random numbers: ***R_i*** and ***R_j*** (*i, j= 0,…, N*)
  - $x_i = a + (b - a)\, R_i$ (distribution range for x)
  - $y_j = y_{max}\, R_j$ (distribution range for y)
  - If $y_j \le f(x_i) \quad then \;\; S = S + 1 \quad else \quad S = S$
  - $\boldsymbol{I} \approx y_{max} \cdot (b - a) \cdot \frac{S}{N}$

Example: $\int_0^\infty \frac{3}{4} x^4 e^{-x^{3/4}}\, dx$ ***Solution with Excel***

3. Random Numbers Generator (RNG)

- The generation of random numbers R uniformly distributed in [0,1) can be obtained physically by throwing a coin or dice, spinning the roulette,…

- Monte Carlo simulations is based on computer generation of ***pseudo random numbers:*** the ***closest random number generator*** that can be obtained by ***computer algorithm***.

- In 1940's, von Neumann proposed to have the computer directly generate the random numbers by means of an appropriate function: ***find the next number X_{i+1} from the preceding one X_i.*** He proposed the **middle-square method:** obtain X_{i+1} by taking the central digits of the square of X_i (Example: 5772156649^2 = 33317<u>7923805949</u>09291).

- ***Linear congruential generators (LCG) :*** $X_{n+1} = (a \cdot X_n + c) \bmod m$

  a, c >= 0 , m > X_0, a, c

Algorithm generates integers between 0 and *m*, map to zero and one.

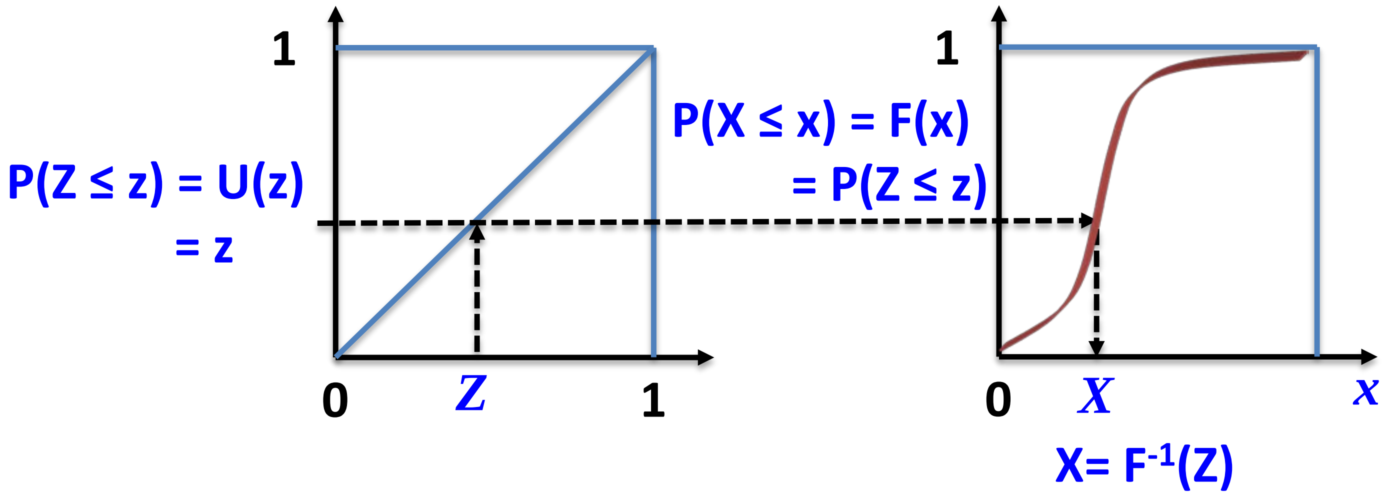
$$R_{n+1} = \frac{X_{n+1}}{m}$$

- ***Example: RANDU generator*** (1960's IBM)

$$X_{n+1} = (65539 \times X_n) \bmod (2^{31})$$

4. Probability distributions simulation: principle

- Any given distribution can be generated from uniform random numbers on [0,1]: **U(0, 1).**

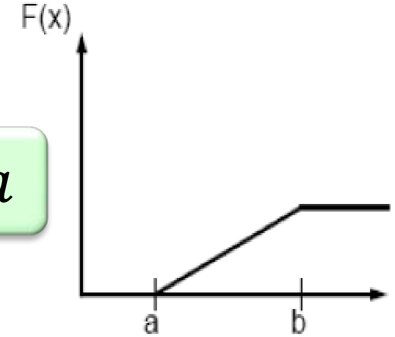


- ***Principle:*** generating samples (x_1, x_2, ..., x_n) of a random variable X (time to failure, to repair, …) obeying any distribution F(x) from a sample (z_1, z_2, ..., z_n) of the variable Z equally distributed between 0 and 1 by performing the transformation: $x_i = F^{-1}(z_i)$.
- If the inversion of the distribution function cannot be performed explicitly, a numeric solution of the equation $F(x_i) = z_i$ can be exploited. In addition, other simulation methods exist (rejection method,…).

4. Probability distributions simulation: some useful distributions

- Uniform distribution: **U(a, b)**

Simulation: $\boldsymbol{X} = (b - a) \cdot \boldsymbol{Z} + a$



$$F(x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & \text{si } a \le x \le b \\ 1 & x > b \end{cases}$$

- Exponential distribution: **Exp(λ)**

$$F(x) = 1 - e^{-\lambda x}$$

Simulation:

$$Z = 1 - e^{-\lambda \cdot X}$$

$$\ln(1 - Z) = -\lambda \cdot X$$

$$X = -\frac{\ln(1 - Z)}{\lambda} \xrightarrow{\text{Z and 1-Z have the same distribution}} \boldsymbol{X} = -\frac{\ln(\boldsymbol{Z})}{\lambda}$$

Example. **Determine the MTTF and MTTR for a component with λ = 1E-4/h and μ=1E-1/h (*solution with Excel*).**

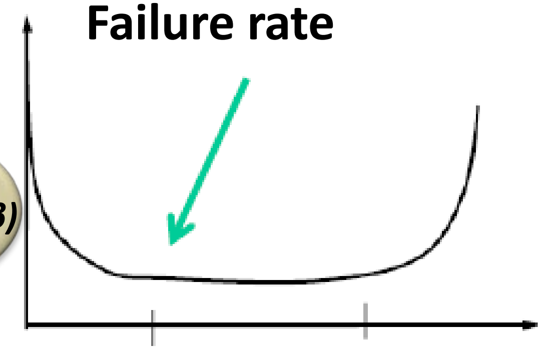
4. Probability distributions simulation: some useful distributions

- Weibull distribution: **W(λ, β)**

$$F(x) = 1 - e^{-\left(\frac{x}{\eta}\right)^{\beta}}$$

Scale parameter (η)

Shape parameter (β)



Simulation:

$$Z = 1 - e^{-\left(\frac{X}{\eta}\right)^{\beta}}$$

$$\ln(1 - Z) = -\left(\frac{X}{\eta}\right)^{\beta}$$

$$\boxed{X = \eta(-\ln(Z))^{\frac{1}{\beta}}}$$

Remark – Marvin book: $F(x) = 1 - e^{-(\lambda x)^{\alpha}}$

- Log Normal distribution: **LN(*μ*, σ) or LN(*m*, EF)**

$$f(x) = \frac{1}{x\sigma\sqrt{2\Pi}} e^{-\frac{(\ln(x) - \mu)^2}{2\sigma^2}}$$

Average

$$\mu = \ln(m) - \frac{\sigma^2}{2}$$

$$\sigma = \frac{\ln(q_{5\%})}{1.64}$$

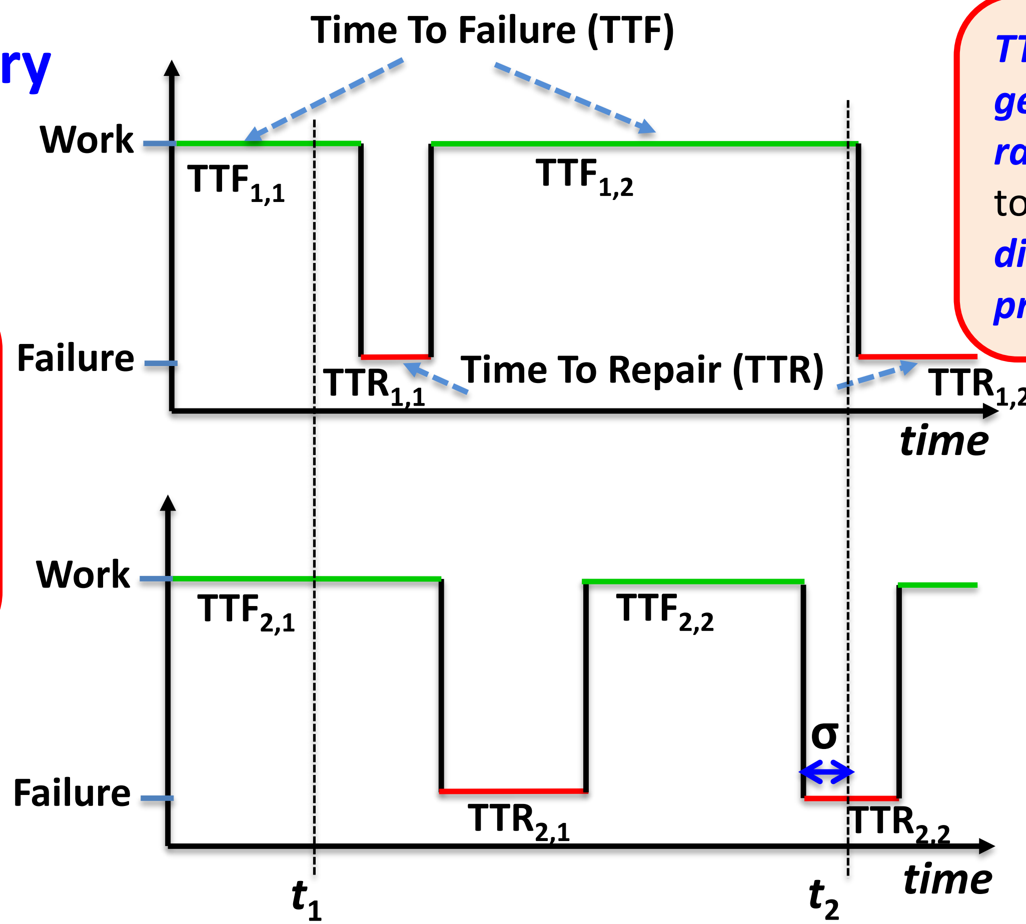
Error factor

$$\boxed{X = e^{(\sigma\nu + \mu)}}$$

$$\nu = -[2\ln(Z_1)]^{\frac{1}{2}} \cos(2\pi Z_2)$$

5. Simulation History

History 1



TTF and *TTR* are ***generated randomly*** according to their ***respective distribution probabilities***

The results accuracy depends on the number of stories: to obtain an accurate result, it is necessary to perform a large number of stories.

History 2

Reliability: R($\boldsymbol{t_1}$) = 1; R($\boldsymbol{t_2}$) = 0

Availaibility: A($\boldsymbol{t_1}$) = 1; A($\boldsymbol{t_2}$) = 0.5;

Mean unavailability: Q(0**,** $\boldsymbol{t_2}$)=**(**TTR_1/t_2+ (TTR_2+σ)/t_2**)**/2

Mean failure number: W(0, $\boldsymbol{t_2}$)=(1+ 2)/2= 1.5

MTTF = ($TTF_{1,1}$+$TTF_{2,1}$)/2

...

6. Output analysis and accuracy

- ***Statistical analysis*** of the ***simulated data sample*** of ***size n*** (number of histories or replications).

n histories

n independent observations: $x_i : i = 1,\ldots,n.$

Point estimation

Sample mean

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Sample variance

$$S^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{X})^2$$

Example: find the confidence interval for ***MTTF*** at a confidence of **90 % (slide 12)**.

Confidence interval quantifies the confidence (probability) that the true (but unknown) statistical parameter falls within an interval whose boundaries are calculated using point estimates.

Confidence interval estimation

$$\boldsymbol{Pr(\bar{X} - e \le \mu \le \bar{X} + e) = 1 - \alpha}$$

Central limit theorem

$$\boldsymbol{Pr\left(\bar{X} - z_{1-\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}} \le \mu \le \bar{X} + z_{1-\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}}\right) = 1 - \alpha}$$

$$\boldsymbol{\bar{X} \pm z_{1-\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}}}$$

Quantiles of the standardized normal distribution

Accuracy is proportional to $\frac{1}{\sqrt{n}}$

7. Example

- Consider a ***periodically tested component***: once occurred, the failures of such component ***remain hidden until the next periodic test***. Failures and repairs follow ***exponential distributions*** with rates **2.5E-3/h** and **0.1/h**, respectively. The duration between two successive periodic tests is **1000 h**.

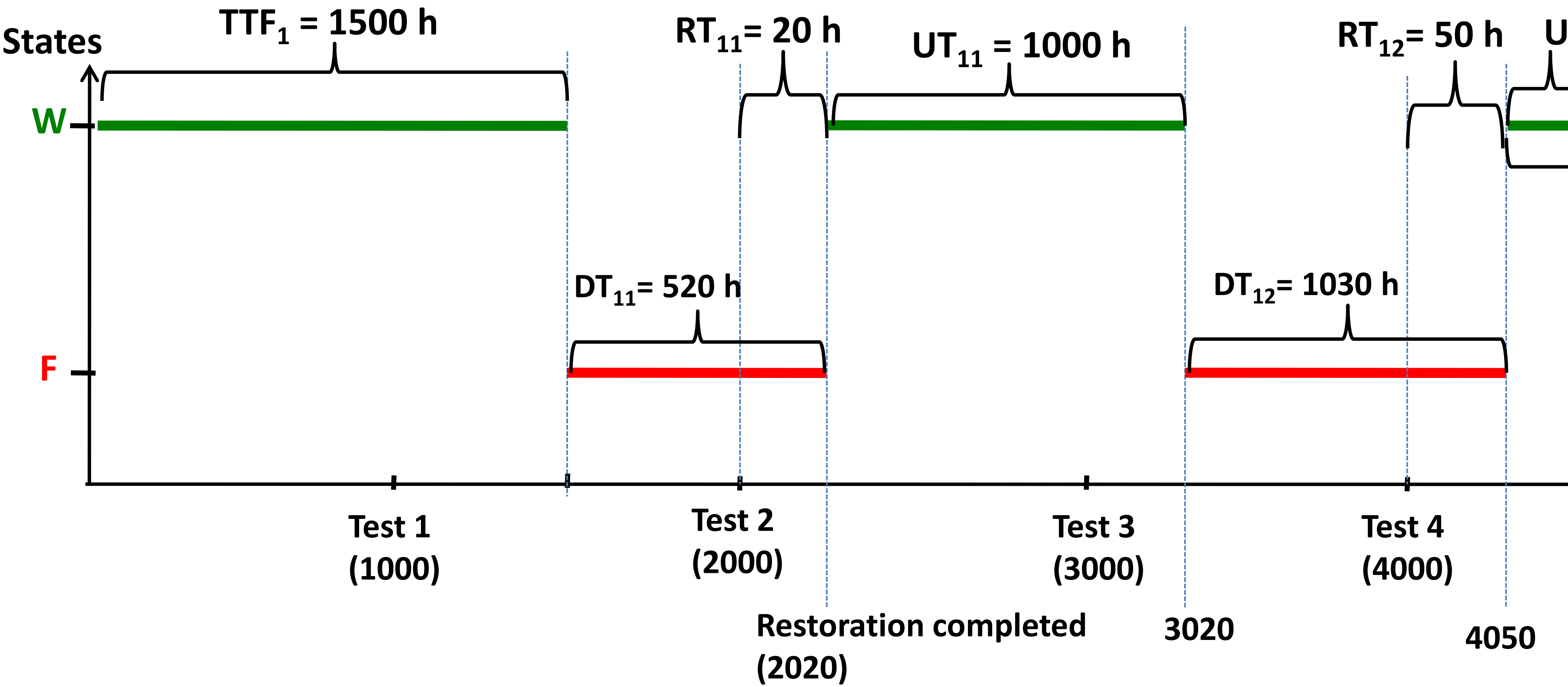
1. Establish 3 histories, each having a duration of 4500 hours, using the following random numbers according to their order of appearance:

- History n°1 : 0.0235177 ; 0.13534 ; 0.082085 ; 0.00674 ; 0.006738.
- History n°2 : 0.28650 ; 0.00674 ; 0.093014 ; 0.36788 ; 0.0024788 ; 0.000045.
- History n°3 : 0.0024788 ; 0.36788 ; 0.0235177.

2. Compute the mean availability of the component.
3. Compute its availability and reliability at t = 3000 h.
4. Compute the MTTF.
5. Estabilsh the confidence interval for the MTTF (confidence = 90 %). What can you deduce ?
6. Compute the MUT and MDT of the component.

A1: histories

History 1:

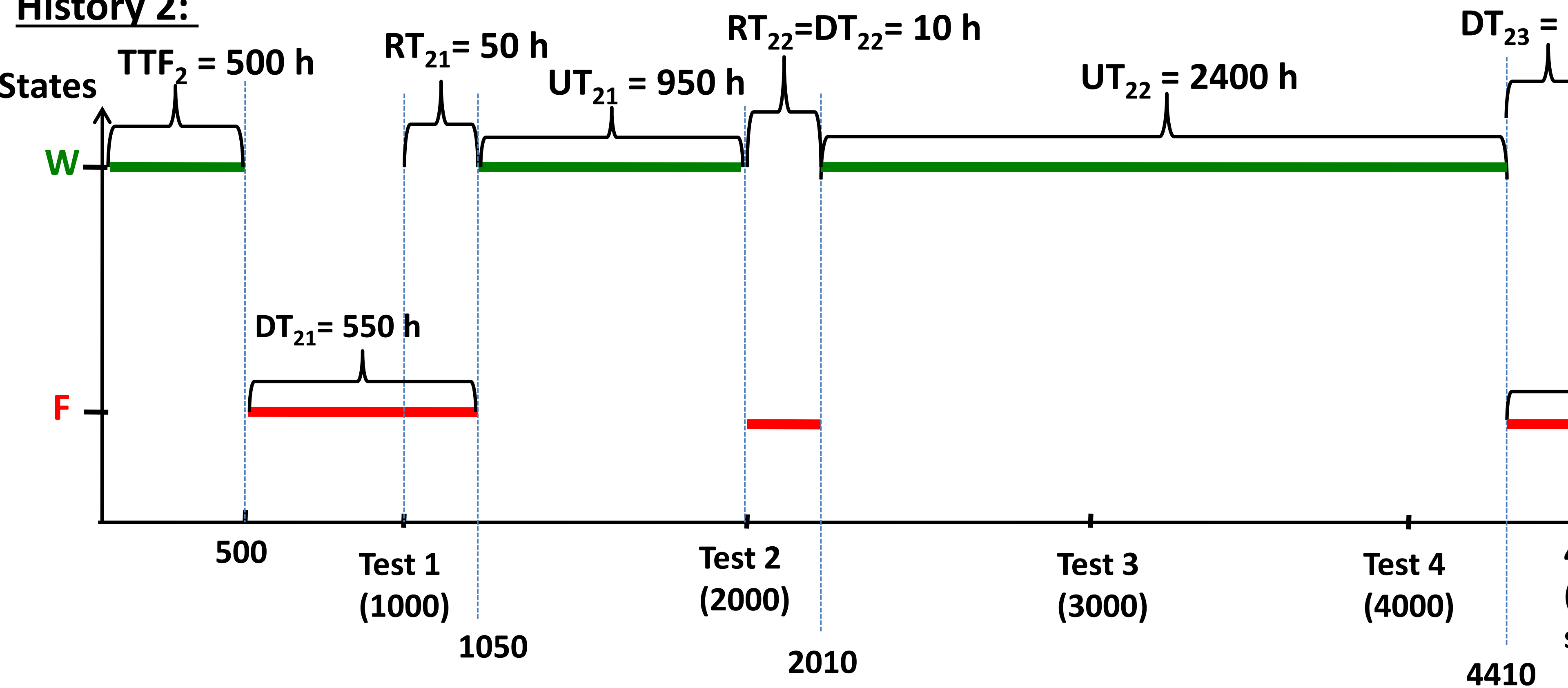


- TTF: Time To first Failure
- RT: Repair Time
- DT: Down Time
- UT: Up Time

- **TTF_1 = - Ln(0.0235177)/2.5E-3 = 1500 h**
- **RT_{11} = - Ln(0.13534)/0.1 = 20 h**
- **DT_{11} = time required for failure detection + repair time = 2000 -1500 + 20 = 520 h**
- **UT_{11} = - Ln(0.082085)/2.5E-3 = 1000 h**
- **RT_{12} = - Ln(0.00674)/0.1 = 50 h**
- **DT_{12} = time required for failure detection + repair time = 4000-3020 + 50 = 1030 h**
- **UT^*_{12} = - Ln(0.006738)/2.5E-3 = 2000 h.**
- **UT_{12} : we only keep from UT^*_{12} the part included in the history duration:**

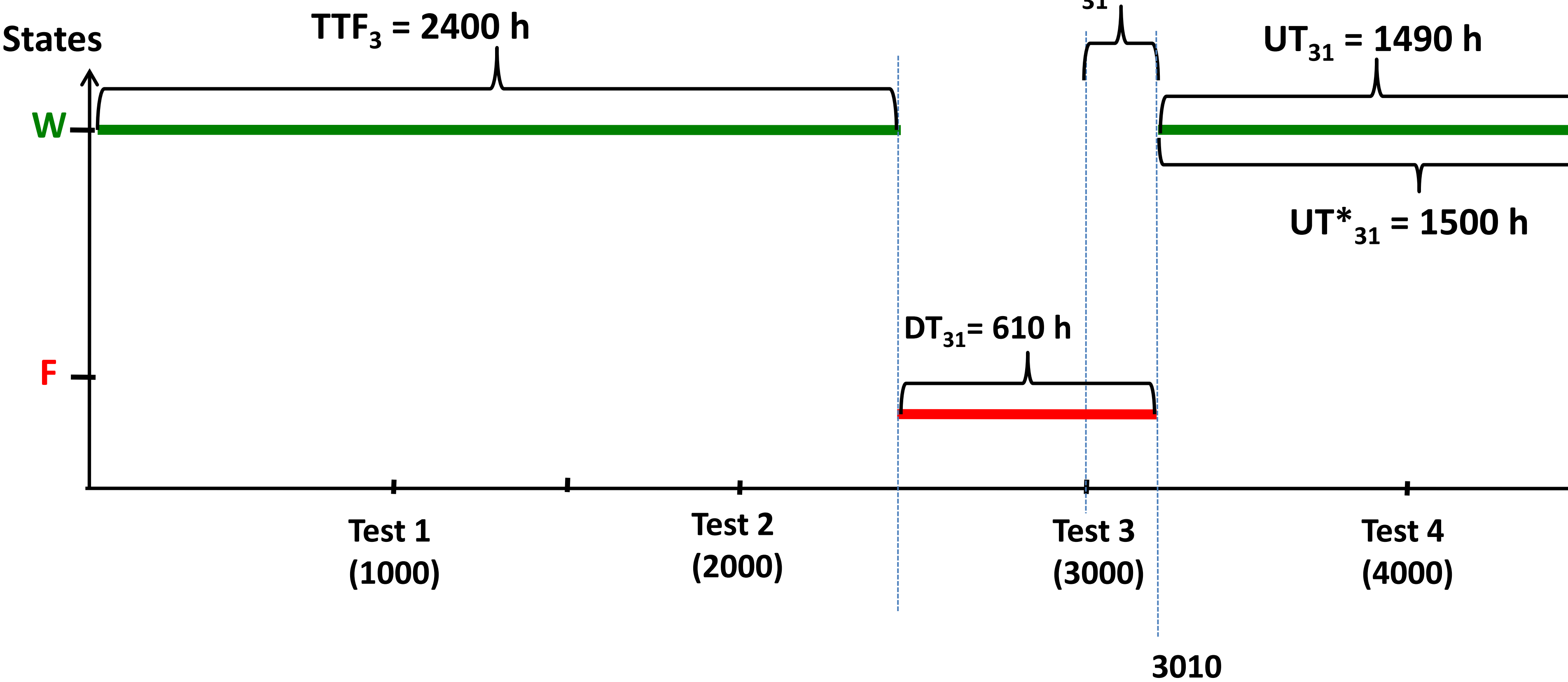
4500 – 4050 = 450 h

History 2:



- TTF_2 = - Ln(0.2865)/2.5E-3 = 500 h
- RT_{21} = - Ln(0.00674)/0.1 = 50 h
- DT_{21} = 1000 -500 + 50 = 550 h
- UT_{21} = - Ln(0.093014)/2.5E-3 = 950 h
- RT_{22} = - Ln(0.36788)/0.1 = 10 h
- DT_{22} = 0+ 10 = 10 h
- UT_{22} = - Ln(0.0024788)/2.5E-3 = 2400 h
- RT_{23} : - Ln(0.000045)/0.1 = 100 h
- DT^*_{23} = 590+ 100= 690 h
- DT_{23} : 4500 – 4410 = 90 h

History 3:



- TTF_3 = - Ln(0.0024788)/2.5E-3 = 2400 h
- RT_{31} = - Ln(0.36788)/0.1 = 10 h
- DT_{31} = 3000 -2400 + 10 = 610 h
- UT^*_{31} = - Ln(0.0235177)/2.5E-3 = 1500 h
- UT_{31} = 4500 – 3010 = 1490 h

A2. Mean availability : A_{avg} [0, 4500 h]

A_{avg} [0, 4500 h] $= [(TTF_1+UT_{11}+UT_{12})/4500 + (TTF_2+UT_{21}+UT_{22})/4500 + (TTF_3+UT_{31})/4500]/3$

$= [(1500 + 1000+ 450)/4500 + (500 + 950 + 2400)/4500+ (2400 + 1490)/4500]/3$

= 0.792

A3:

- **Availability at t = 3000 h** : A (3000) = (1 + 1 + 0) /3 = 2/3 = **0,667**
- **Reliability at t = 3000 h** : R (3000) = (0 + 0 + 0) /3 = **0**

A4: MTTF = $(TTF_1+TTF_2+TTF_3)/3$ = (1500 + 500 + 2400)/3= **1466,667 h.**

A5: confidence interval (CI) at 90 % for the MTTF :

CI = $$\left[MTTF - 1.64\cdot(s/\sqrt{3}),\ MTTF + 1.64\cdot(s/\sqrt{3})\right]$$

with :

$$\sigma = \sqrt{\sum_{i=1}^{3}(TTF_i - MTTF)^2/3} = 776.0298h$$

CI = $$\left[731.880h,\ 2201.454h\right]$$

This interval is very wide. This is due to reduced number of stories (3) (non-representative sample): it is therefore necessary to significantly increase the number of histories.

A6: MUT et MDT :

- **MUT =** $[(UT_{11}+UT_{12})/2 + (UT_{21}+UT_{22})/2 + UT_{31}]/3$

  $= [(1000+ 450)/2+ (950 + 2400)/2+ 1490]/3$

  $=$ **1296.66667 h.**

- **MDT =** $[(DT_{11}+DT_{12})/2 + (DT_{21}+DT_{22}+DT_{23})/3 + DT_{31}]/3$

  $= [(520+ 1030)/2+ (550 + 10+90)/2+ 610]/3$

  $=$ **533.889 h.**

7. MC simulation and uncertainty propagation

- Monte Carlo method is generally used to perform uncertainty study. This technique has become the industry standard for propagating uncertainties. It provides ***an efficient and straightforward*** way for this purpose.

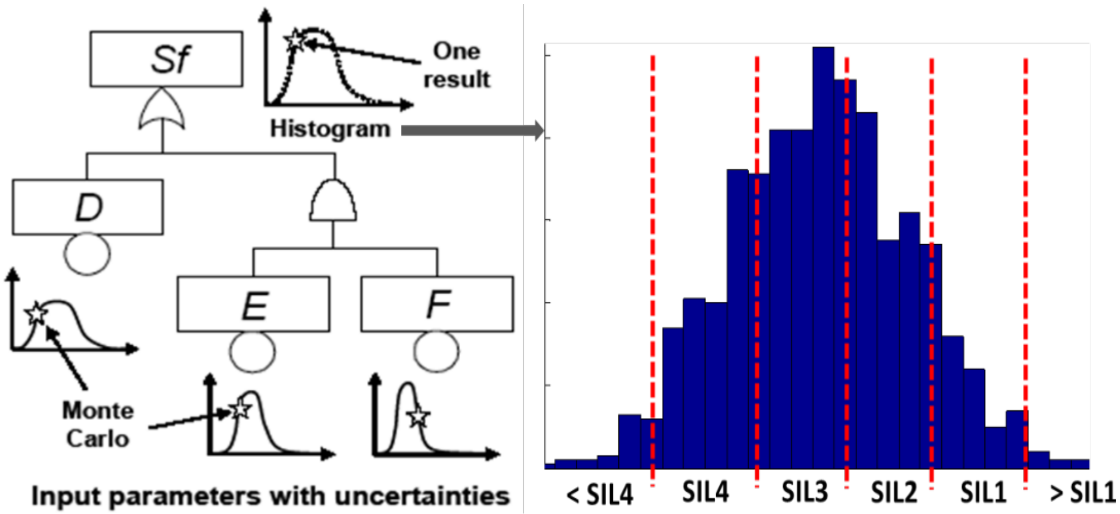
- ***Uncertainty propagation*** shows how the uncertainty of ***input parameters*** (failure rate, for instance) ***spreads onto the output*** of the model at hand.

- **Steps:**

1. Construct a probability density function (*pdf*) for each input parameter (*pdf* reflects state of knowledge about the value of the parameter).
2. Generate one set of input parameters by using random numbers according to pdfs assigned to those parameters.
3. Quantify the output function using the above set of random values. The obtained value is a realization of a random variable (X).
4. Repeat steps 2 to 3 n times (until a sufficient number, e.g. 1000) producing n independent output values. These n output values represent a random sample from the probability distribution of the output function.
5. Generate statistics from the obtained sample for the output result: mean, standard deviation σ, confidence interval, etc.

7. MC simulation and uncertainty propagation: IEC 61508 standard

- **Route 2_H:** the IEC 61508 standard (clause 7.4.4.3.3) stipulates that "*If route 2_H is selected, then the reliability data uncertainties shall be taken into account when calculating the target failure measure (i.e. PFD_{avg} or PFH) and the system shall be improved until there is a confidence greater than 90 % that the target failure measure is achieved*".



- The confidence on the obtained SIL according to the value of PFD_{avg} or *PFH* may be established by checking that the upper limit of the confidence interval is encompassed in the corresponding required SIL zone. Also, a direct measure is the evaluation of the cumulated density function (*cdf*) at the target performance measure (PFD_{max}, PFH_{max}): $\boldsymbol{Pr}\,(X \leq PFD_{max})$.

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7. MC simulation and uncertainty propagation: probability distributions

Probability distribution for the variable x	Probability density function (pdf) and its main properties	Comments and propositions to handle uncertainties
Uniform (a, b)	$f(x) = \begin{cases} \frac{1}{b-a} & if\ a \leq x \leq b \\ 0 & otherwise \end{cases}$ $Mean = Median = (a+b)/2$ $Mode = \text{any value in } [a, b]$ $Std = (b-a)/\sqrt{12}$	- This distribution expresses a big lack of knowledge about the parameter value (non-informative). The source of data provides an interval [a, b]. - Simulation: $x = a + rand \cdot (b-a)$, where rand is a number uniformly distributed between 0 and 1. MATLAB use directly the function unifinv (rand, a, b).
Triangular (a, b, c)	$f(x) = \begin{cases} \frac{2(x-a)}{(b-a)\cdot(c-a)} & if\ \ a \leq x \leq c \\ \frac{2(b-x)}{(b-a)\cdot(b-c)} & if\ \ c \leq x \leq b \\ 0 & otherwise \end{cases}$ $Mean = (a+b+c)/3$ $Mode = c$ $Std = \sqrt{[(b-a)^2 + (c-a)(c-b)]/18}$	- The distribution is more precise than the Uniform distribution. The source of data provides, beside a and b, an estimation of the most likely value of the parameter: c. Using the Mean value, one may compute c (if not available). - Simulation: MATLAB does not include this law. So, one can use the following: $M = \frac{(c-a)}{(b-a)}$ $R = rand$ $x = \begin{cases} a + \sqrt{(c-a)\cdot(b-a)\cdot R}, & if\ R \leq M \\ b - \sqrt{(b-c)\cdot(b-a)\cdot(1-R)}, & otherwise \end{cases}$

7. MC simulation and uncertainty propagation: probability distributions

Lognormal (μ, σ)	$$f(x)=\frac{1}{x\sigma\sqrt{2\pi}}\cdot\exp\left(\frac{-(\ln x-\mu)^2}{2\sigma^2}\right)$$ where: $x>0;\ -\infty<\mu<\infty;\ \sigma>0$ $$\text{Mean}=\exp\left(\mu+\frac{\sigma^2}{2}\right)$$ $$\text{Std}=\exp\left(\mu+\frac{\sigma^2}{2}\right)\cdot\sqrt{\exp(\sigma^2)-1}$$ $$\text{Median}=x_{50}=\exp(\mu)$$ $$x_{95}=\exp(\mu+1.645\,\sigma)$$ $$x_{05}=\exp(\mu-1.645\,\sigma)$$ $$\text{EF}=\frac{x_{50}}{x_{05}}=\frac{x_{95}}{x_{50}}=\exp(1.645\,\sigma)$$	- The lognormal distribution is used frequently in safety and reliability studies to model uncertainty. The distribution is skewed to the right, it allows therefore a pessimistic value of x. - The source of data provides the average value of x or its point estimate ($\hat{x}$) and an uncertainty range (EF: error factor): e.g., EF= 10. In that case, we may assume that $\hat{x}$ = x_{50}. By doing so, one can derivate μ and σ using equations that give the Mean and EF. - The source of data provides the average value of x or its point estimate ($\hat{x}$) and an estimation of the standard deviation σ (e.g. λ = 1E–4 $\pm$ 1E–5). In that situation, one can derivate μ and σ for the lognormal distribution using equations that give the Mean and Std. - The source of data provides the average value of x or its point estimate ($\hat{x}$), lower and upper bounds (x_L, x_U). One could assume that these bounds x_{05} and x_{95}, respectively. The problem been over-determined, the use of two among the three available (the most accurate) value gives μ and σ for the lognormal distribution. - Simulation: MATLAB use the function logninv (rand,μ,σ), by implementing a numeric approach.
Gamma (α, β)	$$f(x)=\frac{x^{\alpha-1}}{\beta^{\alpha}\Gamma(\alpha)}\cdot\exp\left(-\frac{x}{\beta}\right)$$ where: $0<x<\infty;\ \alpha>0;\ \beta>0$ $$Mean=\alpha\beta$$ $$Mode=\beta(\alpha-1)\ for\ \alpha\geq 1$$ $$Std=\beta\sqrt{\alpha}$$	- The Gamma distribution can be a posterior distribution obtained using a Bayesian approach. - All information provided above, especially for the lognormal distribution, may be used to compute α and β, once the Mean and Std are available or calculated. - Simulation: MATLAB use the function gaminv(rand,α,β), by implementing a numeric approach.

7. MC simulation and uncertainty propagation: probability distributions

Chi-square (k): χ_k^2	$$f(x) = \frac{x^{\frac{k}{2}-1}}{2^{\left(\frac{k}{2}\right)}\ \Gamma\left(\frac{k}{2}\right)} \cdot \exp\left(-\frac{x}{2}\right)$$ where: $0 < x < \infty$ $k \in N$ (degree of freedom) $$\text{Mean} = k$$ $$\text{Mode} = \max(k-2, 0)$$ $$\text{Std} = \sqrt{2k}$$	- The χ^2 distribution is a special case of the gamma distribution where $\beta = 2$ and $\alpha = k/2$. - For n observed failures over a cumulated observation time T, it is established that the failure rate (λ) follows the distribution $\chi^2_{2n}/2T$. In that condition: $$\text{Mean} = \frac{k}{2T} = \frac{\alpha}{T} = \frac{n}{T}\ \left(\text{the point estimate of } \lambda\text{: } \hat{\lambda}\right).$$ $\text{Std} = \frac{\sqrt{2k}}{2T} = \left(\frac{1}{T}\right)\sqrt{\frac{k}{2}} = \left(\frac{1}{T}\right)\sqrt{\alpha} = \left(\frac{1}{T}\right)\sqrt{n}$. Therefore, if the Mean and Std are provided, one can easily derivate parameters for the gamma distribution (α=n and β=1/T) and vice versa. Simulation: x = 1/(2T) . chi2inv (rand, 2n).
Beta (α, β)	$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} \cdot x^{\alpha-1}(1-x)^{\beta-1}$$ where: $0 \le x \le 1;\quad \alpha > 0;\quad \beta > 0$ $$Mean = \frac{\alpha}{\alpha+\beta}$$ $$Mode = \frac{\alpha-1}{\alpha+\beta-2} \quad for\ \alpha > 1;\ \beta > 1$$ $$Std = \sqrt{\frac{\alpha\beta}{(\alpha+\beta)^2\cdot(\alpha+\beta+1)}}$$	- The Beta distribution can be a posterior distribution obtained using a Bayesian approach (only for $0 \le x \le 1$). - If the bounds of x are more accurate or out of that range $[x_L, x_U] \neq [0, 1]$, the Mean and Std become: $$Mean = x_L + (x_U - x_L)\cdot\frac{\alpha}{\alpha+\beta}$$ $$Std = (x_U - x_L)\cdot\sqrt{\frac{\alpha\beta}{(\alpha+\beta)^2\cdot(\alpha+\beta+1)}}$$ - If the Mean and Std are provided, the solving of the above equations gives α and β for the Beta distribution. - If Std is not available, one may use the approximation: Std =$(x_U - x_L)$/6). - Simulation: x = x_L+ $(x_U - x_L)$·betainv (rand, α,β).

7. MC simulation and uncertainty propagation: Example

Paper: H. Jin, M. A. Lundteigen & M. Rausand

Uncertainty assessment of reliability estimates for safety instrumented systems

$$\text{PFD}_{2oo3,\text{PTs}} = (\lambda_{\text{DU}} \cdot \tau)^2 + C_{2oo3} \cdot \beta \cdot \lambda_{\text{DU}} \cdot \frac{\tau}{2}$$

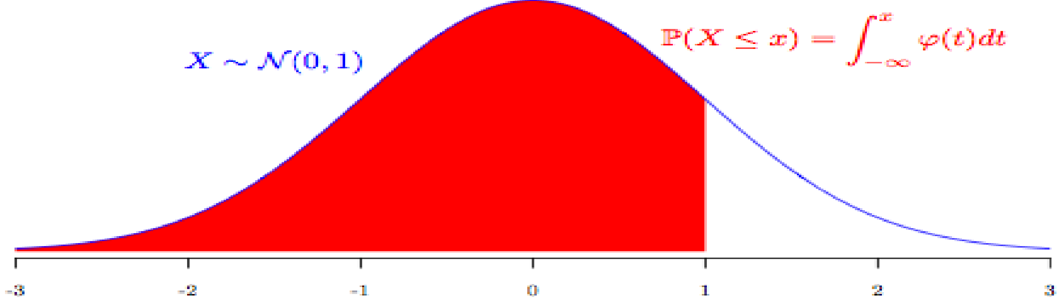
Log-normal :
Median = 3E-7
EF=2

Uniform :
a = 1.5
b=2

Log-normal :
Median = 0.1
EF=2

- Compute the PFD_{avg}
- Establish the confidence interval at 90 %
- Compute the confidence to comply with the different SILs (1, 2, 3, 4)

Excel solution



	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990