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TFE4120 Electromagnetics - Preliminary course

Solution proposal, exercise 3

Oppgave 1

a) Due to symmetry we have $\mathbf{E} = E(r)\hat{\mathbf{r}}$ for this entire task

i) We let the Gaussian surface be the surface of a cylinder with radius r and length l. The charge per unit length on the inner conductor is Q'. Using Gauss' law in its integral from gives

$$\begin{aligned}\oint \epsilon\mathbf{E}\cdot \mathrm{d}\mathbf{S} &= 2\pi r l\epsilon E(r)\\ &= Q_{\text{inside } S}\\ &= Q' l.\end{aligned} \tag{1}$$

Which results in

$$\mathbf{E}(r) = \frac{Q'}{2\pi\epsilon r}\hat{\mathbf{r}}. \tag{2}$$

We find Q' by using the definition of potential (the reference point is set on the outer conductor).

$$\begin{aligned}V(a) - V(b) &= V_0 - 0\\ &= \int_a^b E(r)\mathrm{d}r\\ &= \frac{Q'}{2\pi\epsilon}\int_a^b \frac{\mathrm{d}r}{r}\\ &= \frac{Q'}{2\pi\epsilon}\ln\frac{b}{a}.\end{aligned} \tag{3}$$

The electric field can then be written as

$$\underline{\underline{\mathbf{E} = \frac{V_0}{\ln\frac{b}{a}}\frac{1}{r}\hat{\mathbf{r}}.}} \tag{4}$$

Using the expression for $V(a) - V(b)$, and the definition of capacitance per unit length is $C' \equiv \frac{Q'}{V(a)-V(b)}$, we find

$$\underline{\underline{C' = \frac{2\pi\epsilon}{\ln\frac{b}{a}}.}} \tag{5}$$

ii) Since there are no free charges in the dielectric material we have that

$$\nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon \mathbf{E}) = 0, \text{ for } a \leq r \leq b. \tag{6}$$

The divergence in cylindrical coordinates is

$$\nabla \cdot \mathbf{D} = \frac{1}{r}\frac{\partial [rD(r)]}{\partial r} = 0, \tag{7}$$

where we have assumed $\mathbf{D} = D(r)\hat{\mathbf{r}}$. The solutions for $D(r)$ and $E(r)$ is then

$$D(r) = \frac{\epsilon C_1}{r},$$
$$E(r) = \frac{C_1}{r},$$

where C_1 is a constant that needs to satisfy the boundary conditions on the inner- and outer conductor. C_1 is determined by calculating the potential difference just like we did in the last subtask:

$$V(a) - V(b) = V_0 = \int_a^b \frac{C_1}{r} \mathrm{d}r = C_1 \ln \frac{b}{a}. \tag{8}$$

Thus, we have

$$C_1 = \frac{V_0}{\ln \frac{b}{a}} \tag{9}$$

and

$$\mathbf{E} = \frac{V_0}{\ln \frac{b}{a}} \frac{1}{r} \hat{\mathbf{r}}. \tag{10}$$

Since $C' = \frac{Q'}{V_0}$ we need an expression for Q'. Since all charges on the inner conductor will occur on the surface we can use the boundary condition $D_{\mathrm{n}}^{\text{dielectric}} - D_{\mathrm{n}}^{\text{inner conductor}} = \rho_{\mathrm{s}}$, where ρ_{s} is the surface charge density. Since there are no field inside the inner conductor $D_{\mathrm{n}}^{\text{inner conductor}} = 0$. From $D_{\mathrm{n}}^{\text{dielectric}} = D(a) = \epsilon E(a)$ we find

$$Q' = 2\pi a \rho_{\mathrm{s}} = \frac{2\pi\epsilon V_0}{\ln \frac{b}{a}}, \tag{11}$$

which gives

$$C' = \frac{Q'}{V_0} = \frac{2\pi\epsilon}{\ln \frac{b}{a}}. \tag{12}$$

iii) Since ϵ is constant and there are no free charges in the dielectric material we can use Laplace's equation:

$$\nabla^2 V = 0. \tag{13}$$

(Proof: We have $\nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon \mathbf{E}) = \rho$. During static conditions we have $\mathbf{E} = -\nabla V$ which yields $\nabla \cdot (-\epsilon \nabla V) = \rho$. Since there are no free charges in the material, and ϵ is constant, Gauss' law is reduced to $\nabla \cdot (-\epsilon \nabla V) = \nabla^2 V = 0$.)
Due to symmetry we have $\mathbf{E} = E(r)\hat{\mathbf{r}}$ which implies $V = V(r)$. By using the expression for ∇^2 in cylindrical coordinates we get

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial V(r)}{\partial r}\right) = 0. \tag{14}$$

Integrating once more

$$\frac{\partial V(r)}{\partial r} = \frac{C_2}{r}, \tag{15}$$

where C_2 is a constant. Integration then gives

$$V(r) = C_2 \ln r + C_3, \tag{16}$$

where C_3 is also a constant. Both C_2 and C_3 needs to satisfy the boundary conditions for V. Using the condition $V(b) = 0$ gives

$$V(b) = 0 = C_2 \ln b + C_3, \tag{17}$$

so that $C_3 = -C_2 \ln b$. The boundary condition on $r = a$ is $V(a) = V_0$ which gives

$$V(a) = V_0 = C_2 \ln a - C_2 \ln b = C_2 \ln \frac{a}{b}. \tag{18}$$

Thus, we have $C_2 = V_0 / \ln \frac{a}{b}$ and

$$V(r) = \frac{V_0}{\ln \frac{a}{b}} \ln \frac{r}{b}. \tag{19}$$

We find the electric field by using $\mathbf{E} = -\nabla V$ in cylindric coordinates, so that

$$\underline{\underline{\mathbf{E} = -\nabla V = -\frac{\partial V(r)}{\partial r}\hat{\mathbf{r}} = \frac{V_0}{\ln \frac{b}{a}} \frac{1}{r}\hat{\mathbf{r}}.}} \tag{20}$$

The capacitance can now be found using the same procedure as in the previous subtask.

b) When $\epsilon_\mathrm{r} = 3$ and $\frac{b}{a} = 7$ we get

$$\underline{\underline{C' = \frac{2\pi\epsilon_r\epsilon_0}{\ln \frac{b}{a}} = \frac{6\pi}{\ln 7}\epsilon_0 = 85.8\frac{\mathrm{pF}}{\mathrm{m}}.}} \tag{21}$$

c) i) The stored electrostatic energy per unit length, W_e', of the cable is given by

$$W_\mathrm{e}' = \frac{1}{2} C' V^2. \tag{22}$$

By inserting the expression for C' found in **a)**, and by using $V = V_0$ we get

$$\underline{\underline{W_\mathrm{e}' = \frac{\pi\epsilon}{\ln \frac{b}{a}} V_0^2.}} \tag{23}$$

ii) The energy density w_e in an electric field is given by

$$w_\mathrm{e} = \frac{1}{2}\epsilon E^2. \tag{24}$$

The total energy per unit length of the capacitor is found by integrating the expression for energy density over the cross section between the two conductors:

$$W_\mathrm{e}' = \int_a^b w_e(r) dA = \int_a^b \frac{1}{2}\epsilon E^2 2\pi r dr = \pi\epsilon \int_a^b r E^2 dr. \tag{25}$$

By using the expression for E found in **a)** we find

$$\underline{\underline{W_\mathrm{e}' = \pi\epsilon \frac{V_0^2}{\ln^2 \frac{b}{a}} \int_a^b \frac{r}{r^2} dr = \frac{\pi\epsilon}{\ln \frac{b}{a}} V_0^2.}} \tag{26}$$

d) Due to symmetry the total force acting on the outer conductor is equal to zero.

Oppgave 2

a) We start by determining the electric field inside the capacitor. We place a free charge Q on the inner conductor and another $-Q$ on the outer conductor and use Gauss' law $\oint_S \mathbf{D} \cdot \mathrm{d}\mathbf{S} = Q_{\text{inside } S}$. We chose a spherical cap with radius r as our Gaussian surface. Due to symmetry we have $\mathbf{D} = D(r)\hat{\mathbf{r}}$. Gauss' law gives then

$$\oint \mathbf{D} \cdot \mathrm{d}\mathbf{S} = 4\pi r^2 D(r) = Q. \tag{27}$$

Using $D(r) = \epsilon E(r)$ yields

$$E(r) = \frac{Q}{4\pi\epsilon r^2}. \tag{28}$$

We choose the outer conductor as the point of reference for the potential so that

$$V(a) - V(b) = V_0 = \int_a^b E(r)\mathrm{d}r = \frac{Q}{4\pi\epsilon}\int_a^b \frac{\mathrm{d}r}{r^2} = \frac{Q}{4\pi\epsilon}\left(\frac{1}{a} - \frac{1}{b}\right). \tag{29}$$

The capacitance is given by $C = \frac{Q}{V_0}$:

$$\underline{\underline{C = \frac{4\pi\epsilon ab}{b-a}.}} \tag{30}$$

We can now see that when $b - a = d \ll a$ we can write

$$\underline{\underline{C \simeq \epsilon\frac{A}{d},}} \tag{31}$$

where $A = 4\pi ab$ is the surface area of the capacitor and d is the distance between the plates. This expression is similar for parallel plate capacitors.

b) If we consider a simple conducting sphere from **a)** with the boundary condition $b \to \infty$, and let $\epsilon \to \epsilon_0$, we have

$$\underline{\underline{C = 4\pi\epsilon_0 a,}} \tag{32}$$

for the capacitance for a sphere with radius a.