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TFE4120 Electromagnetics - Crash course

Lecture 6: Maxwell's equations, antenna equations

Thursday August 18th, 9-12 am.

Derivation of the wave equation for E

Maxwell's equations for a linear, homogenenous, isotropic medium with no charges are:

$$\nabla \cdot \mathbf{E} = 0, \tag{1}$$
$$\nabla \cdot \mathbf{B} = 0, \tag{2}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{3}$$
$$\nabla \times \mathbf{B} = \epsilon\mu \frac{\partial \mathbf{E}}{\partial t}. \tag{4}$$

If we take the curl of the third equation, and insert the forth we get

$$\nabla \times \nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = -\frac{\partial}{\partial t}(\epsilon\mu \frac{\partial \mathbf{E}}{\partial t}) = -\epsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2}. \tag{5}$$

From the vector identity

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}, \tag{6}$$

and using the first Maxwell equation we get

$$\nabla^2 \mathbf{E} - \epsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0, \tag{7}$$

where $c = \frac{1}{\sqrt{\epsilon\mu}}$ is the wave velocity. This is the wave equation in three dimensions for $\mathbf{E}$. A solution to the equation is the plane wave:

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}. \tag{8}$$

Inserting this into (7) gives that the following relation between $\mathbf{k}$ and ω must be satisfied:

$$|\mathbf{k}|^2 = \epsilon\mu\omega^2. \tag{9}$$

This is called the "dispersion relation", which relates the wave vector and the frequency of a propagating electromagnetic wave.

Displacement current

In magnetostatics we had

$$\nabla \times \mathbf{H} = \mathbf{J}, \tag{10}$$

where $\mathbf{J}$ is the free current density, which can be measured by an Ampere-meter.

We have already used the "Ampere-Maxwell" equation

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}, \tag{11}$$

but we have not justified why the $\frac{\partial \mathbf{D}}{\partial t}$-term should be added in electrodynamics. To do this, we need the law of charge conservation:

$$\oint_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho \mathrm{d}V, \tag{12}$$

or in differential form

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}. \tag{13}$$

If we insert (10) into (13) we get

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = -\frac{\partial \rho}{\partial t}. \tag{14}$$

This means $\frac{\partial \rho}{\partial t} = 0$, which certainly cannot be true in general in electrodynamics. We must thus modify (10). From Gauss' law ($\nabla \cdot \mathbf{D} = \rho$) we get

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} - \frac{\partial \rho}{\partial t} = \nabla \cdot \mathbf{J} + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}), \tag{15}$$

which gives

$$\nabla \cdot (\nabla \times \mathbf{H}) = \nabla \cdot (\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}). \tag{16}$$

One possible solution to this equation is (11).

The term $\frac{\partial \mathbf{D}}{\partial t}$ is called the "displacement current". Equation (11) has been experimentally verified, all wave phenomena rely on this term, and it adds additional symmetry to Maxwell's equations (varying the magnetic field gives an electric field and varying the electric field gives a magnetic field).

Example:

Parallel plate capacitor.

We want to find $\mathbf{H}$ outside the capacitor. If we use (10) we get for a circular path C centered between the plates:

$$\oint_C \mathbf{H} \cdot \mathrm{d}\mathbf{l} = H 2\pi r = \int_{S_1} \mathbf{J} \cdot \mathrm{d}\mathbf{S} = 0, \tag{17}$$

where S_1 is the circular disk with C as its border, while

$$\oint_C \mathbf{H} \cdot \mathrm{d}\mathbf{l} = H 2\pi r = \int_{S_2} \mathbf{J} \cdot \mathrm{d}\mathbf{S} = I, \tag{18}$$

where S_2 is a circular semi-sphere with C as its border. Since both S_1 and S_2 has C as their border, they should give the same result for H. Let us instead use (11).

Metal plates are (approximately) ideal conductors, so $D = \rho_S = Q/S$ just outside the plates. Between the plates there are no charges, so $\nabla \cdot \mathbf{D} = \frac{\mathrm{d}D}{\mathrm{d}x} = 0$. This means $D = \rho_S$ throughout the volume between the plates. We thus get

$$\frac{\mathrm{d}D}{\mathrm{d}t} = \frac{1}{S}\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{I}{S}. \tag{19}$$

Outside the capacitor $D = 0$. We now have

$$\oint_C \mathbf{H} \cdot \mathrm{d}\mathbf{l} = \int_{S_1} (\mathbf{J} + \frac{\mathrm{d}\mathbf{D}}{\mathrm{d}t}) \cdot \mathrm{d}\mathbf{S} = \int_{S_2} (\mathbf{J} + \frac{\mathrm{d}\mathbf{D}}{\mathrm{d}t}) \cdot \mathrm{d}\mathbf{S} = I. \tag{20}$$

Maxwell's equations (review)

Differential form:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{21}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}, \tag{22}$$

$$\nabla \cdot \mathbf{D} = \rho, \tag{23}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{24}$$

and in integral form:

$$\oint_C \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{S}, \tag{25}$$

$$\oint_C \mathbf{H} \cdot \mathrm{d}\mathbf{l} = \int_S (\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}) \cdot \mathrm{d}\mathbf{S}, \tag{26}$$

$$\oint_S \mathbf{D} \cdot \mathrm{d}\mathbf{S} = \int_V \rho \mathrm{d}V, \tag{27}$$

$$\oint_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0. \tag{28}$$

Remember that in general:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \tag{29}$$

$$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}), \tag{30}$$

where $\mathbf{P}$ is a function of $\mathbf{E}$ and $\mathbf{M}$ is a function of $\mathbf{B}$. For linear media

$$\mathbf{D} = \epsilon \mathbf{E}, \tag{31}$$

$$\mathbf{B} = \mu \mathbf{H}. \tag{32}$$

Maxwell's equations are confirmed by all experiments so far. The boundary conditions for $\mathbf{E}, \mathbf{D}, \mathbf{B}$ and $\mathbf{H}$ from electrostatics are also valid in electrodynamics.

Antenna equations

We want to find the potential $V(\mathbf{r}, t)$ from a general charge distribution $\rho(\mathbf{r}, t)$. First we introduce some tools: The vector potential and scalar potential.

In electrostatics we used that since $\nabla \times \mathbf{E} = 0$ we can introduce a scalar potential V such that

$$\mathbf{E} = -\nabla V. \tag{33}$$

Since $\nabla \cdot \mathbf{B} = 0$ we can introduce a "vector potential" $\mathbf{A}$ (since $\mathbf{B}$ is divergence-free, it may be written as the curl of another vector field) such that

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{34}$$

In electrodynamics, equation (33) must be modified:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) = \nabla \times (-\frac{\partial \mathbf{A}}{\partial t}), \tag{35}$$

which gives

$$\nabla \times (\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}) = 0. \tag{36}$$

This means we may define V such that

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V. \tag{37}$$

We may thus express $\mathbf{E}$ and $\mathbf{B}$ in terms of the scalar potential V and vector potential $\mathbf{A}$:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \tag{38}$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{39}$$

Maxwell's two curl equations are automatically satisfied from these equations.

Gauge transformations (V and $\mathbf{A}$ are not unique!)

Given an arbitrary function $f(x, y, z; t)$, we define

$$\mathbf{A}' = \mathbf{A} + \nabla f, \tag{40}$$

$$V' = V - \frac{\partial f}{\partial t}. \tag{41}$$

Using these modified potentials we get

$$\mathbf{E}' = -\nabla V' - \frac{\partial \mathbf{A}'}{\partial t} = -\nabla V - \frac{\partial}{\partial t}(\nabla f) - \frac{\partial \mathbf{A}}{\partial t} + \frac{\partial}{\partial t}(\nabla f) = \mathbf{E}, \tag{42}$$

$$\mathbf{B}' = \nabla \times \mathbf{A}' = \nabla \times \mathbf{A} + \nabla \times (\nabla f) = \mathbf{B}. \tag{43}$$

The "gauge transformations" (equations (40),(41)) thus leave the electric and magnetic fields unchanged. Since $\mathbf{E}$ and $\mathbf{B}$ are physical, they must be unique! Since $\mathbf{A}$ and V are not physical (only mathematical tools), certain transformations leave the physical quantities $\mathbf{E}$ and $\mathbf{B}$ unchanged!

Lorentz gauge

Given $\mathbf{A}'$ and V' we choose $f(x, y, z; t)$ such that equations (40),(41) satisfy

$$\nabla \cdot \mathbf{A}' + \epsilon\mu \frac{\partial V'}{\partial t} = 0. \tag{44}$$

This is called the "Lorentz gauge". The chosen f must then satisfy

$$\nabla^2 f - \epsilon\mu \frac{\partial^2 f}{\partial t^2} = -(\nabla \cdot \mathbf{A} + \epsilon\mu \frac{\partial V}{\partial t}), \tag{45}$$

which is a wave equation for f with a "source term" on the right hand side given by the original $\mathbf{A}$ and V. For any $V, \mathbf{A}$ it is possible to find a $f(x, y, z; t)$ such that (44) is satisfied.

Finding the antenna equations

We want to find V and $\mathbf{A}$ in a linear, isotropic, homogeneous medium (antenna conditions). Maxwell's equations in such a medium are:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{46}$$

$$\nabla \times \mathbf{B} = \mu\mathbf{J} + \epsilon\mu \frac{\partial \mathbf{E}}{\partial t}, \tag{47}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon}, \tag{48}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{49}$$

Inserting (38) into (48) gives

$$\nabla \cdot (-\nabla V - \frac{\partial \mathbf{A}}{\partial t}) = \frac{\rho}{\epsilon}, \tag{50}$$

which gives

$$-\nabla^2 V - \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = \frac{\rho}{\epsilon}. \tag{51}$$

With $\mathbf{A}$ and V satisfying the Lorentz gauge this gives

$$\nabla^2 V - \epsilon\mu \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon}. \tag{52}$$

Similarly, inserting (39) into (47), and using the vector identity
$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ gives

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu\mathbf{J} + \epsilon\mu \frac{\partial}{\partial t}(-\nabla V - \frac{\partial \mathbf{A}}{\partial t}). \tag{53}$$

With $\mathbf{A}$ and V satisfying the Lorentz gauge we get

$$\nabla^2 \mathbf{A} - \epsilon\mu \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu\mathbf{J}. \tag{54}$$

If we solve (52) and (54) we can obtain $\mathbf{E}$ and $\mathbf{B}$ from (38) and (39).

Solve equation (52)**:** The equation is linear (no V^2, V^3 etc.), so we can solve for one volume element at the time, and then add up all volume elements to find the total potential. We thus first consider a charge $\rho(t)\mathrm{d}V$ placed at the origin, so that the right hand side of (52) is zero for all $\mathbf{r} \neq 0$. The problem has spherical symmetry, so we use a spherical coordinate system, where

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r}) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial V}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r}), \tag{55}$$

where we in the last equation used that $V(\mathbf{r}) = V(r)$, i.e. the potential is independent on θ, ϕ due to the spherical symmetry of the problem. For $\mathbf{r} \neq 0$ equation (52) thus looks like

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r}) - \epsilon\mu\frac{\partial^2 V}{\partial t^2} = 0. \tag{56}$$

Introducing a new variable U such that

$$V = \frac{U}{r} \tag{57}$$

gives

$$\frac{\partial V}{\partial r} = -\frac{U}{r^2} + \frac{1}{r}\frac{\partial U}{\partial r}. \tag{58}$$

Inserting this into (56) gives

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r\frac{\partial U}{\partial r} - U) - \frac{\epsilon\mu}{r}\frac{\partial^2 U}{\partial t^2} = \frac{1}{r}\frac{\partial^2 U}{\partial r^2} - \frac{\epsilon\mu}{r}\frac{\partial^2 U}{\partial t^2} = 0, \tag{59}$$

which gives

$$\frac{\partial^2 U}{\partial r^2} - \epsilon\mu\frac{\partial^2 U}{\partial t^2} = 0. \tag{60}$$

This is a one-dimensional wave equation for U, with general solution

$$U(r,t) = f(t - \frac{r}{c}) + g(t + \frac{r}{c}), \tag{61}$$

where $c = \frac{1}{\sqrt{\epsilon\mu}}$. Equation (57) gives

$$V(r,t) = \frac{1}{r}f(t - \frac{r}{c}) + \frac{1}{r}g(t + \frac{r}{c}). \tag{62}$$

This is a sum of a outward propagating wave $\frac{1}{r}f(t - \frac{r}{c})$ and a inward propagating wave $\frac{1}{r}g(t + \frac{r}{c})$. The inward propagating wave is unphysical, since our source is located at the origin. Neglecting this term we get

$$V(r,t) = \frac{1}{r}f(t - \frac{r}{c}). \tag{63}$$

We now find $f(t - \frac{r}{c})$ from the static solution:

$$V(\mathbf{r}) = \frac{\rho_0 \mathrm{d}V}{4\pi\epsilon r}. \tag{64}$$

Since ρ_0 is time-independent we must have

$$V(r,t) = \frac{1}{r}f(t) = \frac{\rho_0 \mathrm{d}V}{4\pi\epsilon r}. \tag{65}$$

The corresponding solution for a time dependent charge density $\rho(t)$ should thus be

$$V(r,t) = \frac{\rho(t - \frac{r}{c})\mathrm{d}V}{4\pi\epsilon r}. \tag{66}$$

This equation states that the potential value at a given time depends on what the charge was at the origin at the time $\frac{r}{c}$ ago. The equation thus describes "propagation of a signal".

For example, if the charge density is constant ($\rho(t) = \rho_0$ at all times $t > 0$) we get that the potential at a distance $r < c \cdot t$ away from the origin is given by

$$V = \frac{\rho_0 \mathrm{d}V}{4\pi\epsilon r}. \tag{67}$$

Solution for general $\rho(\mathbf{r}, t)$. We now find the solution by adding up the contributions from all volume elements in a volume V:

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon}\int_V \frac{\rho(\mathbf{r}', t - \frac{R}{c})}{R}\mathrm{d}V', \tag{68}$$

where $\mathbf{r}$ is the position of the observation point, $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ is the distance between a source at $\mathbf{r}'$ and the observation point.

The solution to (54) may similarly be found to be

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}', t - \frac{R}{c})\mathrm{d}V'}{R}. \tag{69}$$

We can now use (38) and (39) to find $\mathbf{E}$ and $\mathbf{B}$ from (68) and (69) for any source $\rho(\mathbf{r}, t)$ and $\mathbf{J}(\mathbf{r}, t)$.