TFE4120 Electromagnetics - Crash course

Lecture 5: Ampere's law in media, induction, inductance, Lenz' law

Friday August 12th, 9-12 am.

Magnetic fields in media.

In the same way we defined a free and bound charge density for the electric field, we will now defined free and bound currents.

Magnetic dipole moment

The magnetic dipole moment from a current loop I is

$$\mathbf{m} = I\mathbf{S}_{\mathrm{m}}, \tag{1}$$

where S_{m} is the surface enclosed by the current loop, and $\mathbf{m}$ is pointing along the surface normal of the loop. Magnetic dipoles in a medium will orient along the external magnetic field $\mathbf{B}$-field due to the torque:

$$\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}. \tag{2}$$

Derivation of Ampere's law in media

$$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 \int_S (\mathbf{J}_{\mathrm{free}} + \mathbf{J}_{\mathrm{bound}}) \cdot \mathrm{d}\mathbf{S}. \tag{3}$$

The result will be

$$\oint_C \mathbf{H} \cdot \mathrm{d}\mathbf{l} = \int_S \mathbf{J}_{\mathrm{free}} \cdot \mathrm{d}\mathbf{S}. \tag{4}$$

We want to hide the bound currents in a magnetization vector $\mathbf{M}$, and find the field produced solely by the "free" currents. We thus need to find I_{bound}.

We are interested in the current through S. We define

$$\mathbf{M} = \frac{\sum_i \mathbf{m}_i}{\mathrm{d}V} \tag{5}$$

as an average density of magnetic dipole moment per volume inside the volume element $\mathrm{d}V$. Only the current loops around the boundary C contribute to a net current through the surface. We first consider a line segment $\mathrm{d}\mathbf{l}$ along C.

The magnetization density $\mathbf{M}$ has a related surface element $\mathrm{d}S_{\mathrm{m}}$ normal to it (according to (1)). The "height" of the small cylinder along $\mathbf{M}$ is $\mathrm{d}l\cos\theta$, which gives $\mathrm{d}V = S_{\mathrm{m}}\mathrm{d}l\cos\theta$, so the contribution to I_{bound} from this cylinder is

$$\mathrm{d}I = \frac{|\mathbf{M}|S_{\mathrm{m}}\mathrm{d}l\cos\theta}{S_{\mathrm{m}}} = \mathbf{M}\cdot\mathrm{d}\mathbf{l}. \tag{6}$$

Integration along C gives

$$I_{\mathrm{bound}} = \oint_C \mathbf{M}\cdot\mathrm{d}\mathbf{l}. \tag{7}$$

Inserting this into (3) gives

$$\oint_C (\frac{1}{\mu_0}\mathbf{B} - \mathbf{M})\cdot\mathrm{d}\mathbf{l} = \int_S \mathbf{J}_{\mathrm{free}}\cdot\mathrm{d}\mathbf{S}. \tag{8}$$

Defining

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \tag{9}$$

gives equation (4).

Relationship between H and B

From the definition of $\mathbf{H}$ we have

$$\mathbf{B} = \mu_0\mathbf{H} + \mu_0\mathbf{M}. \tag{10}$$

For linear media, where $\mathbf{M} = \chi_{\mathrm{m}}\mathbf{H}$ we get

$$\mathbf{B} = \mu_0(1 + \chi_{\mathrm{m}})\mathbf{H} = \mu_0\mu_{\mathrm{r}}\mathbf{H} = \mu\mathbf{H}, \tag{11}$$

where μ_{r} is called the relative permeability. For air this parameter is ≈ 1 (non-magnetic), while for iron (magnetic) it 5000 or even higher for some alloys.

Example: Solenoid

Tightly wound conductor over a cylindrical iron core. Want to find $\mathbf{B}$ from Ampere's law for media (equation (4)). We first argue that $\mathbf{B}(r, \phi, z) = B\hat{\mathbf{z}}$, i.e. $\mathbf{B}$ is a constant everywhere (the cosntant is different intside and outside the solenoid).

$\mathbf{B}$ is independent of ϕ due to the cylinder symmetry, and independent on z as we assume the solenoid is very long. The $\hat{\mathbf{r}}$-component of $\mathbf{B}$ is thus zero, since otherwise there would be a flux out through a closed cylinder surface around the solenoid.

The $\hat{\boldsymbol{\phi}}$-component of $\mathbf{B}$ is zero, since (due to no current in the $\hat{\mathbf{z}}$-direction):
$\oint_C \mathbf{B}\cdot\mathrm{d}\mathbf{l} = \oint_C B_\phi \mathrm{d}l = 0$ along a circular curve C. Since $\mathbf{B}$ is independent on ϕ we get $B_\phi = 0$.

We thus have $\mathbf{B} = B(r)\hat{\mathbf{z}}$. From Ampere's law (with no current in the $\hat{\mathbf{z}}$-direction), integrating along a rectangular curve C inside the solenoid, gives $B(a) = B(b)$ for two different radi a and b. The same is true outside the solenoid. $\mathbf{B}$ is thus a constant both inside and outside. The constant outside must be zero, since otherwise there would be an infinite magnetic flux, which is unphysical.

The constant magnitude of $\mathbf{B}$ inside the solenoid is found from Ampere's law, using a rectangular integration path with a cross section starting at the inside and ending at the outside. From $\mathbf{B} = \mu\mathbf{H}$ we have $\mathbf{H} = H\hat{\mathbf{z}}$ inside and $\mathbf{H} = 0$ outside. Equatoin (4) gives

$$\oint_C \mathbf{H}\cdot\mathrm{d}\mathbf{l} = H\cdot l, \tag{12}$$

where l is the length of the solenoid. The right hand side of (4) is

$$\int_S \mathbf{J} \cdot d\mathbf{S} = N \cdot I, \tag{13}$$

which gives

$$H = \frac{NI}{l}. \tag{14}$$

We thus get

$$\mathbf{B} = \mu\mathbf{H} = \frac{\mu NI}{l}\hat{\mathbf{z}} \tag{15}$$

inside the solenoid, and $\mathbf{B} = 0$ outisde. Note that $\mathbf{B}$ is significantly stronger if the core is iron rather than air.

Exercise 5: Problem 1. 15 min + 5 min solution.

Electrodymanics: Faraday's law of induction

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{16}$$

In general we have $\nabla \times \mathbf{E} \neq 0$, since $\mathbf{B}$ may vary in time. From Stoke's theorem we have

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = \int_S (-\frac{\partial \mathbf{B}}{\partial t}) \cdot d\mathbf{S} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}, \tag{17}$$

where in the last equation we assumed that the loop C does is constant in time (not deformed or moved). In (17), $\mathbf{E}$ is the electric force per charge $\frac{\mathbf{F}}{q}$. In general other forces may act on q as well. We define the "electromotoric force" (emf):

$$e = \oint_C \frac{\mathbf{F}}{q} \cdot d\mathbf{l}. \tag{18}$$

Assume we move a closed circuit through a constant magnetic field $\mathbf{B}$. Then

$$e = \oint_C (\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l} = \oint_C (\frac{d\mathbf{r}}{dt}\times\mathbf{B})\cdot d\mathbf{l} = \frac{1}{dt}\oint_C (d\mathbf{r}\times\mathbf{B})\cdot d\mathbf{l} = \frac{1}{dt}\oint_C \mathbf{B}\cdot(d\mathbf{l}\times d\mathbf{r}) = -\frac{1}{dt}\oint_C \mathbf{B}\cdot(d\mathbf{r}\times d\mathbf{l}). \tag{19}$$

The surface element $d\mathbf{r} \times d\mathbf{l}$ is the change of the surface S enclosed by C during the time dt. The integral of $\mathbf{B} \cdot (d\mathbf{r} \times d\mathbf{l})$ is thus the change in the flux through S during the time dt, where S is the surface enclosed by C. We thus get

$$e = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}, \tag{20}$$

which says that the emf equals the time derivative of the magnetic flux through S. It does not matter if the change in flux is due to a time-varying $\mathbf{B}(t)$ or a time varying surface $S = S(t)$. The equation

$$e = -\frac{d\Phi}{dt}, \tag{21}$$

where Φ is the magnetic flux through S is called Faraday's law. The law may also be considered as an experimental fact.

In differential form:
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{22}$$

If there are several "emf's" in a circuit, the following law is valid:
$$\sum \text{emf} = RI, \tag{23}$$

where R is the resistance and I the current in the curciut. For instance, if there are two "emf's": a voltage source V_b and an induced emf due to a time varying $\mathbf{B}$ through the circiut we get
$$V_\text{B} + (-\frac{\text{d}\Phi}{\text{d}t}) = RI. \tag{24}$$

Example

Consider an uniform $\mathbf{B}_\text{ext} = -B_0 \sin(\omega t)\hat{\mathbf{z}}$ through a rectangular circuit with edges a and b. What is I?
$$e = -\frac{\text{d}}{\text{d}t} \int_S \mathbf{B} \cdot \text{d}\mathbf{S} = -\frac{\text{d}}{\text{d}t}(-B_0 \sin(\omega t))ab. \tag{25}$$

Thus
$$e = \omega B_0 ab \cos \omega t. \tag{26}$$

Equation (23) together with Ohm's law gives
$$I = \frac{e}{R} = \frac{\omega B_0 ab \cos(\omega t)}{R}. \tag{27}$$

Inductance

We have so far ignored the following: The induced current I produces it's own $\mathbf{B}$-field! The total $\mathbf{B}$-field is thus given as the sum of the external and the induced field:
$$\mathbf{B}_\text{tot} = \mathbf{B}_\text{ext} + \mathbf{B}_\text{ind}. \tag{28}$$

Consider the induced field separately: $\int_S \mathbf{B}_\text{ind} \cdot \text{d}\mathbf{S}$ is proportional to I. We thus define the inductance
$$L = \frac{\int_S \mathbf{B}_\text{ind} \cdot \text{d}\mathbf{S}}{I} = \frac{\Phi}{I}. \tag{29}$$

The inductance only depends on geometric quantities and the material parameter μ. The "self inductance" of a circiut is the magnetic flux produced solely by the circiut, divided by the current in the curcuit. To solve the problem in the last example correctly the self inductance should be taken into account:
$$e = RI = -\frac{\text{d}}{\text{d}t} \int_S \mathbf{B}_\text{ext} \cdot \text{d}\mathbf{S} - \frac{\text{d}}{\text{d}t} \int_S \mathbf{B}_\text{ind} \cdot \text{d}\mathbf{S} = \omega B_0 ab \cos(\omega t) - \frac{\text{d}}{\text{d}t} LI. \tag{30}$$

This gives the differential equation
$$RI(t) + L\frac{\text{d}I}{\text{d}t} = \omega B_0 ab \cos(\omega t). \tag{31}$$

Solution:
$$I(t) = I_0 e^{-\frac{R}{L}t} + \frac{\omega B_0 ab}{L^2 \omega^2 + R^2}[L\omega \sin(\omega t) + R\cos(\omega t)]. \tag{32}$$

If $L \to 0$ we get back our previous solution (27). If L is significantly small we may therefore for simplicity ignore the self-inductance.

We may also define the "mutual inductance" between two nearby curcuits:

$$L_{ij} = \frac{\Phi_{ij}}{I_j}. \tag{33}$$

Here L_{ij} is the mutual inductance between circuit i and j, Φ_{ij} is the flux in circuit i due to the current in circiut j, and I_j is the current in circuit j.

Lenz' law:

If you change the flux through a conducting loop, there is induced a current in the loop which will try to resist the imposed flux-change.

If the flux is increasing, the induced current will produce a **B**-field in the opposite direction of the increasing external field. The resistance R of the loop decides to which degree the induced current resists the flux change: $R \to \infty$ gives no "counter-flux", while $R \to 0$ (super-conductor) gives NO net flux through the loop!

Exercise 5: Problem 2 and 3. 25 min + 10 min solution.