



TFE4120 Electromagnetics - Crash course

Lecture 3: Electric fields in media, ideal conductors

Wednesday August 10th, 9-12 am.

So far we have described \mathbf{E} and V in vacuum. This description is also valid in any medium, as long as we take all charges into account. For example $NaCl$ in water may be considered as point charges (Na^+ and Cl^- ions) and dipoles (H_2O) randomly ordered. This, however, is impractical. It is simpler to consider the medium as a "background" where we no longer have to consider all the charges separately, only the charges additional to the background medium.

To get there, we divide charges into two categories:

- a) "Bound charges": charges in the medium bound together as dipoles.
- b) "Free charges": all other charges.

Our goal is to modify Gauss' law to only have free charges on the right hand side. In the version of Gauss' law we have considered up until now, all charges are included on the right hand side:

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q_{\text{in } S}}{\epsilon_0} = \frac{Q_{\text{free in } S} + Q_{\text{bound in } S}}{\epsilon_0}. \quad (1)$$

The modified version we will end up with looks like:

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_{\text{free in } S}, \quad (2)$$

where $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$. The bound charge has thus been included in the polarization vector \mathbf{P} .

Dipole moment is defined as

$$\mathbf{p} = Q\mathbf{d}, \quad (3)$$

where the dipole consists of two charges $+Q$ and $-Q$ separated by a position vector \mathbf{d} .

Derivation of Gauss' law in media

In a medium there will be many dipoles, in general with different \mathbf{p} . We want to find $Q_{\text{bound in } S}$ inside S . We first define

$$\mathbf{P} = \frac{\sum_i \mathbf{p}_i}{dV}, \quad (4)$$

where \mathbf{P} is called the *polarization density*, and \mathbf{p}_i are all the dipoles inside a volume element dV . The net dipole moment inside dV is then

$$\mathbf{P}_{\text{net}} = \sum_i \mathbf{p}_i = \mathbf{P}dV. \quad (5)$$

From (3) the net positive charge inside dV may be expressed through the net polarization inside the volume element:

$$Q_{\text{net}} = \frac{|\mathbf{P}|dV}{d}. \quad (6)$$

To find $Q_{\text{bound in } S}$ we must count the dipole-ends belonging to dipoles along the surface. Since the net charge of a dipole is 0, only the dipoles with one end inside and the other outside the surface will contribute to $Q_{\text{bound in } S}$.

Consider a surface element at the boundary:

$$dV = d \cos \theta dS, \quad (7)$$

where dS is the surface element. We have

$$|\mathbf{P}_{\text{net}}| = |\mathbf{P}|dV = (|\mathbf{P}| \cos \theta dS)d = (\mathbf{P} \cdot d\mathbf{S})d. \quad (8)$$

From (6) we then get

$$Q_{\text{tot}} = \frac{|\mathbf{P}|dV}{d} = \mathbf{P} \cdot d\mathbf{S}. \quad (9)$$

The total bound charge is then found by integrating along the entire boundary surface:

$$Q_{\text{bound in } S} = - \oint_S \mathbf{P} \cdot d\mathbf{S}. \quad (10)$$

The sign is negative, because if the dipole points out of the surface, the negative half of the dipole is left inside S . Inserting (10) into (1) gives

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q_{\text{free in } S} + Q_{\text{bound in } S}}{\epsilon_0} = \frac{Q_{\text{free in } S}}{\epsilon_0} + \frac{1}{\epsilon_0} \oint_S \mathbf{P} \cdot d\mathbf{S}. \quad (11)$$

Rearranging gives

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \oint_S (\epsilon_0 \mathbf{E} + \mathbf{P}) \cdot d\mathbf{S} = Q_{\text{free in } S}. \quad (12)$$

where we defined \mathbf{D} as in (2). Equation (12) is sometimes referred to as Gauss' law in media.

Relationship between \mathbf{D} and \mathbf{E}

Many media are "linear", meaning

$$\mathbf{P} = \epsilon_0 \chi \mathbf{E}. \quad (13)$$

Inserting this in (2) gives

$$\mathbf{D} = \epsilon_0(1 + \chi)\mathbf{E} = \epsilon_0 \epsilon_r \mathbf{E} = \epsilon \mathbf{E}, \quad (14)$$

where we introduced the relative permittivity ϵ_r and the total permittivity ϵ . This parameter describes the dielectric properties of a medium, meaning how well the dipoles in the medium align with an applied external electric field. For water $\epsilon_r = 81$ while for air $\epsilon_r \approx 1$, meaning the dielectric properties of air are very similar to vacuum.

Example: Charge Q in water, find \mathbf{E} !

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q, \quad (15)$$

$$\oint_S D(r) dS = Q, \quad (16)$$

$$D(r) \oint_S dS = Q, \quad (17)$$

$$D(r) 4\pi r^2 = Q. \quad (18)$$

From (14) we get

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon} = \frac{Q}{4\pi\epsilon r^2} \hat{\mathbf{r}} = \frac{Q}{4\pi\epsilon_0\epsilon_r r^2} \hat{\mathbf{r}}. \quad (19)$$

We see that \mathbf{E} is $1/\epsilon_r = 1/81$ smaller than it would be in vacuum. This effect is called "screening", the electric field from the charge is screened by the medium.

Gauss' law in differential form

From (1) we previously found Gauss' law in differential form

$$\nabla \cdot \mathbf{E} = \frac{\rho_{\text{tot}}}{\epsilon_0}, \quad (20)$$

where ρ_{tot} is the total charge density (including bound charges). We may similarly find the differential form of (12):

$$\nabla \cdot \mathbf{D} = \rho, \quad (21)$$

where ρ is the *free charge density*.

Electrostatic energy

In the following exercises you may need these two expressions (will not be derived in this course). The stored energy in a capacitor is

$$W = \frac{1}{2} CV^2, \quad (22)$$

and the energy density in an electric field is

$$w = \frac{1}{2} \mathbf{D} \cdot \mathbf{E} = \frac{1}{2} \epsilon E^2. \quad (23)$$

Exercise 3: Problem 1. 25 min + 10 min solution.

Boundary conditions for \mathbf{E} and \mathbf{D}

Are electric fields continuous across boundaries?

$$E_{1t} = E_{2t}, \quad (24)$$

$$D_{1n} - D_{2n} = \rho_s. \quad (25)$$

Proof: We now prove these two formulas.

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{E} \cdot d\mathbf{S} = 0. \quad (26)$$

Integrate along a small square with sides Δl and Δh along the boundary surface, and let $\Delta h \rightarrow 0$. This gives

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \mathbf{E}_1 \cdot \Delta \mathbf{l} - \mathbf{E}_2 \cdot \Delta \mathbf{l} = E_{1t} \Delta l - E_{2t} \Delta l = 0. \quad (27)$$

This gives (24). Similarly we integrate Gauss' law over a cylinder surface at the boundary, and let the height of the cylinder $\Delta h \rightarrow 0$:

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q = \rho_s \Delta S. \quad (28)$$

All free charge must be located at the boundary surface as $\Delta h \rightarrow 0$. Thus

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \mathbf{D}_1 \cdot \hat{\mathbf{n}} \Delta S - \mathbf{D}_2 \cdot \hat{\mathbf{n}} \Delta S = D_{1n} \Delta S - D_{2n} \Delta S = \rho_s \Delta S. \quad (29)$$

This gives (25)

Ideal conductors (charge can move freely around)

Properties of ideal conductors:

- a) $\mathbf{E} = 0$ inside an ideal conductor (otherwise charges would move until $\mathbf{E} = 0$).
- b) $\rho = 0$ inside an ideal conductor. $\rho = \nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = 0$, from a) + there are no bound charges. So all charges are found at the boundary.
- c) $E_t = 0$ outside the boundary, from a) + (24).
- d) The conductor is an equipotential surface/volume. $V_{AB} = \int_A^B \mathbf{E} \cdot d\mathbf{l} = 0$.
- e) The electric field outside the boundary of an ideal conductor is $\mathbf{E} = \frac{\rho_s}{\epsilon_0} \hat{\mathbf{n}}$, as shown below:

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{\rho_s \Delta S}{\epsilon_0}, \quad (30)$$

and use a cylinder as the Gauss' surface. Let $\Delta h \rightarrow 0$ which gives

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \oint_{S_1} \mathbf{E} \cdot d\mathbf{S} + \oint_{S_2} \mathbf{E} \cdot d\mathbf{S} = \oint_{S_1} \mathbf{E} \cdot d\mathbf{S} = E \Delta S = \frac{\rho_s \Delta S}{\epsilon_0}. \quad (31)$$

This gives

$$\mathbf{E} = \frac{\rho_s}{\epsilon_0} \hat{\mathbf{n}}. \quad (32)$$

Application: Faraday cage

A faraday cage is an ideal conductor with a cavity inside. V is constant in the ideal conductor, which means V is constant at the outer boundary of the cavity. Since there are no charges inside the cavity, Poisson's equation gives that V is constant throughout the cavity. This means $\mathbf{E} = 0$ inside the cavity, independent on what \mathbf{E} is outside the conductor.

A Faraday cage thus works as a shield from \mathbf{E} -fields!

Exercise 3: Problem 2. 10 min + 5 min solution.

Current density

An \mathbf{E} -field acts with a force at a free charge q , which starts moving with a drift velocity \mathbf{v} (average, after thermal movements).

Consider a surface with area A . After a time dt all charges which have passed A is inside the volume $vdtA$. If N is the number of charges per volume we have

$$dQ = NqvdtA, \quad (33)$$

which gives

$$\frac{dQ}{dt} = NqvA. \quad (34)$$

From this we define the current

$$I = \frac{dQ}{dt} \quad (35)$$

Current is measured in $[A]$. The current is the total charge passing through a cross section A per unit time.

The current density is defined as:

$$\mathbf{J} = Nq\mathbf{v}, \quad (36)$$

which is measured in $[A/m^2]$. We thus have

$$I = \mathbf{J} \cdot \mathbf{A}. \quad (37)$$

For a constant current normal to a surface A we then have

$$J = \frac{I}{A}. \quad (38)$$

In general \mathbf{J} may vary with position. Then

$$I = \int \mathbf{J} \cdot d\mathbf{S}. \quad (39)$$

For some media *Ohms law* is valid:

$$\mathbf{J} = \sigma\mathbf{E} \quad (40)$$

where σ is the conductivity of the medium. From this one may define the resistance of a wire:

$$R = \frac{V}{I} = \frac{l}{\sigma S}, \quad (41)$$

where l is the length and S is the cross section of the wire.

Charge conservation

Kirchoff's law says that

$$I_1 + I_2 + I_3 = -\frac{dQ}{dt} = 0, \quad (42)$$

in electrostatics. This is also valid for constant currents, since charges cannot pile up over infinite times.

If charges can pile up (over finite time periods), $\frac{dQ}{dt} \neq 0$.

More general:

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{d}{dt} \int_V \rho dV. \quad (43)$$

The divergence theorem gives

$$\int_V \nabla \cdot \mathbf{J} dV = -\frac{d}{dt} \int_V \rho dV. \quad (44)$$

Since this must be valid for any V , we must have

$$\nabla \cdot \mathbf{J} = -\frac{d\rho}{dt}. \quad (45)$$

This equations says that charge is conserved *microscopically*. For constant charge distributions we have $\frac{d\rho}{dt} = 0$, so

$$\nabla \cdot \mathbf{J} = 0. \quad (46)$$