NTNU

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TFE4120 Electromagnetics - Crash course

Lecture 2: Gauss law, scalar potential, capacitance

Tuesday August 9th, 9-12 am.

Properties of the fields: Divergence and curl

Gradient: $\nabla =< \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} >$. "3-dimensional derivative of a scalar function".
The gradient points in the direction of the greatest rate of increase of the function, and its magnitude is the slope of the graph in that direction.

Divergence:

How much does the field flow out of a point?
$\text{div}\mathbf{E} = \nabla \cdot \mathbf{E} = \lim_{\Delta V \to 0} \frac{\oint_S \mathbf{E}\cdot d\mathbf{S}}{\Delta V}$.

Curl:

How much does the field circulate around a point?
$\text{curl}\mathbf{E} = \nabla \times \mathbf{E} = \lim_{\Delta S \to 0} \frac{\oint_C \mathbf{E}\cdot d\mathbf{l}}{\Delta S}$.

Useful theorems

Divergence theorem:

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{E} dV. \tag{1}$$

"Proof": Two ways to find the flux out of a sphere.

a) Flux out of **one** surface.

b) Flux out of many small surfaces.

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \sum_i \oint_{S_i} \mathbf{E} \cdot d\mathbf{S} = \sum_i (\frac{1}{\Delta V_i} \oint_{S_i} \mathbf{E} \cdot d\mathbf{S}_i)\Delta V_i \to \int \nabla \cdot \mathbf{E} dV. \tag{2}$$

Stokes theorem:

$$\oint_C \mathbf{E} \cdot \mathrm{d}\mathbf{l} = \int_V \nabla \times \mathbf{E} \cdot \mathrm{d}\mathbf{S}. \tag{3}$$

"Proof": Two ways to calculate the integral of **E** pointing along a closed loop.

a) Integrate along the edge of the surface.

b) ... or along the edges of many small surfaces (no gaps between them).

Since the integral along any "inner edges" cancel out, these two calculations must be equal.

$$\oint_C \mathbf{E} \cdot \mathrm{d}\mathbf{l} = \sum_i \oint_{C_i} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = \sum_i \left(\frac{\oint_{S_i} \mathbf{E} \cdot \mathrm{d}\mathbf{S}_i}{\Delta S_i}\right)\Delta S_i \rightarrow \int \nabla \times \mathbf{E} \cdot \mathrm{d}\mathbf{S}. \tag{4}$$

This "proof" was for a flat surface, but the theorem is valid in general.

Exercise 1: Problem 2. 5 min + 5 min solution.

Exercise 2: Problem 1. 10 min + 5 min solution.

Application of the divergence theorem: proof of Gauss' law)

Gauss' law: a simpler way to find **E** from a charge distribution (with some degree of symmetry) than from Coulomb's law.

We first consider a single point charge, and draw a spherical surface S around it. The divergence theorem gives:

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \int_V \nabla \cdot \mathbf{E} \mathrm{d}V \tag{5}$$

This problem has spherical symmetry, so we use spherical coordinates. $\nabla \cdot \mathbf{E}$ in spherical coordinates may be found at the formula sheet:

$$\nabla \cdot \mathbf{E} = \frac{1}{r^2}\frac{\partial(r^2 E_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta E_\theta) + \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi}. \tag{6}$$

The symmetry of the problem gives $\mathbf{E}(\mathbf{r}) = \mathbf{E}(r) = E_r(r)\hat{\mathbf{r}}$. That is $E_\theta = E_\phi = 0$.

Coulomb's law gives

$$E_r(r) = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}. \tag{7}$$

From Eqs. (6) and (7) we get

$$\nabla \cdot \mathbf{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{Q}{4\pi\epsilon_0 r^2}\right), \tag{8}$$

which gives $\nabla \cdot \mathbf{E} = 0$ for all r except at $r = 0$.

We may thus split the last integral in (5) into two parts:

$$\int_V \nabla \cdot \mathbf{E}\mathrm{d}V = \int_{r>a} \nabla \cdot \mathbf{E}\mathrm{d}V + \int_{r\le a} \nabla \cdot \mathbf{E}\mathrm{d}V = \int_{r\le a} \nabla \cdot \mathbf{E}\mathrm{d}V, \tag{9}$$

where a is some small radius (can in fact be made arbitrary small). We here used that $\int_{r>a} \nabla \cdot \mathbf{E}\mathrm{d}V = 0$ since $\nabla \cdot \mathbf{E} = 0$ everywhere except at $r = 0$. We therefore have

$$\oint_S \mathbf{E}\cdot\mathrm{d}\mathbf{S} = \oint_{S_a} \mathbf{E}\cdot\mathrm{d}\mathbf{S} = E(a)\oint_{S_a}\mathrm{d}S = E(a)\cdot 4\pi a^2 = \frac{Q}{4\pi\epsilon_0 a^2}4\pi a^2 = \frac{Q}{\epsilon_0}. \tag{10}$$

What did we find from this? we ended where we started, just with a smaller sphere.

Note: We never used the assumption that the outer surface was a sphere. The outer surface could have been arbitrary in shape and size, as long as it contains Q and is closed, and we would still get the same result!

The fact that

$$\oint_S \mathbf{E}\cdot\mathrm{d}\mathbf{S} = \frac{Q}{\epsilon_0}, \tag{11}$$

for an arbitrary closed surface surrounding a charge Q is called *Gauss' law*. In words: The flux out of a closed surface is always proportional to the charge inside it. Generalized to more charges:

$$\oint_S \mathbf{E}\cdot\mathrm{d}\mathbf{S} = \int_V \nabla\cdot\mathbf{E}\mathrm{d}V = \int_{V_1}\nabla\cdot\mathbf{E}\mathrm{d}V + \int_{V_2}\nabla\cdot\mathbf{E}\mathrm{d}V + \int_{V_3}\nabla\cdot\mathbf{E}\mathrm{d}V + \int_{V_4}\nabla\cdot\mathbf{E}\mathrm{d}V = \frac{Q_1}{\epsilon_0}+\frac{Q_2}{\epsilon_0}+\frac{Q_3}{\epsilon_0}+\frac{Q_4}{\epsilon_0}. \tag{12}$$

The general form of Gauss' law is thus

$$\oint_S \mathbf{E}\cdot\mathrm{d}\mathbf{S} = \frac{Q_{\text{in S}}}{\epsilon_0}, \tag{13}$$

where S is any arbitrary closed surface, and $Q_{\text{in S}}$ is the total charge inside S.

Gauss' law in differential form

Express $Q_{\text{in S}}$ in terms of the charge distribution ρ:
$\oint_S \mathbf{E}\cdot\mathrm{d}\mathbf{S} = \int_V \nabla\cdot\mathbf{E}\mathrm{d}V = Q_{\text{in S}}/\epsilon_0 = \int_V \rho\mathrm{d}V/\epsilon_0$.

Consider a small volume element $\mathrm{d}V$:

$$\nabla\cdot\mathbf{E}\mathrm{d}V = \frac{\rho}{\epsilon_0}\mathrm{d}V, \tag{14}$$

which gives

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}. \tag{15}$$

This is called Gauss' law in differential form.

Exercise 2: Problem 2+3. 20 min + 10 min solution.

Wrong usage of Gauss' law: $\oint_S \mathbf{E}\cdot\mathrm{d}\mathbf{S} = \frac{+Q-Q}{\epsilon_0} = 0$. Does this mean $\mathbf{E} = 0$? NO!!

This means the flux out of the surface S is zero, meaning as much field is flowing OUT of S is also flowing INTO S (at different areas of S).

Scalar potential

One of Maxwell's equations says that $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = 0$ in electrostatics. This means in electrostatics we may define a scalar function V such that

$$\mathbf{E} = -\nabla V. \tag{16}$$

It is possible to define such a V since

$$\nabla \times (-\nabla V) = -\nabla \times (\nabla V) = 0, \tag{17}$$

since the curl of any gradient is zero! (no circulation)

We now solve for V:

$$\int_{\mathbf{r}_{\mathrm{ref}}}^{\mathbf{r}} (\nabla V) \cdot \mathrm{d}\mathbf{l} = V(\mathbf{r}) - V(\mathbf{r}_{\mathrm{ref}}), \tag{18}$$

which follows from the second fundamental theorem of calculus (the integral of a derivative of a function equals the difference of the functions values at the boundary values of the integral). It is common to let $\mathbf{r}_{\mathrm{ref}} \to \infty$ and set $V(\mathbf{r}_{\mathrm{ref}}) = 0$. Inserting (16) into (18) gives

$$V(\mathbf{r}) = -\int_{\infty}^{\mathbf{r}} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = \int_{\mathbf{r}}^{\infty} \mathbf{E} \cdot \mathrm{d}\mathbf{l}. \tag{19}$$

Interpretation: $V(\mathbf{r}) = \int_{\mathbf{r}}^{\infty} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = \frac{1}{q} \int_{\mathbf{r}}^{\infty} \mathbf{F} \cdot \mathrm{d}\mathbf{l} = \frac{\mathrm{Work}}{q}$. That is, the potential $V(\mathbf{r})$ is the amount of work required to move a charge q from $\mathbf{r}$ to our reference point at ∞, divided by the charge q. The electric potential is thus related to potential energy (i.e. the name).

Example: Potential from a point charge.

$$V(\mathbf{R}) = \int_{\mathbf{R}}^{\infty} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = \int_{R}^{\infty} E(r)\mathrm{d}r = \int_{R}^{\infty} \frac{Q}{4\pi\epsilon_0 r^2}\mathrm{d}r = \frac{Q}{4\pi\epsilon_0 R}. \tag{20}$$

Potential difference: What is the potential difference between A and B?

$$V_{AB} = \int_{A}^{B} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = \int_{A}^{\infty} \mathbf{E} \cdot \mathrm{d}\mathbf{l} + \int_{\infty}^{B} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = \int_{A}^{\infty} \mathbf{E} \cdot \mathrm{d}\mathbf{l} - \int_{B}^{\infty} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = V(A) - V(B). \tag{21}$$

For a point charge:

$$V_{AB} = \frac{Q}{4\pi\epsilon_0 A} - \frac{Q}{4\pi\epsilon_0 B} = \frac{Q}{4\pi\epsilon_0}(\frac{1}{A} - \frac{1}{B}). \tag{22}$$

The potential is independent of the integration path Introducing the potential V is an advantage: it is often easier to calculate than $\mathbf{E}$.

$$\oint_{C_1 \& C_2} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = \int_{C_1} \mathbf{E} \cdot \mathrm{d}\mathbf{l} - \int_{C_2} \mathbf{E} \cdot \mathrm{d}\mathbf{l}. \tag{23}$$

Stokes theorem gives

$$\oint_{C_1 \& C_2} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = \int_{S} \nabla \times \mathbf{E} \cdot \mathrm{d}\mathbf{S} = 0, \tag{24}$$

which again gives

$$\int_{C_1} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = \int_{C_2} \mathbf{E} \cdot \mathrm{d}\mathbf{l}. \tag{25}$$

This is true for arbitrary integration paths C_1 and C_2 from A to B. Potential differences are therefore independent of the path we integrat $\mathbf{E} \cdot \mathrm{d}\mathbf{l}$ along.

Exercise 2: Problem 4. 15 min + 5 min solution.

Poisson's equation

Inserting $\mathbf{E} = -\nabla V$ into Maxwell's equation $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ gives

$$-\nabla \cdot (\nabla V) = \frac{\rho}{\epsilon_0}, \tag{26}$$

which gives

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}. \tag{27}$$

This is called Poisson's equation. Here

$$\nabla^2 = < \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2} > \tag{28}$$

in cartesian coordinates.

Capacitance

Consider two conductors where a voltage source has moved a charge Q from the lower to the upper conductor. There is thus a voltage difference V_0 between them. We define a quantity known as capacitance:

$$C = \frac{Q}{V_0}. \tag{29}$$

Since V is proportional to Q we will end up with an expression for C which is independent of Q and V_0. The capacitance only depends on geometrical quantities and ϵ (material parameter).

Example 1: Parallel plate capacitor. We want to find the capacitance between two parallel plate conductors with air between them. Our strategy is to first find Q and V_0 in terms of $\mathbf{E}$

a) Find Q: Gauss' law on a cylinder surface placed through one of the plates gives

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{\rho_s A}{\epsilon_0}, \tag{30}$$

where $Q = \rho_s A$. Symmetry gives that $\mathbf{E} = E\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is the normal vector of the plate. Since there only is flux through the top and bottom of the syllinder, and the flux through both of them must be equal (again due to symmetry) we get

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{S} = 2AE = \frac{\rho_s A}{\epsilon_0}. \tag{31}$$

This gives $\mathbf{E} = \frac{\rho_s}{2\epsilon_0}\hat{\mathbf{n}}$. Between the plates the total field is given as the sum of the field from each of the plates, i.e. $\mathbf{E} = -\frac{Q}{A\epsilon_0}\hat{\mathbf{x}}$. This gives

$$Q = -EA\epsilon_0. \tag{32}$$

b) Find V_0. Poisson's equation $\nabla^2 V = \frac{\partial^2}{\partial x^2} = 0$ (between the plates there are no charges, i.e. $\rho = 0$). This means $V(x) = C_1 x + C_2$. From the boundary conditions $V(0) = 0$ and $V(d) = V_0$ we get

$$V(x) = \frac{V_0}{d}x, \tag{33}$$

which gives $\mathbf{E} = -\nabla V = -V_0/d$. We have thus found

$$V_0 = -Ed. \tag{34}$$

Inserting 1 and 2 into (29) gives

$$C = \frac{\epsilon_0 A}{d}. \tag{35}$$

Potential from point/line/surface/space charge

We let $\mathbf{r}_{\text{ref}} \to \infty$. We list the potentials from

a) A point charge:

$$V = \int_R^\infty \frac{Q}{4\pi\epsilon_0 r^2}\mathrm{d}r = \frac{Q}{4\pi\epsilon_0 R}, \tag{36}$$

b) A line charge (line charge density Q'):

$$V = \int_C \frac{Q'}{4\pi\epsilon_0 R}\mathrm{d}l, \tag{37}$$

c) A surface charge (surface charge density ρ_s):

$$V = \int_S \frac{\rho_s}{4\pi\epsilon_0 R}\mathrm{d}S, \tag{38}$$

d) A space charge (charge density ρ):

$$V = \int_V \frac{\rho}{4\pi\epsilon_0 R}\mathrm{d}V. \tag{39}$$

The last three is found by superposition from the point charge expression.

Exercise 2: Problem 5