Exercise Session 2

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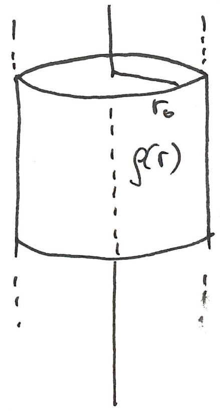
Given a cylindrical charge distribution $\rho(r) = \begin{cases} \rho_0(a-br) \ \frac{C}{m^3} & \text{for } r \leq r_0 \\ 0 \ \frac{C}{m^3} & \text{for } r > r_0 \end{cases}$ having an indefinite height, express analitically the form of the electrostatic field $\bar{E}$ produced.

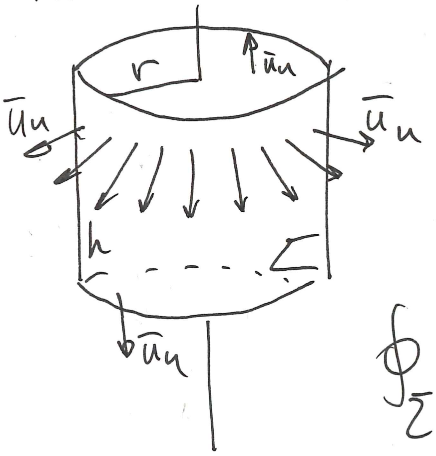
Solution

Thanks to the geometry (axial symmetry and indefinite height), the field $\bar{E}(\bar{r})$ is oriented radially with respect to the cylinder axis and depends only on the distance $r = \|\bar{r}\|$ from the axis, i.e. no dependence on the polar angle φ.



We apply the Gauss law to a cylinder having arbitrary height h and radius r. Two cases must be considered.

$r \leq r_0$



We call Σ the surface of the Gauss cylinder. The versor $\bar{u}_n$ indicates the normal direction of Σ, pointing outward.

$$\oint_\Sigma \bar{E} \cdot \bar{u}_n \, d\Sigma = \frac{Q}{\varepsilon_0}$$ where Q is the charge included in Σ.

$$\oint_\Sigma \bar{E} \cdot \bar{u}_n \, d\Sigma = \int_{base\,1} \bar{E} \cdot \bar{u}_n d\Sigma + \int_{base\,2} \bar{E} \cdot \bar{u}_n d\Sigma + \int_{lat.\,sup.} \bar{E} \cdot \bar{u}_n \, d\Sigma = \int_{lat.\,sup} \bar{E} \cdot \bar{u}_n \, d\Sigma$$ since on base 1 and 2, $\bar{E} \cdot \bar{u}_n = 0$.

$$\int_{lat.\,Sup} \bar{E} \cdot \bar{u}_n \, d\Sigma = \int_{lat.\,sup.} E \, d\Sigma = E \cdot \int_{lat.\,sup.} d\Sigma = 2\pi r \cdot h \cdot E$$ since E is constant at a given r.

$$\frac{Q}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_{volume} \rho(r) \, d\tau = \frac{1}{\varepsilon_0} \int_0^h dz \cdot \int_0^{2\pi} d\varphi \cdot \int_0^r \rho(r) \cdot r \, dr = \frac{2\pi h}{\varepsilon_0} \rho_0 \left(\frac{1}{2} a r^2 - \frac{1}{3} b r^3 \right)$$

Therefor $$2\pi r \cdot h \cdot E = \frac{2\pi h}{\varepsilon_0} \rho_0 \left(\frac{1}{2} a r^2 - \frac{1}{3} b r^3 \right) \Rightarrow E(r) = \frac{\rho_0}{\varepsilon_0} \left(\frac{1}{2} a r - \frac{1}{3} b r^2 \right)$$

In vectorial form, $$\bar{E}(r) = \frac{\rho_0}{\varepsilon_0} \left(\frac{1}{2} a r - \frac{1}{3} b r^2 \right) \cdot \bar{u}_r$$ where $\bar{u}_r$ is the radial direction versor.

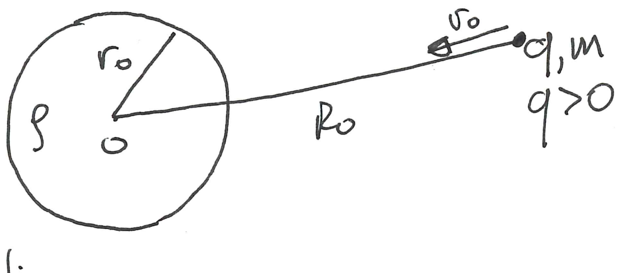
$r > r_0$

Similar approach when the Gauss cylinder is chosen with a radius larger than r_0. Attention: when calculating the charge Q included within the Gauss cylinder, we must recall that $\rho = 0$ for $r > r_0$!

$$\oint_\Sigma \bar{E} \cdot \bar{u}_n \, d\Sigma = 2\pi r h \cdot E = \frac{Q}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_0^h dz \cdot \int_0^{2\pi} d\varphi \cdot \int_0^{r_0} \rho_0 (a - br) \, r \, dr$$

$$2\pi r h \cdot E = \frac{2\pi h}{\varepsilon_0} \rho_0 \left(\frac{1}{2} a r_0^2 - \frac{1}{3} b r_0^3 \right) \mapsto \bar{E}(r) = \frac{\rho_0}{\varepsilon_0 r} \left(\frac{1}{2} a r_0^2 - \frac{1}{3} b r_0^3 \right) \cdot \bar{u}_r .$$

Given the spherical volumetric distribution of charge ρ illustrated in the figure, determine the initial velocity v_0 the charged particle must have in order to reach the center of the sphere O. The charge density ρ = constant and the particle has charge q and mass m.



Solution

The particle is subjected to an electrostatic force due to the field produced by the sphere. Such a force is oriented radially, pointing outward, as the electrostatic field.

$$\overline{F} = q\cdot\overline{E} = q\cdot E(r)\cdot\overline{u}_r$$

The electrostatic field depends on the distance r from the center O, because of the symmetry of the geometry.

Since the particle is moving radially, toward the sphere center, its displacement vector is anti-parallel to $\overline{u}_r$ and $\overline{E}$.

By imposing energy conservation: $\frac{1}{2}mv_0^2 = \int_{R_0}^{0} \overline{F}\cdot d\overline{s}$ where $d\overline{s}$ is the infinitesimal displacement

$\int_{R_0}^{0} \overline{F}\cdot d\overline{s}$ is the work done by $\overline{F}$ in moving the particle from R_0 to O

Recall that $d\overline{s} = -dr\cdot\overline{u}_r$, so

$$\int_{R_0}^{0} \overline{F}\cdot d\overline{s} = -\int_{R_0}^{0} \overline{F}\cdot\overline{u}_r\,dr = -\int_{R_0}^{0} F\,dr = \int_{0}^{R_0} F\cdot dr = q\int_{0}^{R_0} E\cdot dr$$

where E is the electrostatic field produced by the spherical charge distrib.

$q\int_{0}^{R_0} E(r)\cdot dr$ requires the analytical form of $E(r)$.

From the application of the Gauss law, we find that

$$E(r) = \begin{cases} \dfrac{\rho r_0^3}{3\varepsilon_0 r^2} & \text{for } r \geq r_0 \\ \dfrac{\rho r}{3\varepsilon_0} & \text{for } r \leq r_0 \end{cases}$$

$$q\int_{0}^{R_0} E(r)\cdot dr = q\left[\int_{0}^{r_0} \frac{\rho r}{3\varepsilon_0}\,dr + \int_{r_0}^{R_0} \frac{\rho r_0^3}{3\varepsilon_0 r^2}\,dr\right] =$$

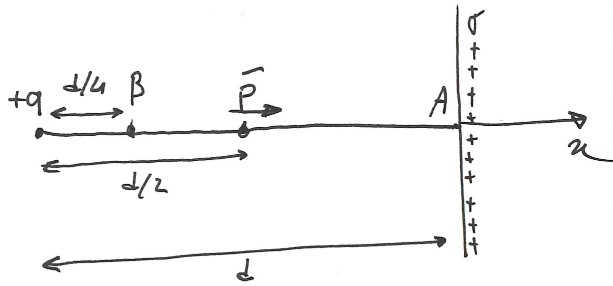
$$= \frac{q\rho}{3\varepsilon_0}\left[\frac{1}{2}r_0^2 + r_0^2 - \frac{r_0^3}{R_0}\right] = \frac{q\rho}{3\varepsilon_0}\left[\frac{3}{2}r_0^2 - \frac{r_0^3}{R_0}\right] = \frac{1}{2}mv_0^2$$

$$v_0 = \sqrt{\frac{2q\rho}{3m\varepsilon_0}\left[\frac{3}{2}r_0^2 - \frac{r_0^3}{R_0}\right]}.$$

A point-like charge $+q$ is separated by a distance $d = 40\,cm$ from a uniform planar distribution with charge density $\sigma = 8.86 \cdot 10^{-10} \frac{C}{m^2}$ - An electric dipole $|\bar{p}| = 10^{-12}\,Cm$ is located at a distance $\frac{d}{2}$ from the point-like source - The dipole is oriented parallel to the x-axis, i.e. $\bar{p} = |\bar{p}| \cdot \bar{u}_x$. On the dipole, a force $|\bar{F}| = 2.25 \cdot 10^{-9}\,N$ is exerted along the x-direction.

① Calculate q

② Calculate the velocity v_B an electron will have in point B if it leaves from point A with $v_A = 3 \cdot 10^6 \frac{m}{s}$.



Solution ①

Where the dipole is located, the electrostatic field resolting from the sum of the field produced by the charge q and the field produced by the distribution σ has only x-component, from symmetry considerations.

$$\bar{E} = E_x \cdot \bar{u}_x \quad \text{where} \quad E_x = \frac{q}{4\pi\varepsilon_0 x^2} - \frac{\sigma}{2\varepsilon_0}$$

Note that the field produced by σ is anti-parallel to the x-axis in the point where the dipole is located

It is known that

$$\bar{F} = (\bar{p} \cdot \nabla)\,\bar{E}, \quad \text{whic means} \quad \bar{F} = |\bar{p}| \cdot \frac{dE_x}{dx} \cdot \bar{u}_x \quad \text{since } p_y = p_z = 0;\ E_y = E_z = 0.$$

Therefore $$|\bar{F}| = F_x = -\frac{2q|\bar{p}|}{4\pi\varepsilon_0 x^3}$$

- The force is pointing toward the point-like source.
- The planar charge distribution does not contribute to the force on the dipole because it produces an uniform field (with null gradient).

In $x = \frac{d}{2}$

$$F_x = -\frac{2|\bar{p}|q}{4\pi\varepsilon_0 x^3}\Bigg|_{x=\frac{d}{2}} \Longrightarrow q \simeq 10^{-9}\,C.$$

Solution (2)

From an energy balance: $q_e \cdot V_A + \frac{1}{2} m_e v_A^2 = q_e \cdot V_B + \frac{1}{2} m_e v_B^2$

the total energy is conserved when the electron (charge q_e and mass m_e) is moving from A to B.

$$q_e (V_B - V_A) = \frac{1}{2} m_e (v_A^2 - v_B^2)$$

$$V_B - V_A = \frac{q}{4\pi\varepsilon_0}\left(\frac{4}{d} - \frac{1}{d}\right) - \frac{\sigma}{2\varepsilon_0}\frac{3}{4}d$$

where $\frac{q}{4\pi\varepsilon_0}\left(\frac{4}{d} - \frac{1}{d}\right)$ is the potential difference from B to A due to the field generated by q only. and $\frac{\sigma}{2\varepsilon_0}\frac{3}{4}d$ is the potential difference due to the planar distribution only.

$$V_B - V_A = \Delta V$$

$$v_B = \sqrt{v_A^2 - \frac{2 q_e \Delta V}{m_e}} = 5.26 \cdot 10^6 \ \frac{m}{s}$$

where $|q_e| = 1.6 \cdot 10^{-19}$ C and $m_e = 0.9 \cdot 10^{-30}$ kg.

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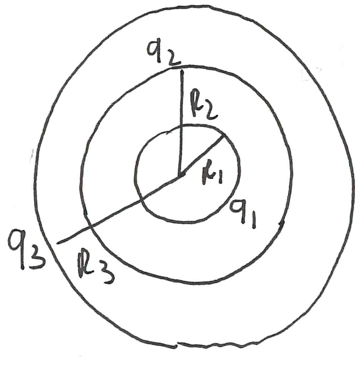
Express the electrostatic field generated by 3 homocentric spherical distributions of charge having radii $R_1 < R_2 < R_3$ and corresponding charges q_1, q_2, q_3 on each sphere surface. Find out the analytical expression for the potential too.

Solution



Each sphere carries a charge q on it surface. The electrostatic field and potential produced by each sphere are:

$$\bar{E}(r) = \begin{cases} 0, & r < R \\ \frac{\sigma R^2}{\varepsilon r^2} \bar{u}_r, & r \geq R \end{cases} ; \quad V(r) = \begin{cases} \frac{q}{4\pi\varepsilon_0 R} & r < R \\ \frac{q}{4\pi\varepsilon_0 r}, & r \geq R \end{cases}$$

This result is found by applying the Gauss law to a generic sphere of radius R and surface charge density σ with $\sigma = \frac{q}{4\pi R^2}$

Therefore, by involving the superposition of the effects:

$0 \leq r \leq R_1$ $\quad \bar{E} = (0+0+0)\bar{u}_r ; \quad V = \frac{q_1}{4\pi\varepsilon_0 R_1} + \frac{q_2}{4\pi\varepsilon_0 R_2} + \frac{q_3}{4\pi\varepsilon_0 R_3}$

$R_1 < r \leq R_2$ $\quad \bar{E} = \left(\frac{q_1}{4\pi\varepsilon r^2} + 0 + 0\right)\bar{u}_r ; \quad V = \frac{q_1}{4\pi\varepsilon r} + \frac{q_2}{4\pi\varepsilon_0 R_2} + \frac{q_3}{4\pi\varepsilon_0 R_3}$

$R_2 < r \leq R_3$ $\quad \bar{E} = \left(\frac{q_1}{4\pi\varepsilon r^2} + \frac{q_2}{4\pi\varepsilon r^2} + 0\right)\bar{u}_r ; \quad V = \frac{q_1 + q_2}{4\pi\varepsilon_0 r} + \frac{q_3}{4\pi\varepsilon R_3}$

$r > R_3$ $\quad \bar{E} = (q_1 + q_2 + q_3)/4\pi\varepsilon r^2 \cdot \bar{u}_r ; \quad V = \frac{q_1 + q_2 + q_3}{4\pi\varepsilon r}$

In the special case where $q_1 = q$; $q_2 = -q$; $q_3 = q$, the following plots are obtained: