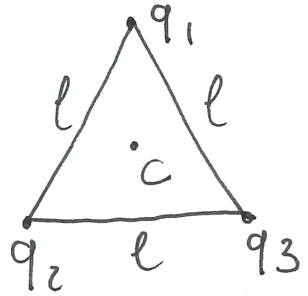
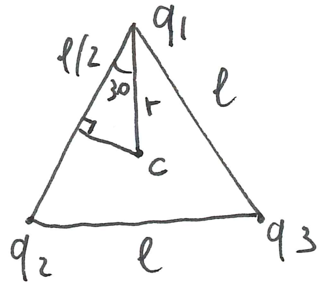
Exercise Session 1

-1-

Calculate the electrostatic potential in C.



Solution.



$V_1 = \dfrac{q_1}{4\pi\varepsilon_0 r}$ potential in C due to charge q_1

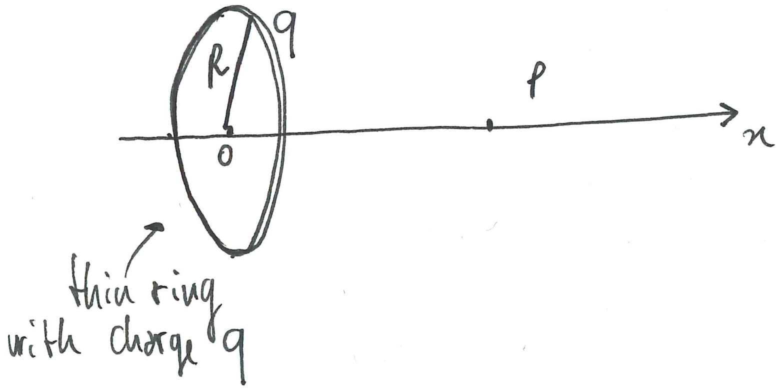
where $r = \dfrac{l}{\sqrt{3}}$, so $V_1 = \dfrac{\sqrt{3}}{4\pi\varepsilon_0 l} \cdot q_1$

Thanks to the superposition rule, the total potential in C is given by the algebraic sum of all contributions from each single charge.

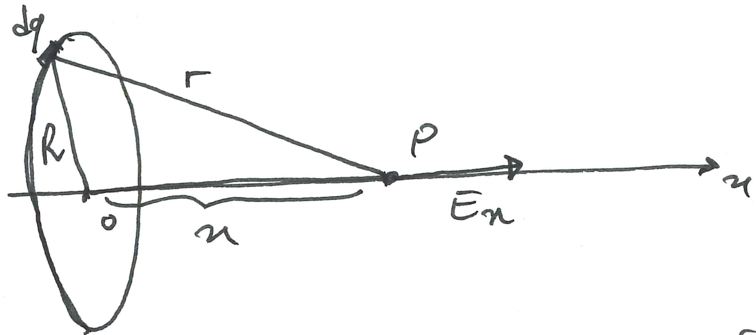
$$V = \sum_{i=1}^{3} V_i = \frac{\sqrt{3}}{4\pi\varepsilon_0 l} \sum_{i=1}^{3} q_i$$

-2-

Calculate the electrostatic field $\bar{E}$ and potential V at an arbitrary point P along the x-axis. The thin ring has radius R and a total charge q.



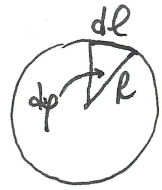
Solution



The charge distribution is one-dimensional, so that a linear charge density can be defined as $\lambda = \frac{q}{2\pi R} \quad \frac{C}{m}$

Each infinitesimal element dl holds a charge $dq = \lambda \cdot dl$.
Each element dl is treated as a point-like source, producing an infinitesimal potential dV at a distance r.

$$dV = \frac{dq}{4\pi\varepsilon_0 r} = \frac{\lambda\, dl}{4\pi\varepsilon_0 r} \, * \qquad \text{where } r = \sqrt{R^2 + x^2}, \quad dl = R\, d\varphi$$



After integration:

$$V = \int_0^{2\pi} \frac{\lambda R\, d\varphi}{4\pi\varepsilon_0 r} = \frac{q}{4\pi\varepsilon_0 \sqrt{R^2 + x^2}}$$

If $x \gg R$, $\sqrt{R^2 + x^2} = x \cdot \sqrt{1 + R^2/x^2} \sim x\left(1 + \frac{1}{2}\frac{R^2}{x^2} + \dots\right) \sim x$

That is: the potential is the same as the charge q would be concentrated in O.

* <u>note</u>: the integration constant $c = 0$ because this expression for dV is equivalent to the potential of a single point-like charge.

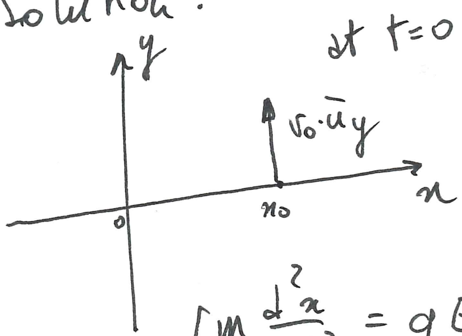
The electrostatic field is calculated upon derivation of V.

$$\begin{cases} E_x = -\dfrac{\partial V}{\partial x} = -\dfrac{q}{4\pi\varepsilon_0}\dfrac{\partial}{\partial x}\left(\dfrac{1}{\sqrt{x^2 + R^2}}\right) = \dfrac{q}{4\pi\varepsilon_0}\dfrac{x}{(R^2 + x^2)^{3/2}} \\ E_y = -\dfrac{\partial V}{\partial y} = 0 \\ E_z = -\dfrac{\partial V}{\partial z} = 0 \end{cases}$$

$\vec{E}$ is oriented parallel to the x-axis.

Express the equation of motion for a particle with electric charge q and mass m, such that $\frac{q}{m} = 5\cdot 10^6 \frac{C}{kg}$, within an electrostatic potential defined over the xy plane as $V(x,y) = V_0(x^2+y^2)$, where $V_0 = 10^7 \frac{V}{m^2}$. At time $t=0$, the particle is in point $P(x_0, 0)$ and has a velocity $\bar{v} = v_0 \bar{u}_y$, where $x_0 = 1$ cm and $v_0 = 10^5 \frac{m}{s}$ -

Solution:



$$\bar{E} = -\nabla V = \bar{u}_x\left(-\frac{\partial V}{\partial x}\right) + \bar{u}_y\left(-\frac{\partial V}{\partial y}\right)$$

$$\bar{E} = -2V_0 x \cdot \bar{u}_x - 2V_0 y \cdot \bar{u}_y$$

From the Newton's 2nd law:

$$\begin{cases} m\dfrac{d^2x}{dt^2} = qE_x = -2V_0 x \cdot q & \text{motion along } x \\ m\dfrac{d^2y}{dt^2} = qE_y = -2V_0 y \cdot q & \text{motion along } y \end{cases}$$

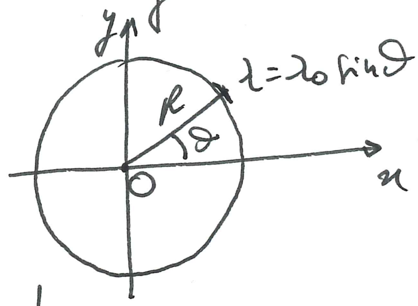
The two equations (differential) allow for harmonic solutions:

$$\begin{cases} x(t) = x_0 \cos wt & \text{where } x_0 \text{ is known } (x(0) = x_0) \\ y(t) = y_0 \sin wt & \text{where } y_0 \text{ must be calculated using the information on the velocity.} \end{cases}$$

$v_y(t) = \frac{dy}{dt} = w y_0 \cos wt$. At $t=0$, $v_y(0) = v_0$, therefore: $w y_0 = v_0$

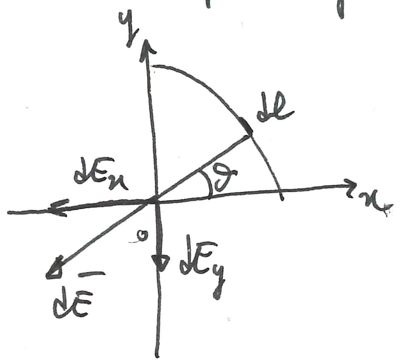
$y(t) = \frac{v_0}{w} \sin wt$.

Calculate the expression of the electrostatic field in the center O of an annular distribution of electric charge having linear density $\lambda = \lambda_0 \sin\vartheta$, as depicted in the figure.



Solution.

The total electric field in the center is given by the sum (integral) of all the infinitesimal contributions $d\vec{E}$ from corresponding infinitesimal arcs dl on the ring.



Each arc dl brings an infinitesimal charge

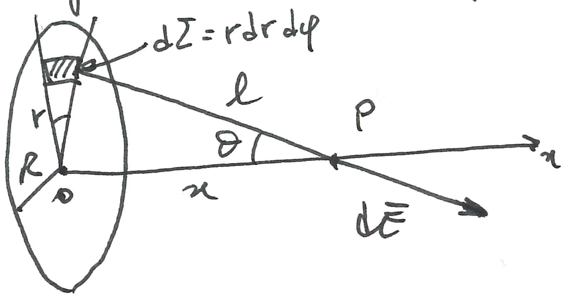
$$dq = \lambda \cdot dl = \lambda_0 \sin\vartheta \cdot dl$$

$$\begin{cases} dE_x = -\dfrac{dq}{4\pi\varepsilon_0 R^2}\cos\vartheta = -\dfrac{\lambda_0 \sin\vartheta \cdot dl}{4\pi\varepsilon_0 R^2}\cos\vartheta \\[2ex] dE_y = -\dfrac{dq}{4\pi\varepsilon_0 R^2}\sin\vartheta = -\dfrac{\lambda_0 \sin\vartheta \cdot dl}{4\pi\varepsilon_0 R^2}\sin\vartheta \end{cases}$$

Taking in mind that $dl = R\,d\vartheta$

$$\begin{cases} E_x = -\dfrac{\lambda_0}{4\pi\varepsilon R}\displaystyle\int_0^{2\pi} \sin\vartheta \cdot \cos\vartheta \, d\vartheta = 0 \\[2ex] E_y = -\dfrac{\lambda_0}{4\pi\varepsilon R}\displaystyle\int_0^{2\pi} \sin^2\vartheta \, d\vartheta = -\dfrac{\lambda_0}{4\varepsilon_0 R} \end{cases}$$

Express the electrostatic field produced by a disk-shaped planar distribution of electric charge with constant density σ $\left(\frac{C}{m^2}\right)$ at an arbitrary distance x along its axis.



Solution

Each infinitesimal surface $d\Sigma$ contributes with a field $d\bar{E}$ to the total field in P.

For symmetry reasons, the field $\bar{E}$ will only have x-component as all the transverse components will be cancelled out pairwise.

$$dE_x = \frac{dq}{4\pi\varepsilon_0 l^2}\cos\theta \quad \text{where} \quad dq = \sigma \cdot d\Sigma = \sigma \cdot r\,dr\,d\varphi$$

$$l = \sqrt{x^2 + r^2}$$

$$\cos\theta = \frac{x}{\sqrt{x^2+r^2}}$$

Integrating on r and φ leads to:

$$E_x = \int_0^{2\pi} d\varphi \int_0^R dr \, \frac{\sigma x}{4\pi\varepsilon_0} \frac{r}{(x^2+r^2)^{3/2}} = \frac{\sigma x}{2\varepsilon_0}\left(\frac{1}{x} - \frac{1}{\sqrt{x^2+R^2}}\right)$$

Alternative approach: Calculate $E_x = -\dfrac{\partial V}{\partial x}$.

$$dV = \frac{dq}{4\pi\varepsilon_0 l} = \frac{\sigma\, r\,dr\,d\varphi}{4\pi\varepsilon_0\sqrt{x^2+r^2}}$$

$$V = \frac{\sigma}{4\pi\varepsilon_0}\int_0^{2\pi} d\varphi \int_0^R dr\,\frac{r}{\sqrt{x^2+r^2}} = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{x^2+r^2}\right]_0^R = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{x^2+R^2} - x\right)$$

$$E_x = -\frac{\partial V}{\partial x} = \frac{\sigma}{2\varepsilon_0} - \frac{\sigma x}{2\varepsilon_0\sqrt{x^2+R^2}}$$