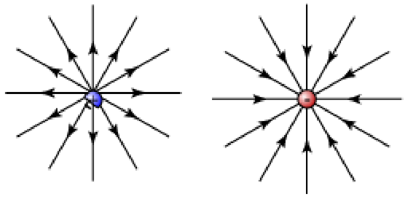
Lecture 2: Stationary electric field (Electrostatic)

- Electric field , electric displacement field, and electric potential
- Conservative vector field: E in stationary electric field
- Gauss’s law

Divergence and Stoke’s theorem

Gradient: fastest rate of increase in spatial.

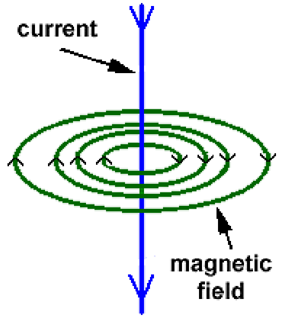
$$\nabla \boldsymbol{f} = \frac{\partial \boldsymbol{f}}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial \boldsymbol{f}}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial \boldsymbol{f}}{\partial z}\hat{\boldsymbol{z}}$$



Field pattern of
a pointed electrode

Divergence: Flux out of a point

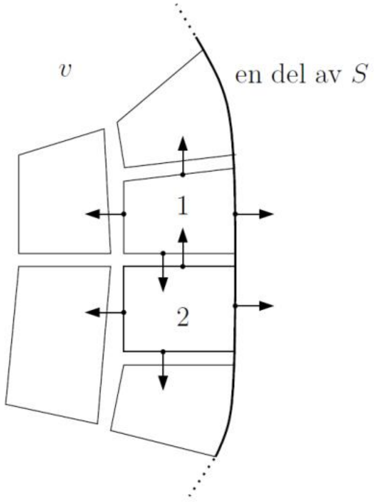
$$\nabla \cdot E = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$



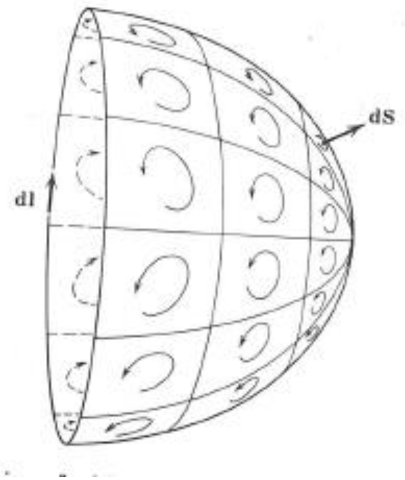
how much does a field circulate around a point

$$\nabla \times A = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}$$

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{A}\, dv$$

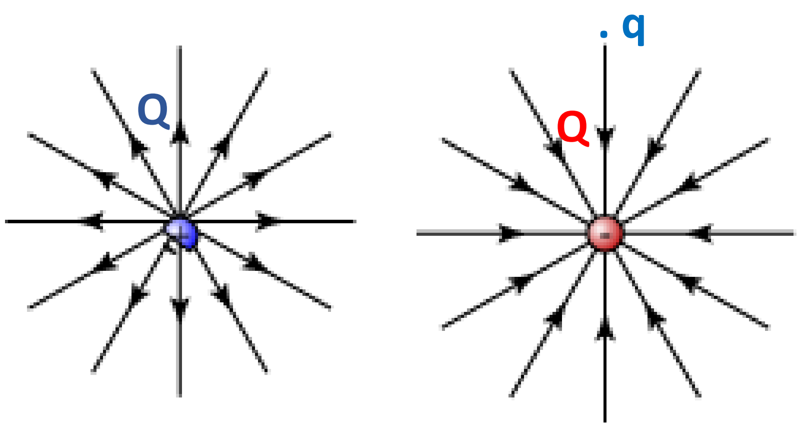


$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{A} \cdot d\mathbf{S}.$$



Electric field and density (vector)

A stationary distribution of charges produces an electric field **E** in vacuum



Vector: $\vec{\boldsymbol{E}} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}}$

Vector: $\vec{\boldsymbol{F}} = \frac{qQ}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}}$: *Coulomb's law*

$\vec{\boldsymbol{E}}=\vec{\boldsymbol{F}}/q$

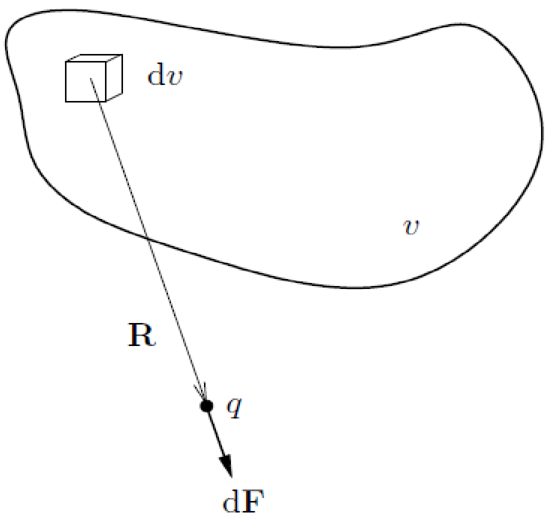
$\boldsymbol{\varepsilon_0}$ is vacuum permittivity (physical Property) ε_0 = 8.85e-12 F/m

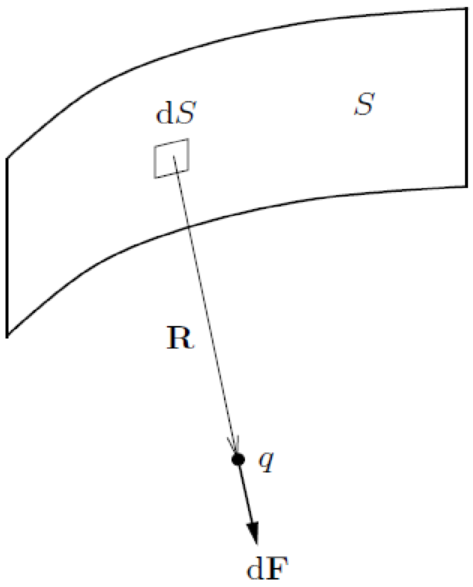
Electric displacement field D and D= ε_0E.

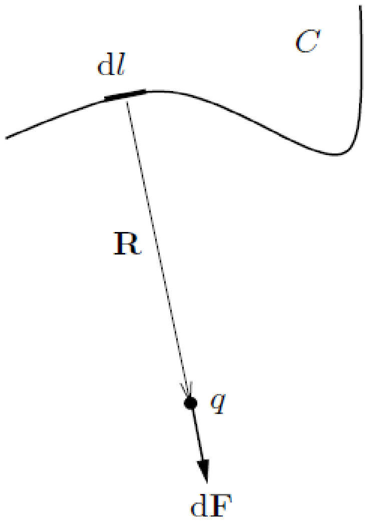
Vector: $\vec{\boldsymbol{D}} = \frac{Q}{4\pi r^2}\hat{\boldsymbol{r}}$

Electric displacement field: the equation is material independant.

Superposition: Vector **E**







$$\mathbf{E} = \int_v \frac{\hat{\mathbf{R}}\rho \mathrm{d}v}{4\pi\epsilon_0 R^2}.$$

$$\mathbf{E} = \int_S \frac{\hat{\mathbf{R}}\rho_\mathrm{s} \mathrm{d}S}{4\pi\epsilon_0 R^2}.$$

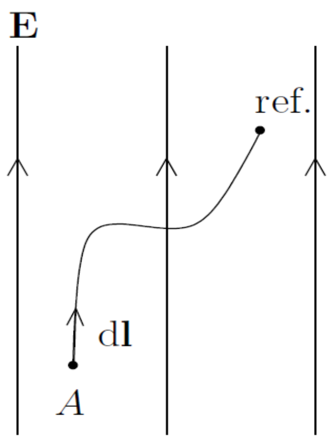
$$\mathbf{E} = \int_C \frac{\hat{\mathbf{R}}Q' \mathrm{d}l}{4\pi\epsilon_0 R^2}.$$

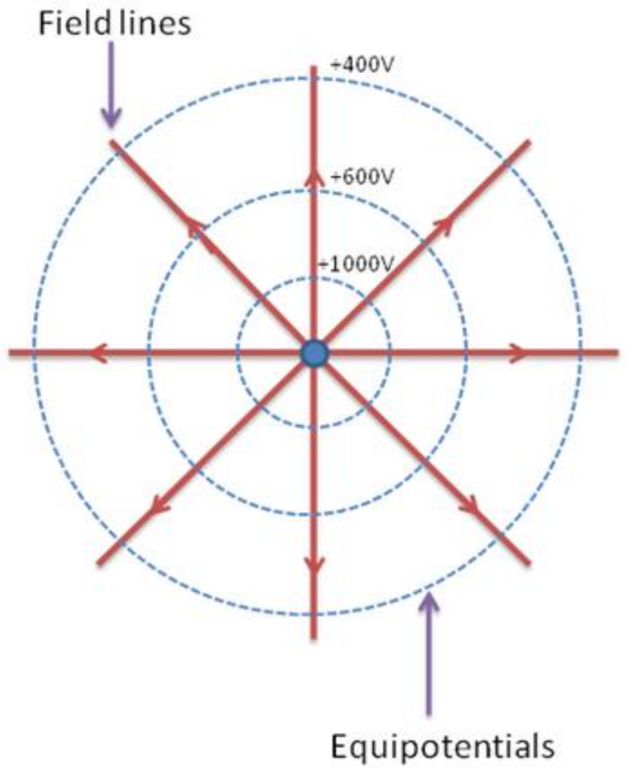
Electric potential

Electric potential definition $V = \frac{W}{q} = \frac{\int \boldsymbol{F.dl}}{q} = -\int \boldsymbol{E.dl}$

Vector: $\vec{\boldsymbol{F}} = \frac{qQ}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}}$: $\boldsymbol{Coulomb's\ law}$

Vector: $\vec{\boldsymbol{E}} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}}$





$V_A = \int_A^{\text{ref.}} \mathbf{E} \cdot d\mathbf{l}.$ The potential difference between two point A and B:

$$V_{AB} = -\int_{rA}^{rB} \boldsymbol{E}.d\boldsymbol{l} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\varepsilon_0 {r_i}^2} dr_i = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$

Electric field and potential (voltage)

- Potential is a scalar.
- $V = -\int \boldsymbol{E}.\boldsymbol{dl}$

$$\nabla \boldsymbol{f} = \frac{\partial f}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}}$$

$$\mathbf{E} = -\nabla V, \ \mathrm{V/m}$$

$$\nabla V \equiv \mathrm{grad}\, V$$

$$V = \int_{\boldsymbol{r}}^{\infty} \boldsymbol{E} d\boldsymbol{l} = \int_{\boldsymbol{r}}^{\infty} \frac{Q}{4\pi\varepsilon_0 l^2} d\boldsymbol{l} = \frac{Q}{4\pi\varepsilon_0 r}$$

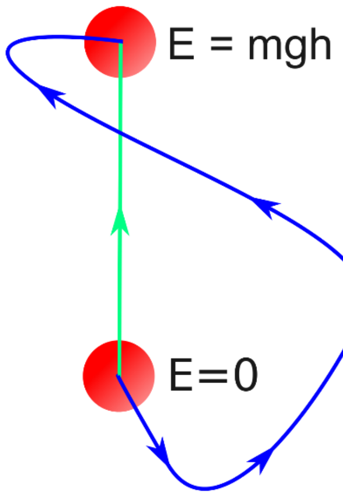
$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$

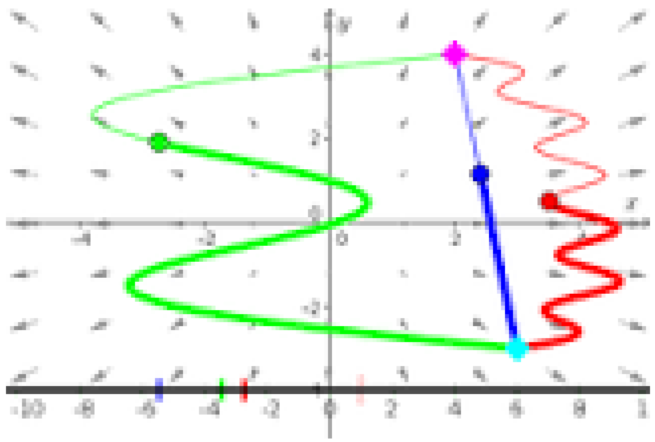
Conservative vector field: stationary electric field

Stationary Electric field is a conservative vector.

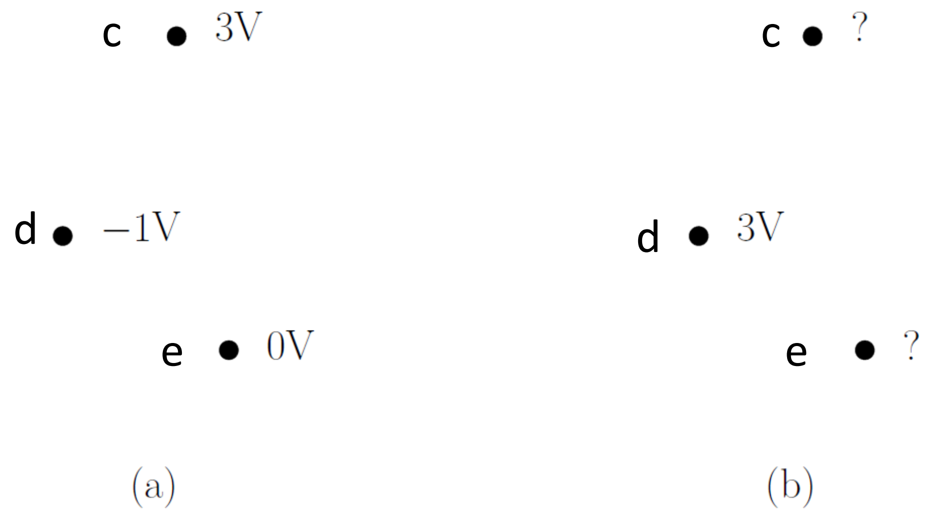
. Conservative vector fields have the property that the line integral is path independent.

. A conservative vector field is also Ir-rotational. In three dimensions , it has vanishing curl. $\nabla \times f$=0.





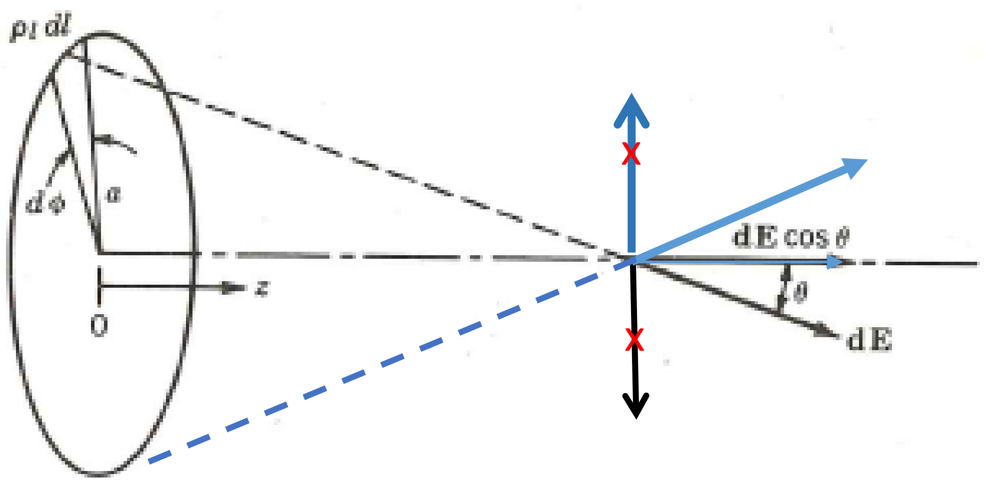
Example: find out the uknown potentials



in a stationaly electric field, the potential at each point is listed in a)
in the same stationary field and the potential at point d is 3V with another reference point. what are the potential that are missed in b).

Example: field of a ring of charge

Charge ring with radius *a* and line charge density: ρl
Let calculate E at pont on the z axis.



$$\vec{E} = \sum_{i=1}^{n} \frac{q_i}{4\pi\varepsilon_0 r_i^2}\hat{r}_i$$

$$\sum_{i=1}^{n} q_i = \oint \rho_l dl = \int_0^{2\pi a} \rho_l a d\phi$$

$dl = a d\phi$

$$E = \int_0^{2\pi a} \rho_l a d\phi \frac{.\cos\theta}{4\pi\varepsilon r^2} = \int_0^{2\pi a} \frac{\rho_l z a \; d\phi}{4\pi\varepsilon (a^2+z^2)^{3/2}} = \frac{\rho_l a z}{2\varepsilon (a^2+z^2)^{3/2}}$$

$r^2 = a^2 + z^2$

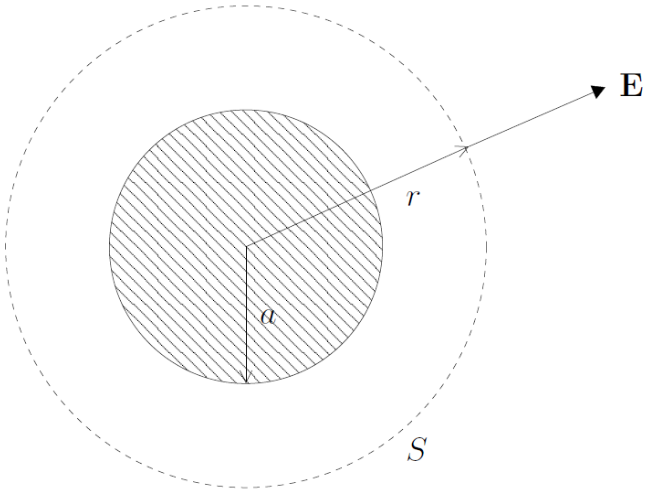
$\cos\theta = z \Big/ (a^2+z^2)^{\frac{1}{2}}$

The resultant E at z-axis is also in the direction of z-axis.

Example: electric field around a charged ball

What is the electric field outside a charged ball with total charge Q? (r>>a)

$$\text{dvs. } \mathbf{E} = E\hat{\mathbf{r}}$$



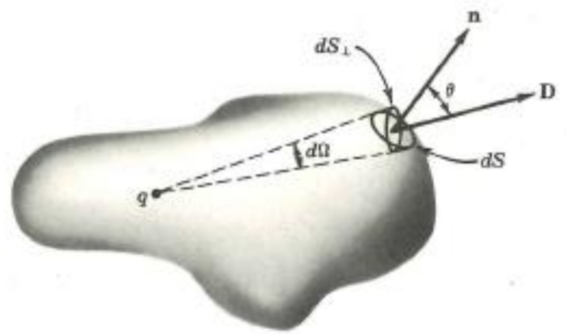
Vector: $\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}, \quad$ (r>>a)

Gauss' law

Electric flux flowing out of a closed surface = Total charge enclosed divided by the permittivity $\oint Dds = Q$ **Vector:** $\vec{E} = \frac{Q}{4\pi r^2 \varepsilon_0}$ in vacuum, $\vec{D} = \frac{Q}{4\pi r^2}$

Electric displacement field D and D= εE.

D is independent of medium:

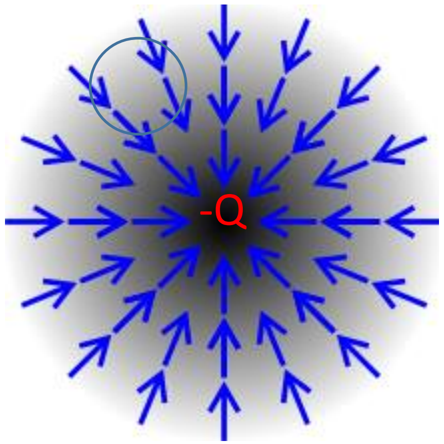


- $\nabla \cdot E = \frac{\rho}{\varepsilon}$
- $\nabla \cdot D = \rho$

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{E} dV.$$

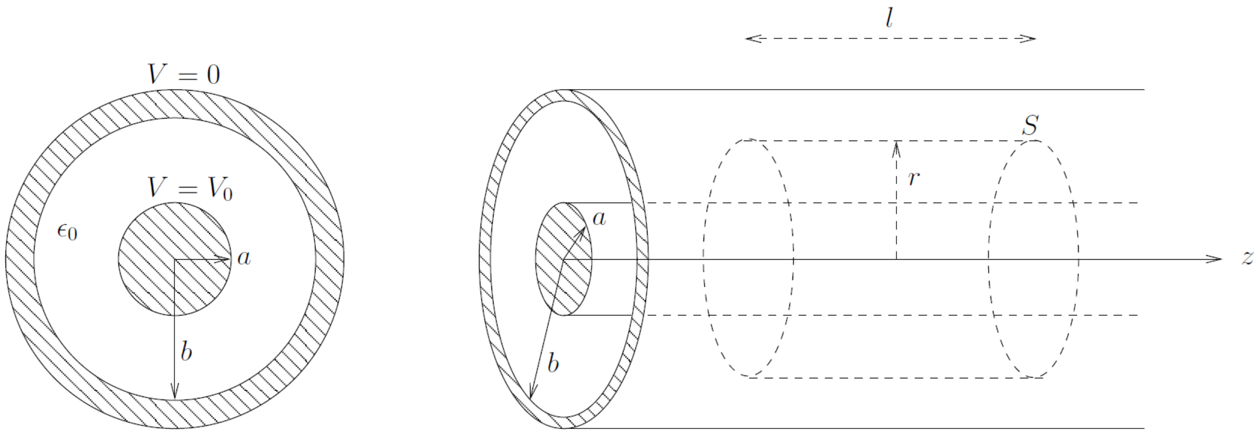
$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \sum_i \oint_{S_i} \mathbf{E} \cdot d\mathbf{S} = \sum_i \left(\frac{1}{\Delta V_i} \oint_{S_i} \mathbf{E} \cdot d\mathbf{S}_i\right) \Delta V_i \to \int \nabla \cdot \mathbf{E} dV.$$

$$\epsilon_0 \oint_S \mathbf{E} \cdot d\mathbf{S} = Q_{\text{total i } S}$$



$$\oint E ds = \oint \frac{-Q}{4\pi r^2 \varepsilon_0} ds = \frac{-Q}{\varepsilon_0}$$

Example:coaxial cable



Assumming : infinite long cable
Calculating electric field E between the out surface of inner cable and inner surface of the outer cable.

Assuming Unit length and the surface charge density is ρ_s

$$\epsilon_0 \oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \epsilon_0 E 2\pi r l$$

Circumference $\rho_s 2\pi a l = Q' l = Q$

$$\mathbf{E} = \begin{cases} \frac{Q'}{2\pi\epsilon_0 r}\hat{\mathbf{r}}, & \text{for } a < r < b \\ 0, & \text{ellers.} \end{cases}$$

$$V_0 = \int_a^b \mathbf{E} \cdot \mathrm{d}\mathbf{l} = \int_a^b E \mathrm{d}r = \frac{Q'}{2\pi\epsilon_0} \int_a^b \frac{\mathrm{d}r}{r} = \frac{Q'}{2\pi\epsilon_0} \ln \frac{b}{a}.$$

$$\mathbf{E} = \begin{cases} \frac{V_0}{r \ln \frac{b}{a}}\hat{\mathbf{r}}, & \text{for } a < r < b \\ 0, & \text{ellers.} \end{cases}$$

Poisson equation

Inserting $\mathbf{E} = -\nabla V$ into Maxwell's equation $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ gives

$$-\nabla \cdot (\nabla V) = \frac{\rho}{\epsilon_0},$$

which gives

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}.$$

This is called Poisson's equation. Here

$$\nabla^2 = < \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2} >$$

Point, line, surface and space charge

Potential from point/line/surface/space charge

We let $\mathbf{r}_{\text{ref}} \to \infty$. We list the potentials from

a) A point charge:

$$V = \int_R^{\infty} \frac{Q}{4\pi\epsilon_0 r^2} \mathrm{d}r = \frac{Q}{4\pi\epsilon_0 R},$$

b) A line charge (line charge density Q'):

$$V = \int_C \frac{Q'}{4\pi\epsilon_0 R} \mathrm{d}l,$$

c) A surface charge (surface charge density ρ_s):

$$V = \int_S \frac{\rho_s}{4\pi\epsilon_0 R} \mathrm{d}S,$$

d) A space charge (charge density ρ):

$$V = \int_V \frac{\rho}{4\pi\epsilon_0 R} \mathrm{d}V.$$

The last three is found by superposition from the point charge expression.

Electric field in dielectric

Force in vaccum: $$\vec{F}_{tot} = \sum_{i=1}^{n} \frac{qq_i}{4\pi\varepsilon_0 r_i^2}\hat{r}_i$$

Vector: $\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$

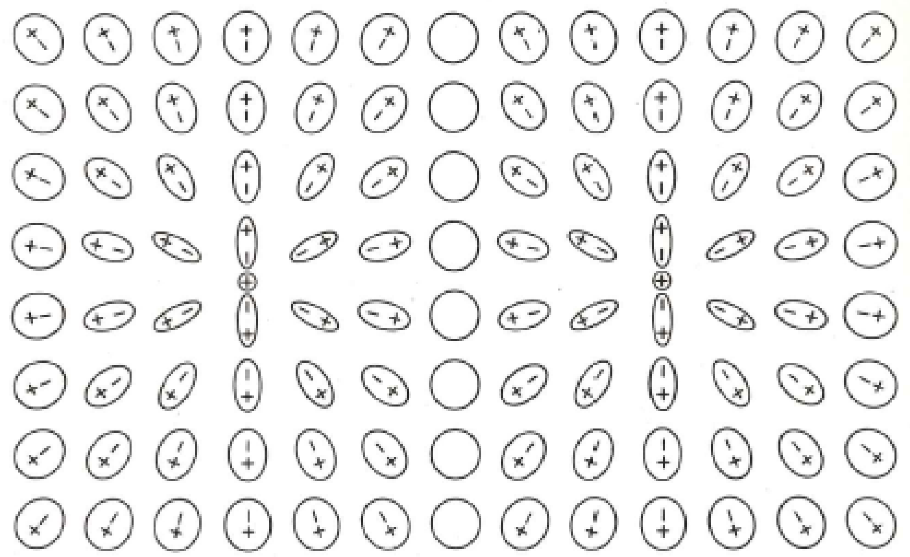


Fig. 1.3c Polarization of the atoms of a dielectric by a pair of equal positive charges.

In dielectric media, the force between charges depends upon the presence of media/ dielectric:

1: Once electric filed exists in dielectric media, the atoms are distorted or polarized, similar distortions can occur in molecules

2: Electric polarization produced by the electric field in a material depends upon material properties.

3: Polarization influence the force between charges.

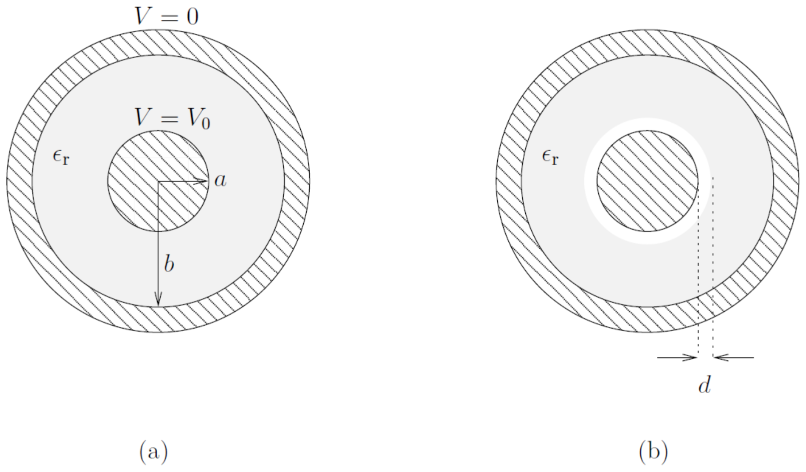
Electric polarization and material permittivity

The influence of electric polarization: $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$

- $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$
- $\boldsymbol{P} = \varepsilon_0 \chi_e \boldsymbol{E}$
- $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \varepsilon_0 \chi_e \boldsymbol{E} = (1 + \chi_e) \varepsilon_0 \boldsymbol{E}$ **:** χ_e *called* ***electric susceptibility***

- $(1 + \chi_e) = \varepsilon_r$ relative permittivity:
- $\varepsilon = \varepsilon_r \varepsilon_0$ electric permittivity .(dielectric material property)

Example: coaxial cable with dielectric material and airgap



Assumming : infinite long cable
Calculating electric field E between the out surface of inner cable and inner surface of the outer cable.

a) The permittivity is $\epsilon = \epsilon_r \epsilon_0$

b) when
$$\epsilon(r) = \begin{cases} \epsilon_0, & \text{for } a < r < a + d, \\ \epsilon_r \epsilon_0, & \text{for } a + d < r < b. \end{cases}$$

a) D is material independant:
$$\mathbf{D} = \frac{Q'}{2\pi r}\hat{\mathbf{r}} \quad \text{for } a < r < b,$$

$$\mathbf{E} = \frac{Q'}{2\pi \epsilon r}\hat{\mathbf{r}} \quad \text{for } a < r < b.$$

$$\mathbf{E} = \frac{V_0}{r \ln \frac{b}{a}}\hat{\mathbf{r}} \quad \text{for } a < r < b.$$

b) D is material independant:

$$\epsilon(r) = \begin{cases} \epsilon_0, & \text{for } a < r < a + d, \\ \epsilon_r \epsilon_0, & \text{for } a + d < r < b. \end{cases}$$

$$V_0 = \int_a^b E(r)\mathrm{d}r = \frac{Q'}{2\pi}\int_a^b \frac{\mathrm{d}r}{\epsilon(r) r} = \frac{Q'}{2\pi\epsilon_0}\left(\ln\frac{a+d}{a} + \frac{1}{\epsilon_r}\ln\frac{b}{a+d}\right)$$

$$\mathbf{E} = \begin{cases} \frac{\epsilon_r V_0}{r\left(\epsilon_r \ln\frac{a+d}{a} + \ln\frac{b}{a+d}\right)}\hat{\mathbf{r}}, & \text{for } a < r \leq a + d \\ \frac{V_0}{r\left(\epsilon_r \ln\frac{a+d}{a} + \ln\frac{b}{a+d}\right)}\hat{\mathbf{r}}, & \text{for } a + d < r < b. \end{cases}$$