Lecture 6: electro-magnetic wave

- Electric potential in electro-dynamic
- Electro-magnetic wave
- Energy stored in electro-magnetic field and Poynting’s theorem.

Electric potential in electric-dynamic field

In stationary electric field, conservative E field: $\nabla \times \mathbf{E} = 0$ $V = \oint \mathbf{E} \cdot d\mathbf{l} = 0$

A scaler, magnetic potential, based on $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Electro-dynamic field

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) = \nabla \times (-\frac{\partial \mathbf{A}}{\partial t}) \Rightarrow \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0$$

Stationary electric field:

$$\nabla \times \mathbf{E} = 0$$

$$\mathbf{E} = -\nabla V$$

Electro-dynamic field:

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0$$

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V$$

Maxwell's equations in vacuum

$$\nabla \cdot \mathbf{E} = \rho/\varepsilon$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \boldsymbol{B} = \mu \mathbf{J} + \frac{\mu \partial \mathbf{D}}{\partial t} = \mu \mathbf{J} + \frac{\mu \varepsilon \partial \mathbf{E}}{\partial t}$$

In vacuum

$$\nabla \cdot \mathbf{E} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t} \nabla \times \mathbf{B} = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$
$$\nabla \times (\nabla \times \mathbf{B}) = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \nabla \times \mathbf{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

Vector identity:

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla (\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}$$

In vacuum

$$\nabla \cdot \mathbf{E} = 0$$
$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla^2 \boldsymbol{V} = \nabla \cdot \nabla \boldsymbol{V} = \mathrm{div}(\mathrm{grad}\, \boldsymbol{V})$$

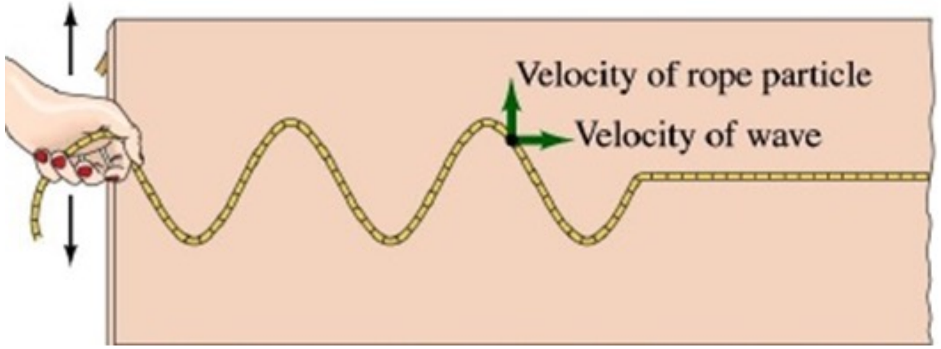
$$\frac{1}{c_0^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = \mathbf{0}$$
$$\frac{1}{c_0^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = \mathbf{0}$$

$$c_0 = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

Typical wave equation in one dimension

The wave equation for a plane wave traveling in the x direction is

$$\nabla^2 \mathbf{E}(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{E}(\mathbf{r}, t) = 0$$



where *v* is the phase velocity of the wave and *u* represents the variable which is changing as the wave passes. This is the form of the wave equation which applies to a stretched string.

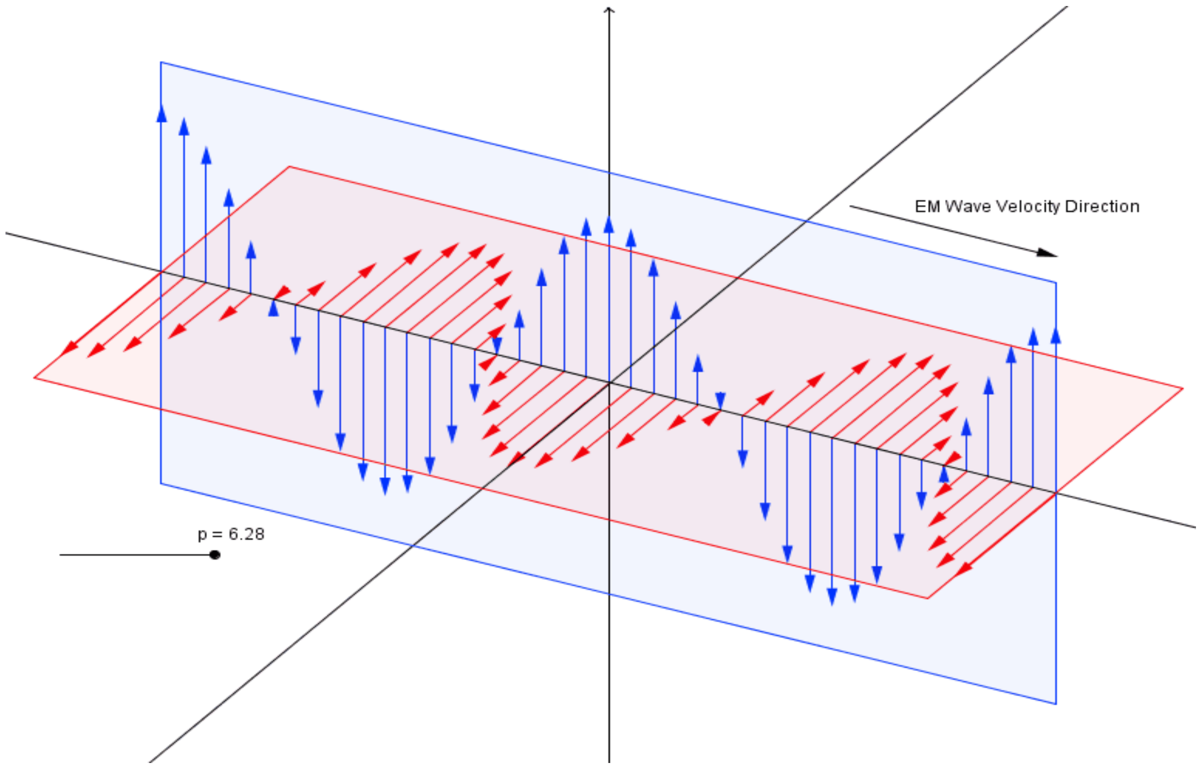
$$\frac{\partial^2 E_y}{\partial x^2} = \epsilon_0 \mu_0 \frac{\partial^2 E_y}{\partial t^2}$$

$$c_0 = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 299{,}792{,}500 \text{ m/s}$$

Electro-magnetic wave in 3D

$$\frac{1}{c_0^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = \mathbf{0}$$

$$\frac{1}{c_0^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = \mathbf{0}$$



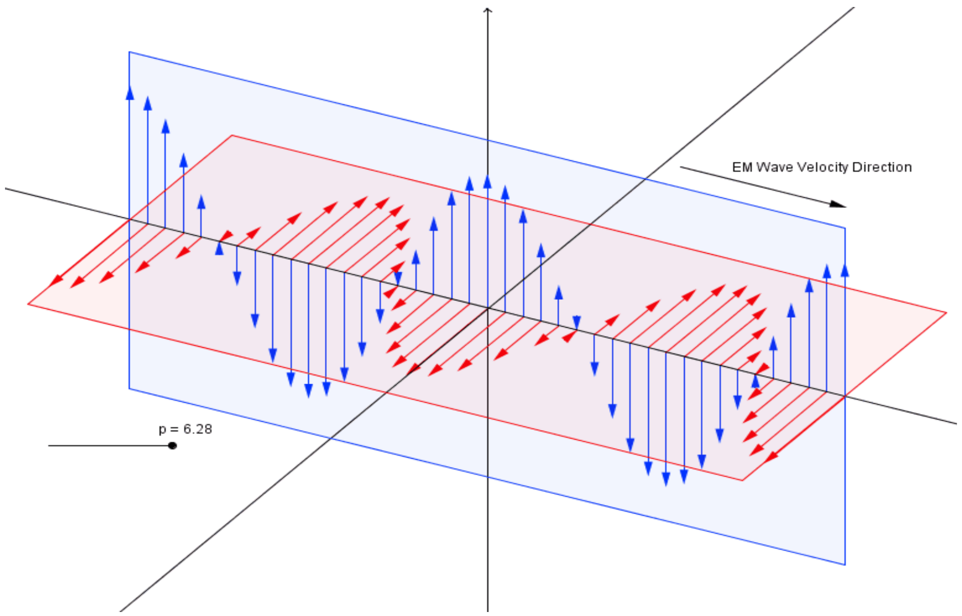
Three dimension wave:

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = 0$$

$$c_0 = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 299{,}792{,}500 \text{ m/s}$$

Electro-magnetic radiation

Electromagnetic radiation consists of electromagnetic waves, which are synchronized oscillations of electric and magnetic fields that propagate at the speed of light through a vacuum. The oscillations of the two fields are perpendicular to each other and perpendicular to the direction of energy and wave propagation, forming a transverse wave.



Poynting's theorem: Power flow in electromagnetic fields

The energy transfer due to time-varying electric and magnetic fields is perpendicular to the fields

$$W = -\oint (\boldsymbol{E} \times \boldsymbol{H}) \cdot d\boldsymbol{s}$$

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{A} dv$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$-\nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) = \boldsymbol{E} \cdot \nabla \times \boldsymbol{H} - \boldsymbol{H} \cdot \nabla \times \boldsymbol{E} = \boldsymbol{J} \cdot \boldsymbol{E} + \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t}$$

$$\int_v \left(\boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{E} \cdot \boldsymbol{J} \right) dv = -\oint_S (\boldsymbol{E} \times \boldsymbol{H}) \cdot d\boldsymbol{s}$$

Either power loss or power required to accelerate charges

$$\frac{1}{2}\frac{\partial (\boldsymbol{B} \cdot \boldsymbol{H})}{\partial t} = \frac{1}{2}\frac{\partial (\mu \boldsymbol{H}^2)}{\partial t} = \boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t}$$

Energy density stored in magnetic field

$$\frac{1}{2}\frac{\partial (\boldsymbol{D} \cdot \boldsymbol{E})}{\partial t} = \frac{1}{2}\frac{\partial (\varepsilon \boldsymbol{E}^2)}{\partial t} = \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t}$$

Energy density stored in electric field

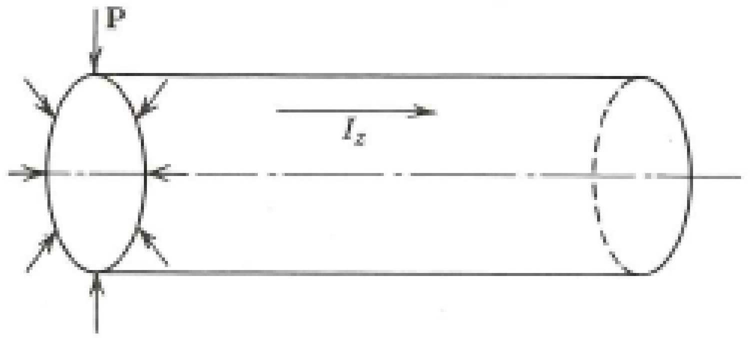
Example:

One conductor with DC current I_z, if R is the resistane per unit length.

Calculating the loss over the conductor in unit length.

$$E_z = I_z R$$

$$H_\phi = \frac{I_z}{2\pi r}$$



Poynting vector directed radially inward

$$P_r = -E_z H_\phi = -\frac{R I_z^2}{2\pi r}$$

P has no component normal to the end surfaces

$$W = 2\pi r(-P_r) = I_z^2 R$$

Surface integral over the conductor

Example:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (1)$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \quad (2)$$

$$\nabla \cdot \mathbf{D} = \rho, \quad (3)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (4)$$

Use (3) to show Gauss' law on integral form, i.e,

$$\oint_S \mathbf{D} \cdot \mathrm{d}\mathbf{S} = Q_{\text{free in } S}.$$

Additionally find the field from a point charge Q located in the origin. Show that force acting on a small charge q with a distance r, is given by Coulomb's law:

$$\mathbf{F} = \frac{Qq}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}},$$

where $\hat{\mathbf{r}}$ is a unit vector in the $\mathbf{r}$-direction.

$$\int_v \nabla \cdot \mathbf{D} \mathrm{d}v = \oint_S \mathbf{D} \cdot \mathrm{d}\mathbf{S}$$

$$\oint_S \mathbf{D} \cdot \mathrm{d}\mathbf{S} = \int_v \rho \mathrm{d}v$$

$$\int_v \rho \mathrm{d}v = Q = \oint_S \mathbf{D} \cdot \mathrm{d}\mathbf{S} = \oint_S D \mathrm{d}S = 4\pi r^2 D$$

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon_0} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$$

$$\mathbf{F} = \frac{qQ}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}.$$

Example

Assume stationary conditions. Show that the scalar potential V in a linear, isotropic and homogeneous material satisfies Poisson's equation

$$\nabla^2 V = -\frac{\rho}{\epsilon}.$$

$$\mathbf{E} = -\nabla V$$

$$\nabla \cdot \mathbf{D} = \epsilon \nabla \cdot \mathbf{E} = \rho.$$

$$\epsilon \nabla \cdot (-\nabla V) = \rho,$$

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}.$$

Summary

Maxwell's Equations:

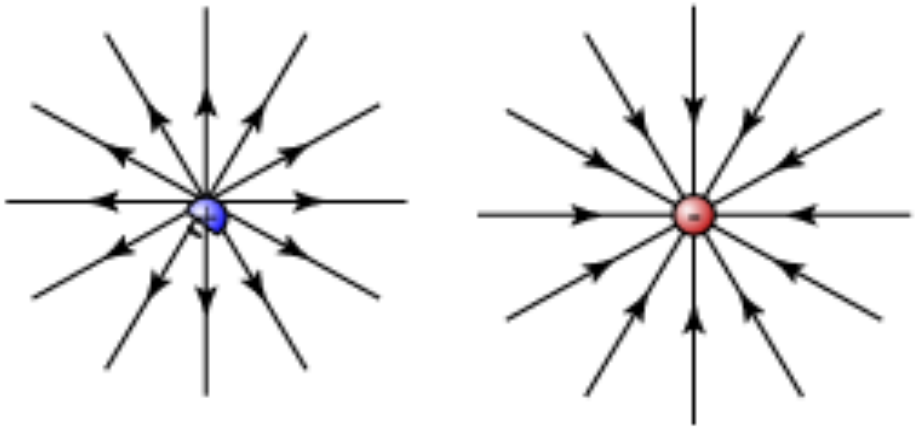
Name	Integral equations (SI convention)	Differential equations (SI convention)
Gauss's law	$\oint\!\!\!\oint_{\partial\Omega} \mathbf{D} \cdot \mathrm{d}\mathbf{S} = \iiint_{\Omega} \rho_{\mathrm{f}} \, \mathrm{d}V$	$\nabla \cdot \mathbf{D} = \rho_{\mathrm{f}}$
Ampère's circuital law (with Maxwell's addition)	$\oint_{\partial\Sigma} \mathbf{H} \cdot \mathrm{d}\boldsymbol{\ell} = \iint_{\Sigma} \mathbf{J}_{\mathrm{f}} \cdot \mathrm{d}\mathbf{S} + \frac{d}{dt} \iint_{\Sigma} \mathbf{D} \cdot \mathrm{d}\mathbf{S}$	$\nabla \times \mathbf{H} = \mathbf{J}_{\mathrm{f}} + \frac{\partial \mathbf{D}}{\partial t}$
Gauss's law for magnetism	$\oint\!\!\!\oint_{\partial\Omega} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0$	$\nabla \cdot \mathbf{B} = 0$
Maxwell–Faraday equation (Faraday's law of induction)	$\oint_{\partial\Sigma} \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} = -\frac{d}{dt} \iint_{\Sigma} \mathbf{B} \cdot \mathrm{d}\mathbf{S}$	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$

Electric field produced by charges

Electric field: $\vec{\boldsymbol{E}} = \frac{\boldsymbol{Q}}{\boldsymbol{4\pi\varepsilon_0 r^2}}\hat{\boldsymbol{r}}$ in vacuum

Electric displacement field **D** and **D**= $\varepsilon_0 \boldsymbol{E}$.

Coulomb's law: $\vec{\boldsymbol{F}} = \frac{\boldsymbol{qQ}}{\boldsymbol{4\pi\varepsilon_0 r^2}}\hat{\boldsymbol{r}}$



Field pattern of
a pointed electrode

Operators

$$\nabla = \frac{\partial}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial}{\partial z}\hat{\boldsymbol{z}}$$

$$\nabla^2 D = \nabla \cdot \nabla D = \frac{\partial^2 D}{\partial x^2} + \frac{\partial^2 D}{\partial y^2} + \frac{\partial^2 D}{\partial z^2}$$

$$\nabla D = \frac{\partial D}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial D}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial D}{\partial z}\hat{\boldsymbol{z}}$$

Vector identity:

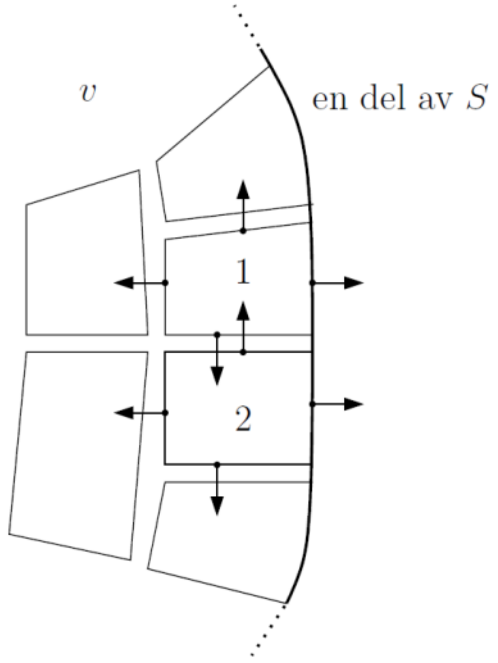
$$\nabla \times (\nabla \times \mathbf{V}) = \nabla (\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}$$

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

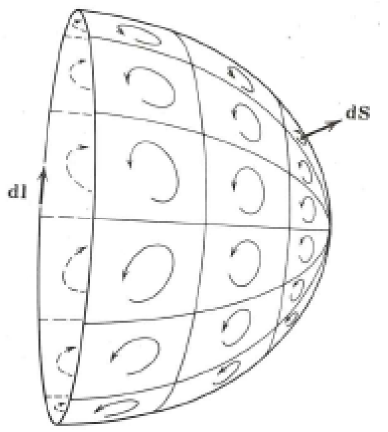
$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\boldsymbol{z}}$$

Divergence theorem and Stokes' theorem

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{A} dv$$



$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{A} \cdot d\mathbf{S}.$$



Electric field and potential (voltage)

- Potential is a scalar.
- $V = -\int \boldsymbol{E}.\boldsymbol{dl}$

$$\boxed{\mathbf{E} = -\nabla V, \ \mathrm{V/m}}$$

$$\nabla V \equiv \mathrm{grad}\, V$$

Poisson equation

Inserting $\mathbf{E} = -\nabla V$ into Maxwell's equation $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ gives

$$-\nabla \cdot (\nabla V) = \frac{\rho}{\epsilon_0},$$

which gives

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}.$$

Gauss's law

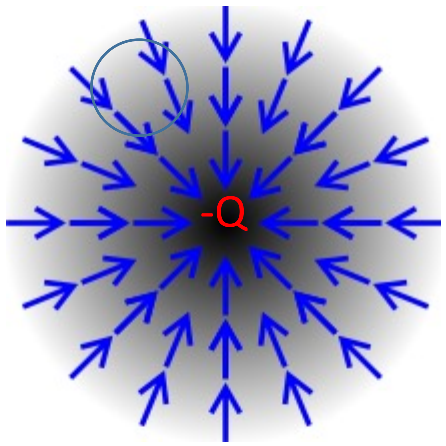
Electric flux flowing out of a closed surface = Enclosed total charges divided by the permittivity

Gauss's law

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon}$$

$$\nabla \cdot \mathbf{D} = \rho$$

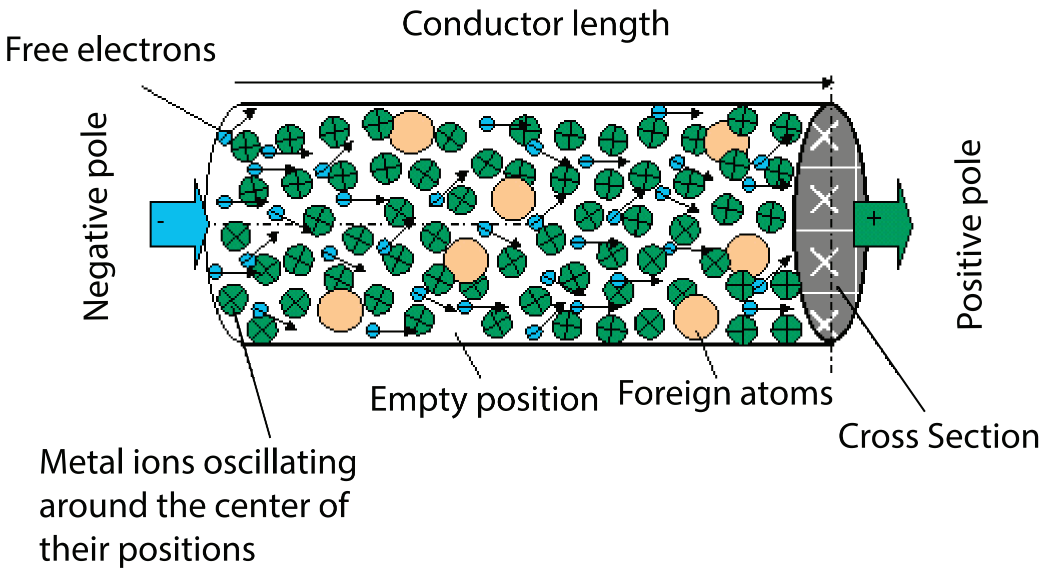
$$\mathbf{D} = \varepsilon \mathbf{E}$$

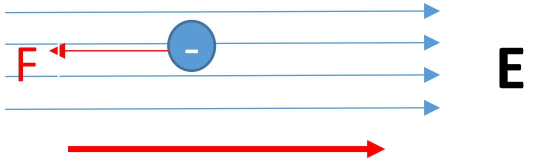


$$\oint \boldsymbol{E} d\boldsymbol{s} = \oint \frac{-Q}{4\pi r^2 \varepsilon_0} d\boldsymbol{s} = \frac{-Q}{\varepsilon_0}$$

Current and electric field

Free charges move along electric field direction





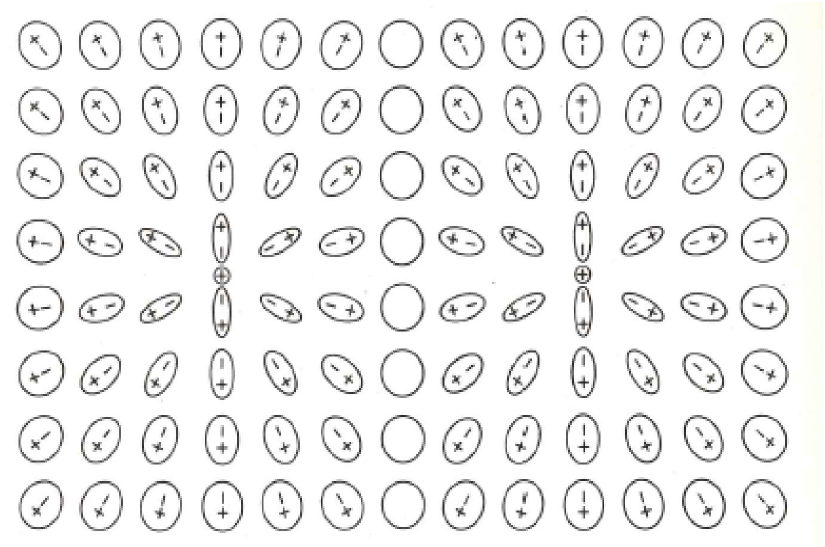
$$\vec{J} = \sigma \vec{E}$$

Ohm's law

$$\vec{F} = q\vec{E}$$

Electric polarization and material permittivity

The influence of electric polarization: $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$



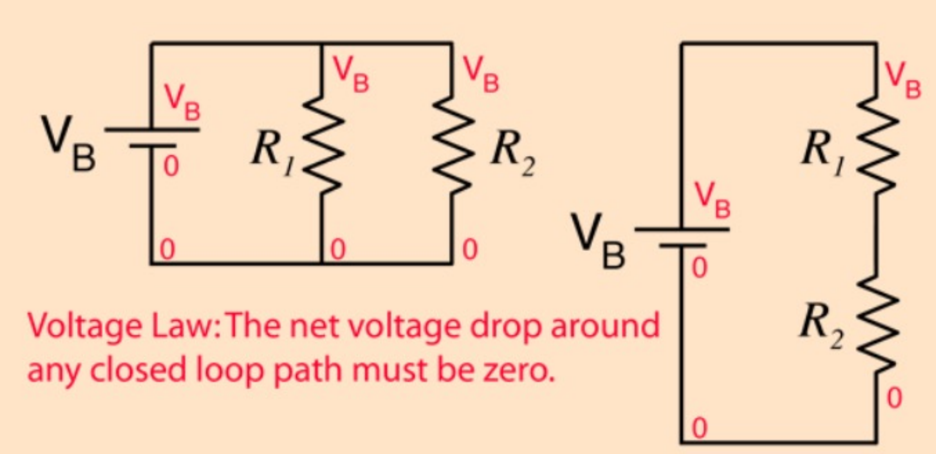
- $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$
- $\boldsymbol{P} = \varepsilon_0 \chi_e \boldsymbol{E}$
- $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \varepsilon_0 \chi_e \boldsymbol{E} = (1 + \chi_e) \varepsilon_0 \boldsymbol{E}$

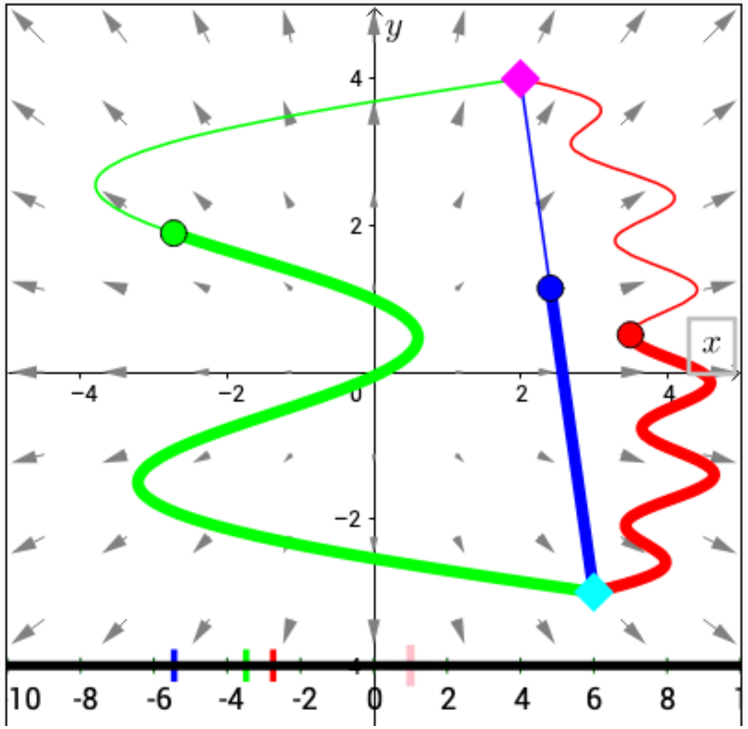
- $(1 + \chi_e) = \varepsilon_r$ relative permittivity
- $\varepsilon = \varepsilon_r \varepsilon_0$ electric permittivity (dielectric material property)

Conservative vector field: stationary electric field

Stationary Electric field is a conservative vector.

- Conservative vector fields have the property that the line integral is path independent.
- A conservative vector field is also irrotational. In three dimensions , it has vanishing curl. $\nabla\times\mathbf{E}=0$.
- $V = -\oint_c \mathbf{E}\cdot d\boldsymbol{l} = 0$





Boundary conditions in electrostatics

Electric field involves more than one materials

$$D_{n1}\,\Delta S - D_{n2}\,\Delta S = \rho_s\,\Delta S$$

Gauss's law:

$$D_{n1} - D_{n2} = \rho_s$$

Conservative field : electrostatic electric field

$$\oint \mathbf{E}\cdot d\mathbf{l} = E_{t1}\,\Delta l - E_{t2}\,\Delta l = 0$$

$$E_{t1} = E_{t2}$$

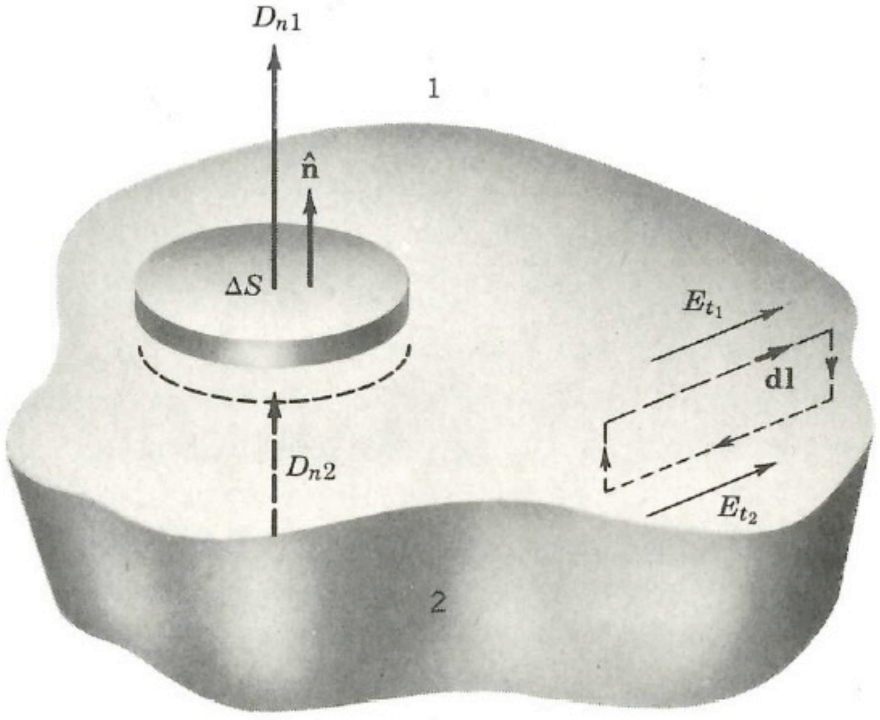
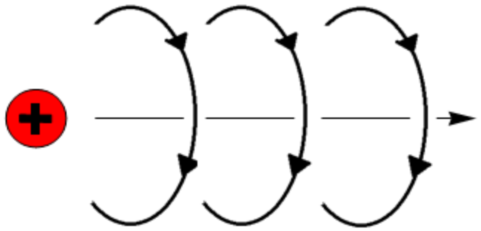


Fig. 1.14*a* Boundary between two different media.

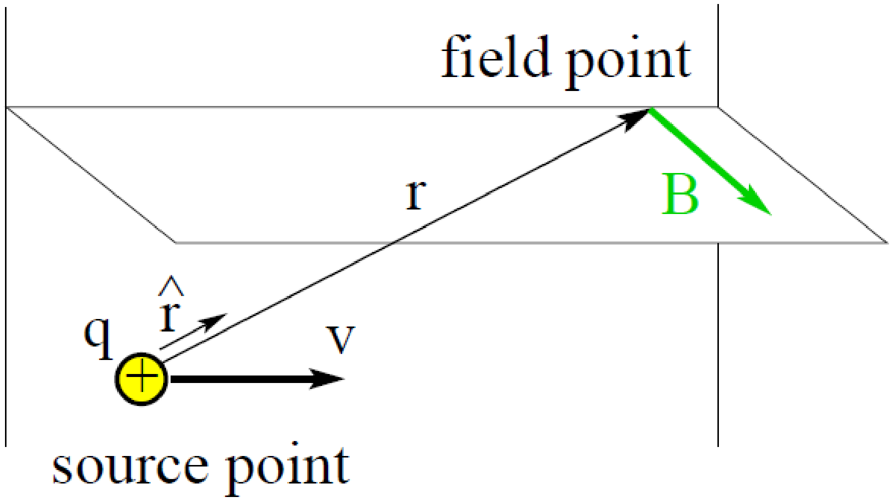
Charges in motion

Charges in *motion* (electrical current) produce a magnetic field .

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\frac{q\boldsymbol{v}\times\hat{\boldsymbol{r}}}{r^2}$$



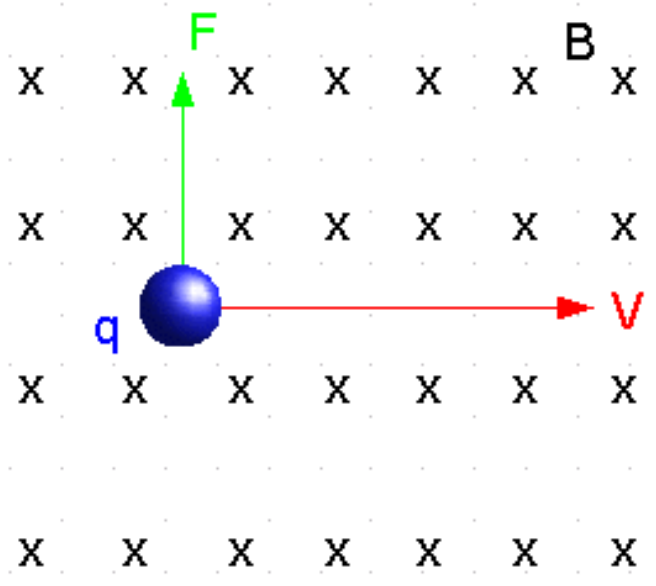
$$\boldsymbol{H} = \frac{1}{4\pi}\frac{q\boldsymbol{v}\times\hat{\boldsymbol{r}}}{r^2} = \frac{B}{\mu_0}$$



Electro-magnetic force

Force exerted by magnetic field **B** on a moving point charge Q is:

Lorentz law $$\mathbf{F} = q\mathbf{E} + q(\mathbf{v} \times \mathbf{B})$$

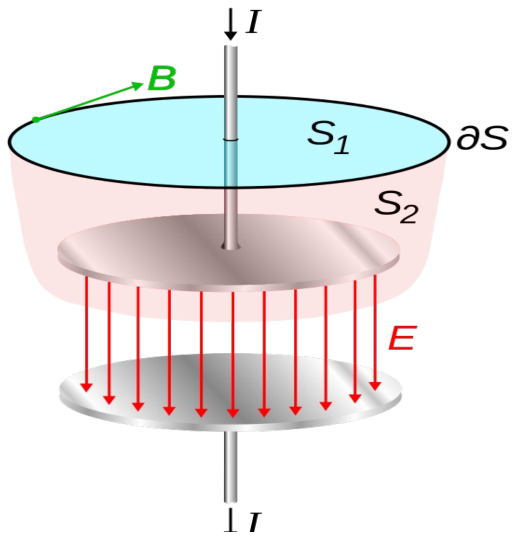


$$\boldsymbol{F} = Q\ \boldsymbol{v} \times \boldsymbol{B}$$

Magnetic force acting on a moving charge is always perpendicular to it's moving direction, so magnetic force does not work on the charges, but changing the charges' moving direction

Magnetostatic: Ampere's law and Kirchhoff's law

At magnetostatic: No change of electric field involved, $\frac{\partial D}{\partial t} = 0$



Ampere's law

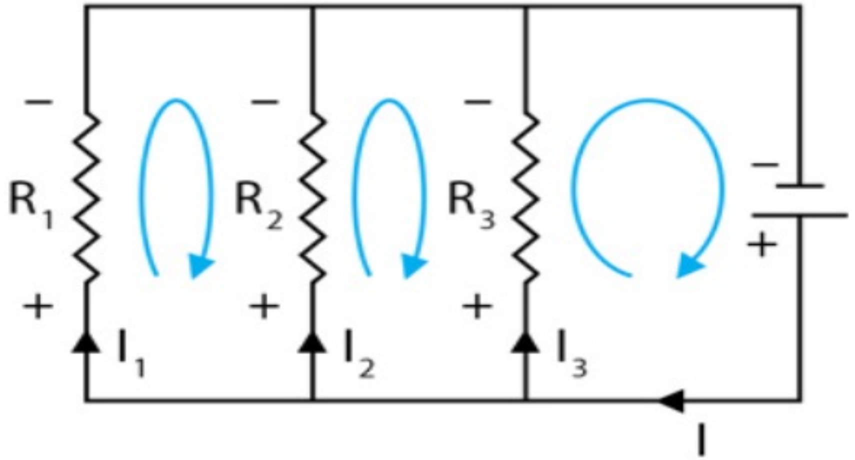
The integral of magnetic field around a closed curve is equal to the total electric current enclosed by the loop

$$\nabla \times \mathbf{H} = \mathbf{J} \qquad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$$

Kirchhoff's Law

At any point in an electric circuit, the sum of the current is zero.

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = 0.$$



$$I = I_1 + I_2 + I_3$$

Magnetization and material permeability

Magnetic field in material:

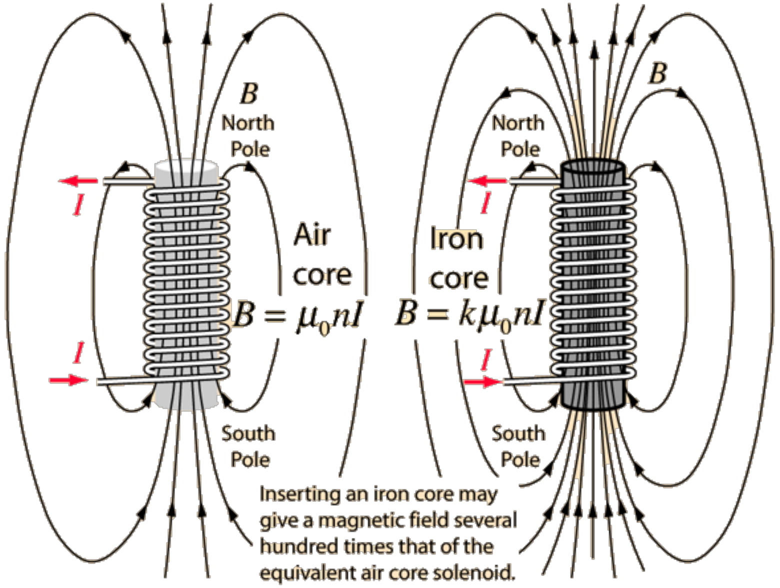
$$\mathbf{B} = \mu_0 \mathbf{H} \text{ in vacuum}$$

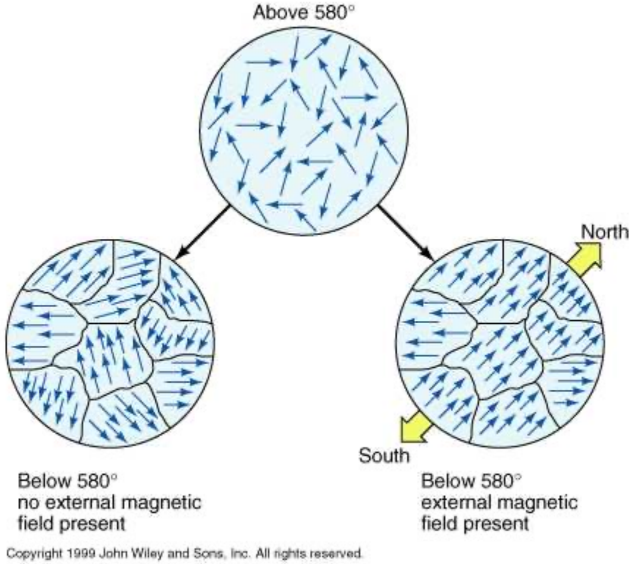
Once there is magnetic field applied to medium, the electronic spin motions in the atoms can be thought of as circulating current that produce a field **M**, magnetization, which adds to magnetic field **H**.

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$$

Magnetic susceptibility χ_m is used to quantify the additional field **M**.

$$\boldsymbol{B} = \mu_0(1+\chi_m)\boldsymbol{H} = \mu\boldsymbol{H} = \mu_0\mu_r\boldsymbol{H}$$





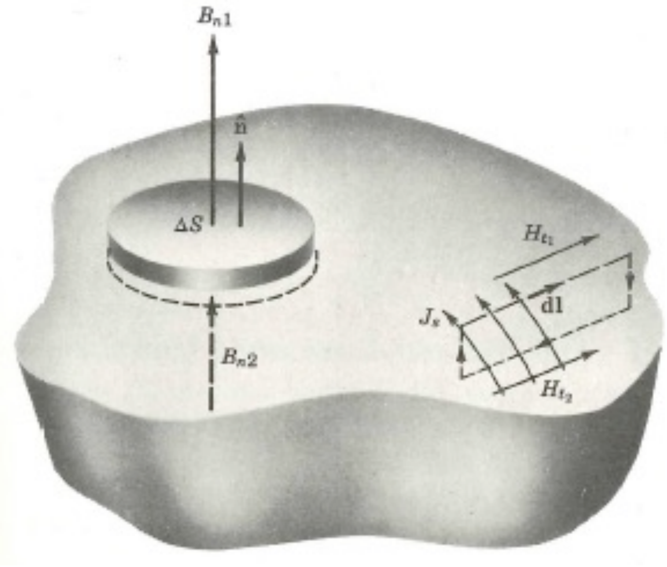
Boundary condition for static magnetic field

$$B_{n1}\,\Delta S = B_{n2}\,\Delta S$$

$$B_{n1} = B_{n2}$$

$$\oint \mathbf{H}\cdot d\mathbf{l} = H_{t1}\,\Delta l - H_{t2}\,\Delta l = J_s\,\Delta l$$

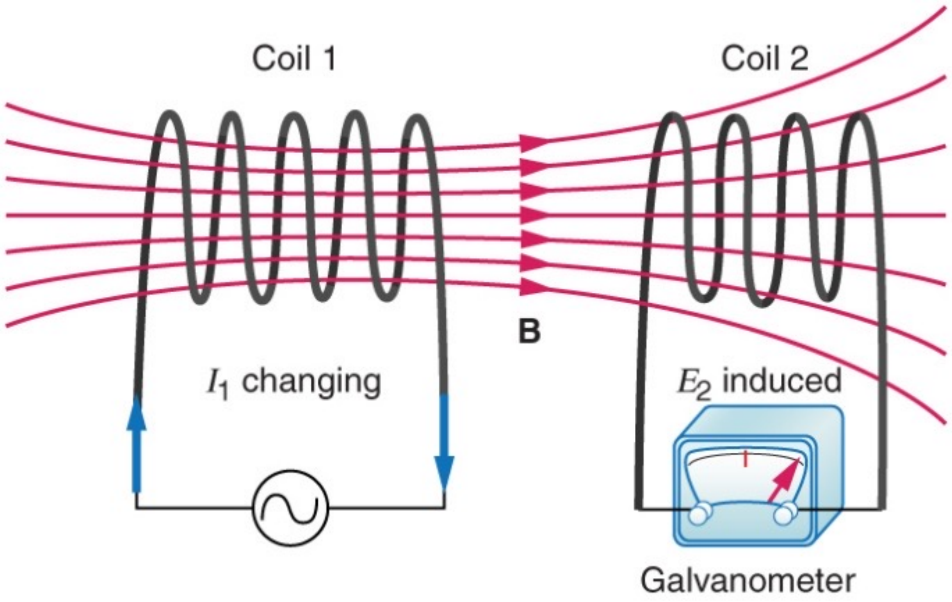
$$H_{t1} - H_{t2} = J_s$$



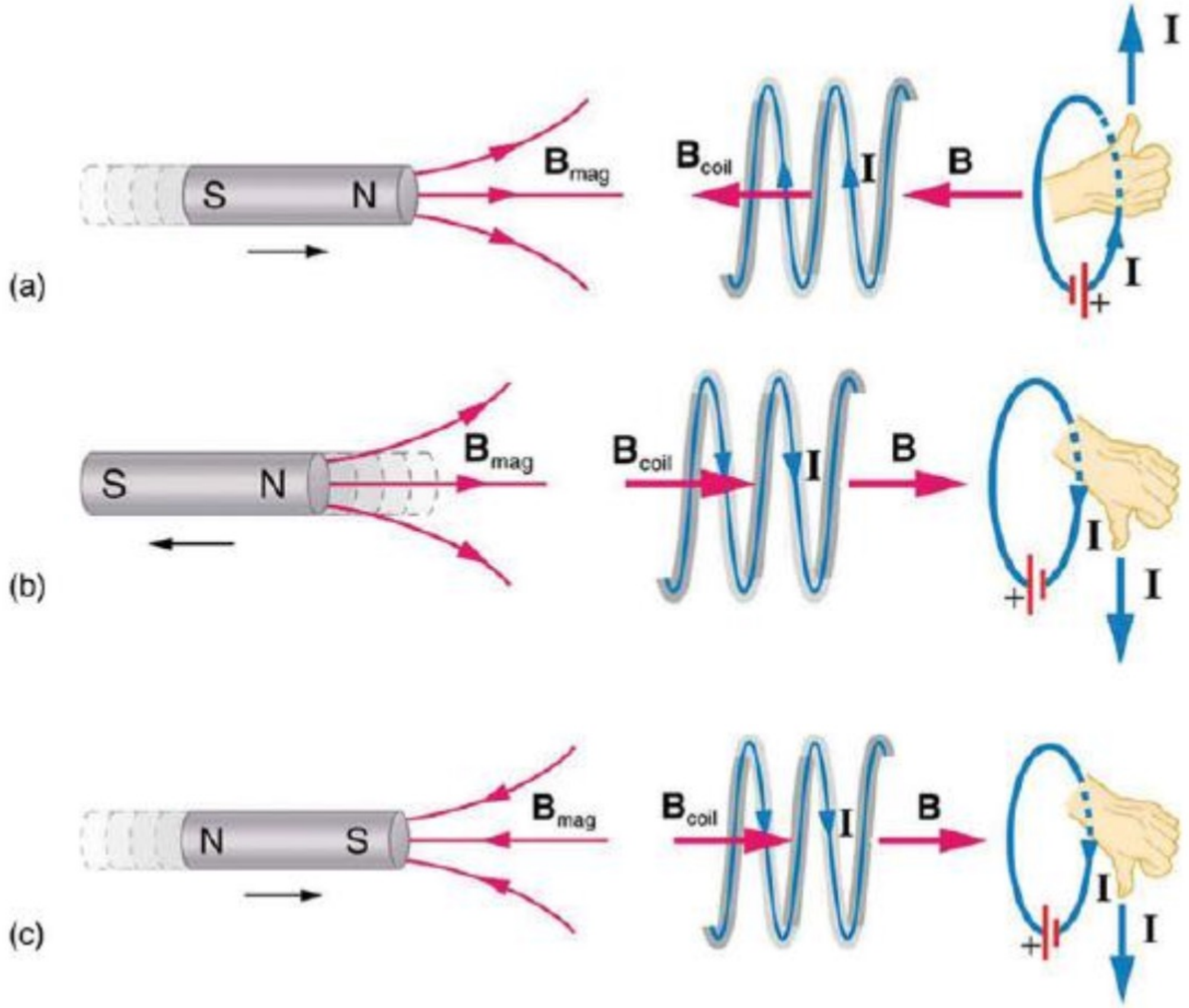
Faraday's law

$\boldsymbol{\varepsilon} = -\frac{\boldsymbol{d}\phi}{\boldsymbol{dt}}$ **Faraday's law,** $\phi = \int_{s} \boldsymbol{B} \cdot \boldsymbol{dS}$ **is the magnetic flux.**

$$\nabla \times \boldsymbol{E} = -\frac{\partial B}{\partial t}$$



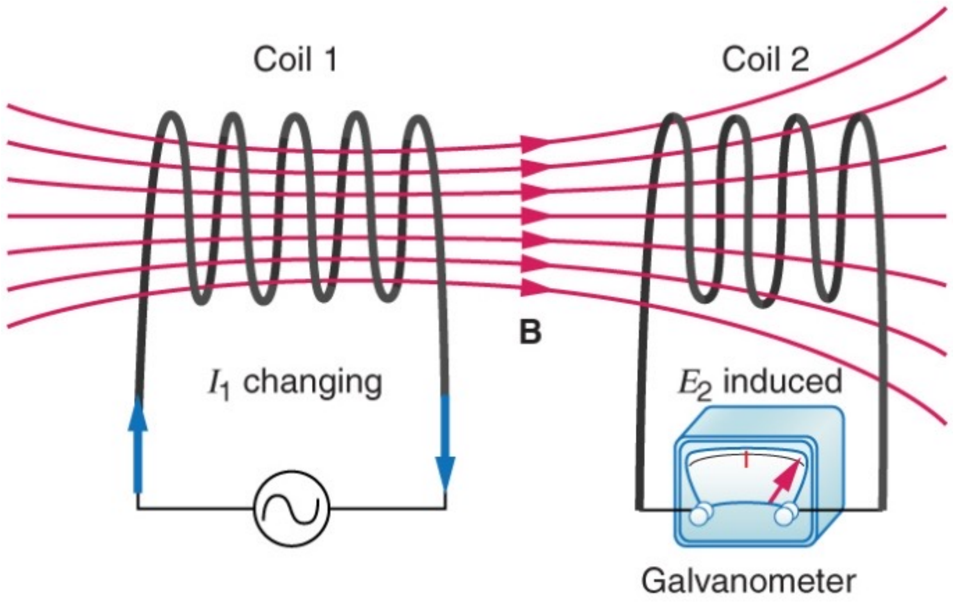
Lenz's law



Energy stored in inductance

$$L = \frac{\int_S \boldsymbol{B} \cdot d\boldsymbol{S}}{I}$$

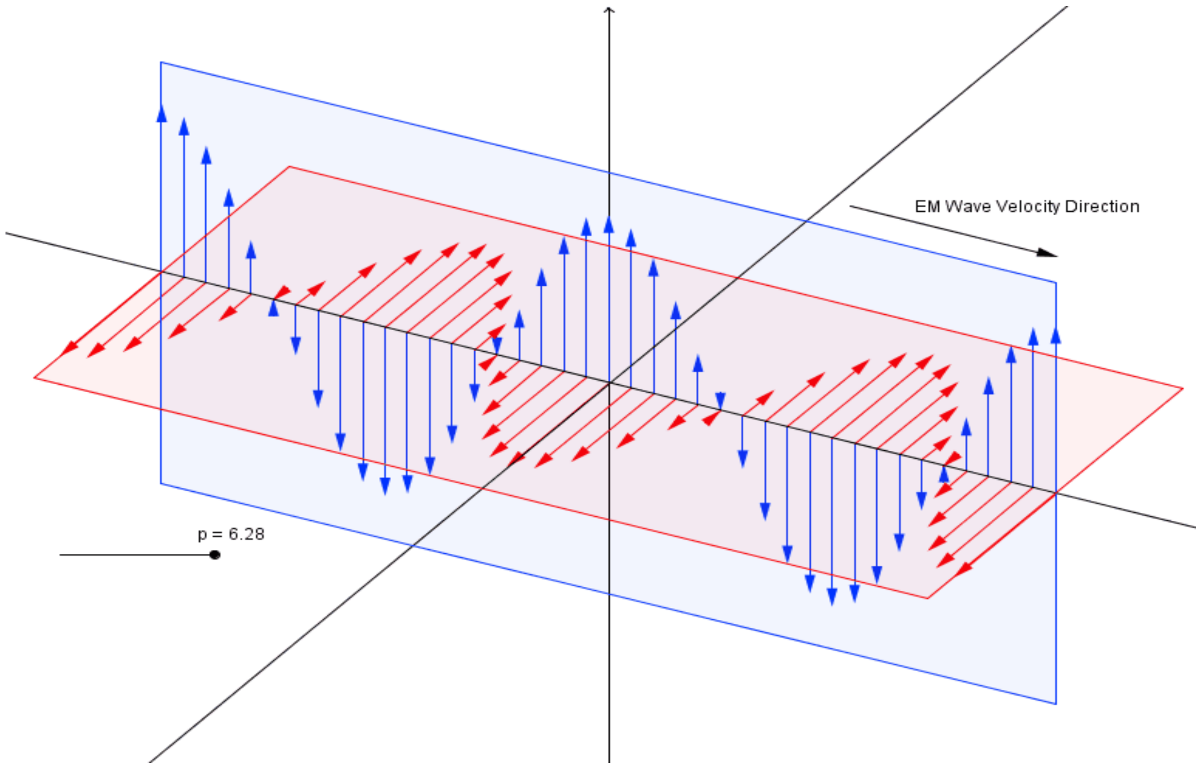
$$P = \int_0^I LI \frac{dI}{dt} dt = \int_0^I LIdI = \frac{1}{2} LI^2$$



Electro-magnetic wave in 3D

$$\frac{1}{c_0^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = \mathbf{0}$$

$$\frac{1}{c_0^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = \mathbf{0}$$



Three dimension wave:

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} - \mu\varepsilon\frac{\partial^2 E}{\partial t^2} = 0$$

$$c_0 = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 299{,}792{,}500 \text{ m/s}$$

Poynting's theorem

The energy transfer due to time-varying electric and magnetic fields is perpendicular to the fields

$$\mathbf{W} = -\oint (\boldsymbol{E} \times \boldsymbol{H}) \cdot d\boldsymbol{s}$$

$$-\nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) = \boldsymbol{E} \cdot \nabla \times \boldsymbol{H} - \boldsymbol{H} \cdot \nabla \times \boldsymbol{E} = \boldsymbol{J} \cdot \boldsymbol{E} + \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t}$$

$$\int_v \left(\boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{E} \cdot \boldsymbol{J} \right) dv = -\oint_s (\boldsymbol{E} \times \boldsymbol{H}) \cdot d\boldsymbol{s}$$

Either power loss or power required to accelerate charges

Energy density stored in magnetic field

Energy density stored in electric field

Maxwell's equations

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{H} = 0$$

$$\nabla \times \boldsymbol{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\mathbf{B} = \mu \mathbf{H}$$

$$\mathbf{D} = \varepsilon \mathbf{E}$$

$$\mathbf{J} = \sigma \mathbf{E}$$

Lorentz law

$$\mathbf{F} = q\mathbf{E} + q(\mathbf{v} \times \mathbf{B})$$

Thank you !