Lecture 6: electro-magnetic wave

- Electric potential in electro-dynamic
- Electro-magnetic wave
- Energy stored in electro-magentic field and poynting’s theorem.

Elelctric potential in electric-dynamic field

In stationaly electric field, conservative E field: $\nabla \times \mathbf{E} = 0$ $V = \oint \boldsymbol{Edl} = 0$

A scaler, magnetic potential, based on $\nabla . B = 0$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Electro-dynamic:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) = \nabla \times (-\frac{\partial \mathbf{A}}{\partial t}) \Longrightarrow \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0$$

Stationary electric field:

$$\nabla \times \mathbf{E} = 0$$

$$\mathbf{E} = -\nabla V$$

Electro-dynamic field:

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0$$

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V$$

Maxwell's equations in vacuum

- $\nabla . E = \rho/\varepsilon$
- $\nabla \times E = -\frac{\partial B}{\partial t}$
- $\nabla . B = 0$
- $\nabla \times B = \mu J + \frac{\mu \partial D}{\partial t} = \mu J + \frac{\mu \varepsilon \partial E}{\partial t}$

In vacuum:

$$\nabla \cdot \mathbf{E} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t} \nabla \times \mathbf{B} = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$
$$\nabla \times (\nabla \times \mathbf{B}) = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \nabla \times \mathbf{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

Vector identity: (mathematics)

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla (\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}$$

$$\nabla \cdot \mathbf{E} = 0$$
$$\nabla \cdot \mathbf{B} = 0$$

In vacuum

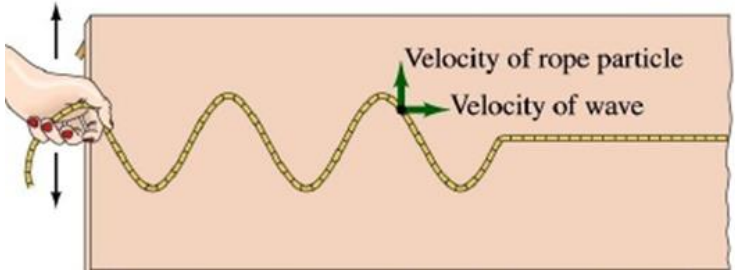
$$\nabla^2 \mathbf{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2},$$
$$\nabla^2 \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2},$$

$$\nabla^2 \boldsymbol{V} = \nabla \cdot \nabla \boldsymbol{V} = \mathrm{div}(\mathrm{grad}\, \boldsymbol{V})$$

Typical wave equation in one dimension.

The wave equation for a plane wave traveling in the x direction is

$$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2},$$



where *v* is the phase velocity of the wave and *u* represents the variable which is changing as the wave passes. This is the form of the wave equation which applies to a stretched string.

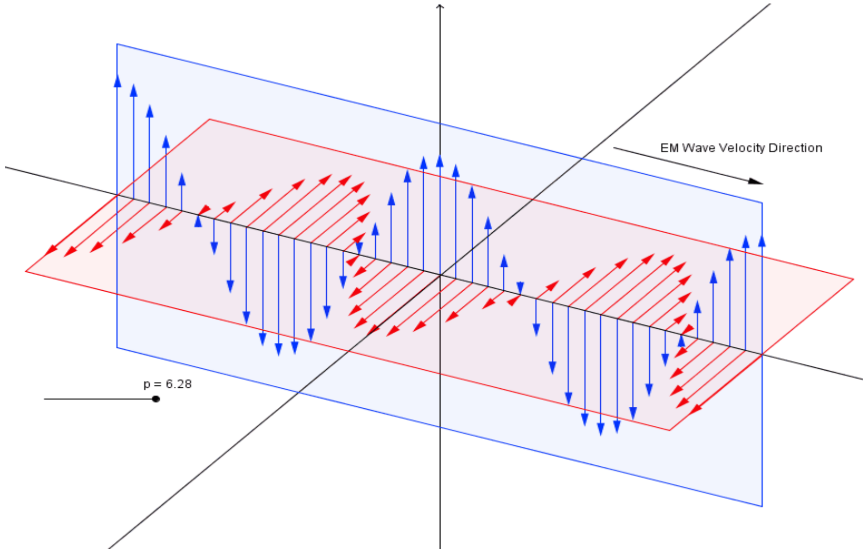
Re-write the equations of E and B

$$\nabla^2 \mathbf{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2},$$
$$\nabla^2 \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2},$$

$$\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2},$$
$$\nabla^2 \mathbf{B} = \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2},$$

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}},$$

c = 299,792,500 m/s.

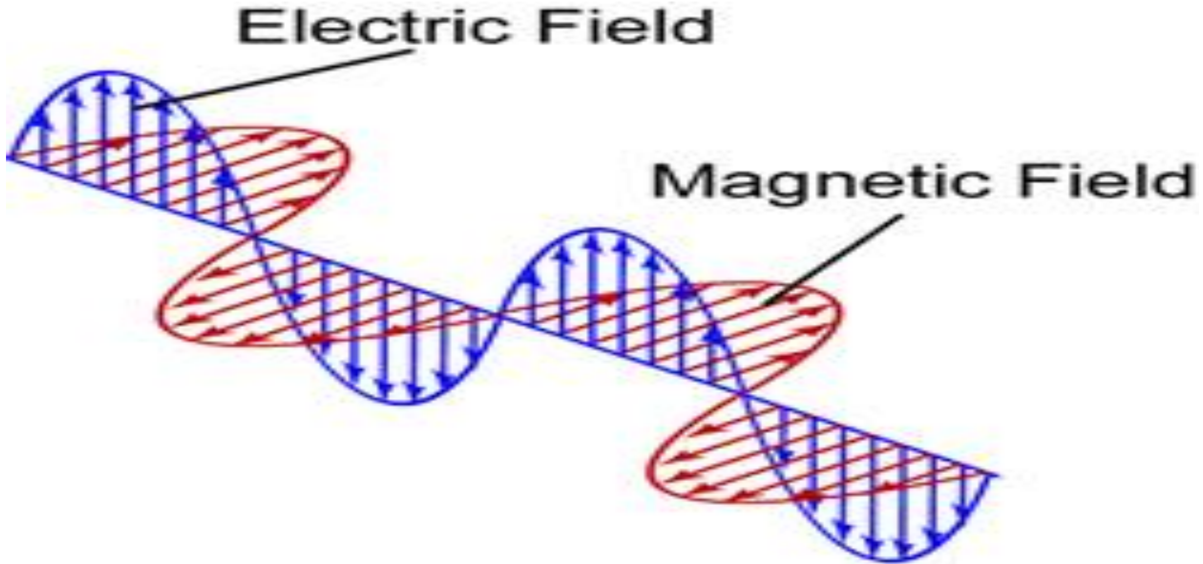


Electro-magnetic radiation.

Electromagnetic radiaton consists of electromagnetic waves, which are synchronized oscillations of electric and magnetic fields that propagate at the speed of light through a vacuum. The oscillations of the two fields are perpendicular to each other and perpendicular to the direction of energy and wave propagation, forming a transverse wave

$$\frac{1}{c_0^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = 0$$

$$\frac{1}{c_0^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = 0$$



Poynting's theorem: Power flow in electromagentic fields

Poynting's theorem states that the rate of energy transfer per unit volume from a region of space equals the rate of work done on the charge distribution in the region, plus the energy flux leaving that region. The energy transfer due to time-varying electric and magnetic fields is perpendicular to the fields

$$-\frac{\partial}{\partial t}\left(\frac{BH}{2}+\frac{DE}{2}\right)=\boldsymbol{J}\cdot\boldsymbol{E}+\nabla\cdot\boldsymbol{S}$$

$$\nabla\times H=J+\frac{\partial D}{\partial t}$$

$$\nabla\times E=-\frac{\partial B}{\partial t}$$

$$\oint_S \mathbf{A}\cdot d\mathbf{S}=\int_v \nabla\cdot\mathbf{A}\,dv$$

$$-\nabla\cdot(\boldsymbol{E}\times\boldsymbol{H})=\boldsymbol{E}\cdot\nabla\times\boldsymbol{H}-\boldsymbol{H}\cdot\nabla\times\boldsymbol{E}=\boldsymbol{J}\cdot\boldsymbol{E}+\boldsymbol{E}\cdot\frac{\partial\boldsymbol{D}}{\partial t}+\boldsymbol{H}\cdot\frac{\partial\boldsymbol{B}}{\partial t}$$

$$\int_v\left(\boldsymbol{H}\cdot\frac{\partial\boldsymbol{B}}{\partial t}+\boldsymbol{E}\cdot\frac{\partial\boldsymbol{D}}{\partial t}+\boldsymbol{E}\cdot\boldsymbol{J}\right)dv=-\oint_s(\boldsymbol{E}\times\boldsymbol{H})ds$$

Either power loss or power required to accelerate charges

$$\frac{1}{2}\frac{\partial(\boldsymbol{B}\cdot\boldsymbol{H})}{\partial t}=\frac{1}{2}\frac{\partial(\mu\boldsymbol{H}^2)}{\partial t}=\boldsymbol{H}\cdot\frac{\partial\boldsymbol{B}}{\partial t}$$

$$\frac{1}{2}\frac{\partial(\boldsymbol{D}\cdot\boldsymbol{E})}{\partial t}=\frac{1}{2}\frac{\partial(\varepsilon\boldsymbol{E}^2)}{\partial t}=\boldsymbol{E}\cdot\frac{\partial\boldsymbol{D}}{\partial t}$$

$$\mathbf{W}=-\oint(\boldsymbol{E}\times\boldsymbol{H})ds$$

Energy density stored in magnetic field

Energy density stored in electric field

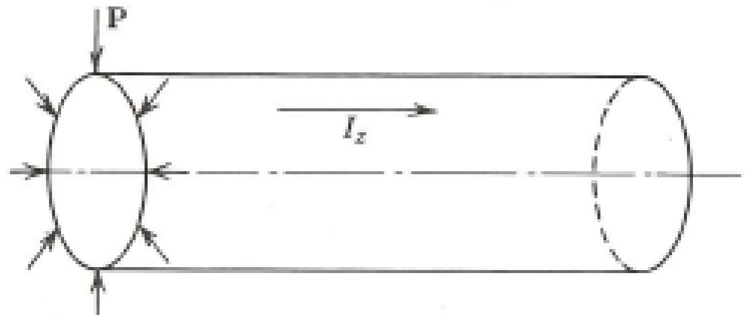
Example:

One conductor with DC current I_z, if R is the resistane per unit length.

Calculating the loss over the conductor in unit length.

$$E_z = I_z R$$

$$\mathbf{J} = \sigma \mathbf{E}$$



Poynting vector directed radially inward

$$H_\phi = \frac{I_z}{2\pi r}$$

$$P_r = -E_z H_\phi = -\frac{R I_z^2}{2\pi r}$$

P has no component normal to the end surfaces

$$W = 2\pi r(-P_r) = I_z^2 R$$

Surface integral over the conductor

Summary

Maxwell Equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho_v}{\varepsilon} \qquad \text{(Gauss' Law)}$$

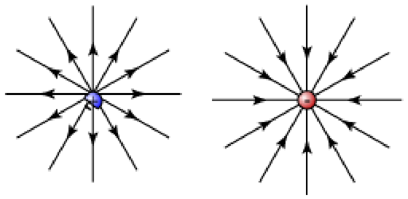
$$\nabla \cdot \mathbf{H} = 0 \qquad \text{(Gauss' Law for Magnetism)}$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \qquad \text{(Faraday's Law)}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \qquad \text{(Ampere's Law)}$$

Divergence and Stoke's theorem

Gradient: fastest rate of increase in spatial. $\nabla \boldsymbol{f} = \frac{\partial f}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}}$

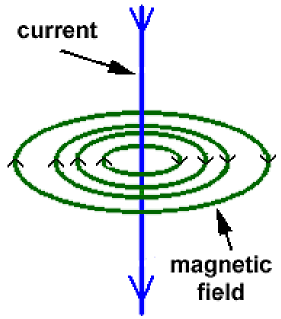


Divergence: Flux out of a point

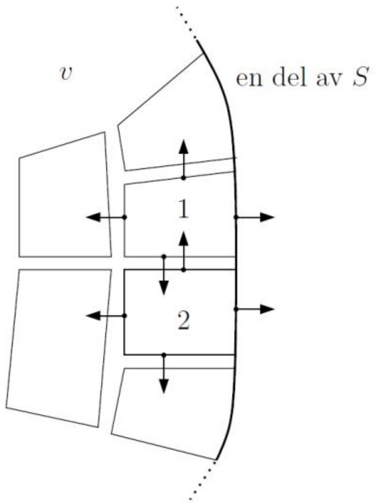
$$\nabla \cdot E = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

Curl: field circulating around a point

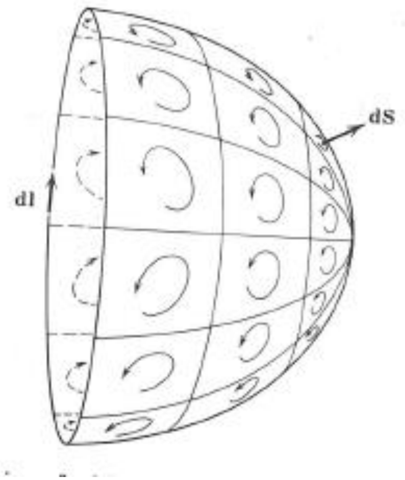
$$\nabla \times A = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}$$



$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{A} dv$$



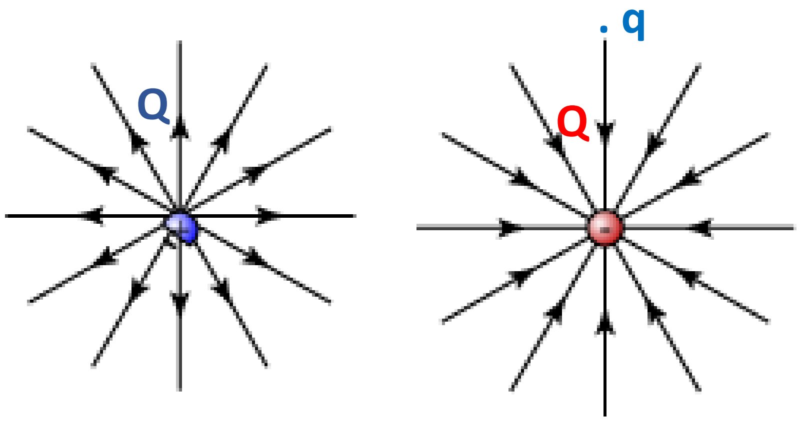
$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{A} \cdot d\mathbf{S}.$$



Electric field and coulomb force

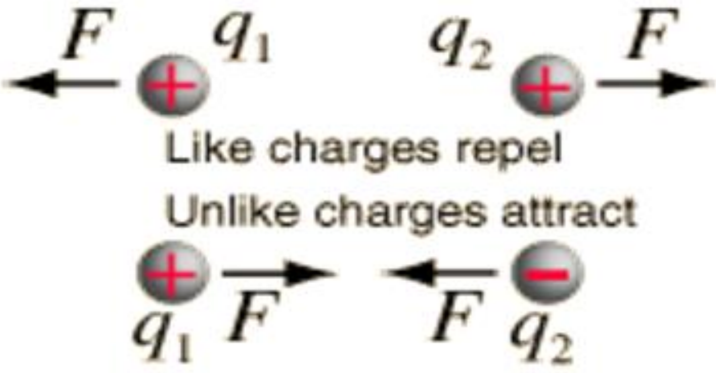
A stationary distribution of charges produces an electric field **E** in vacuum

Vector: $\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$



Field pattern of
a pointed electrode

Vector: $\vec{F} = \frac{qQ}{4\pi\varepsilon_0 r^2}\hat{r}$: $Coulomb's\ law$



$\vec{E}=\vec{F}/q$

Electric field and potential (voltage)

- Potential is a scalar.
- $V = -\int \boldsymbol{E}.\boldsymbol{dl}$

$$\nabla \boldsymbol{f} = \frac{\partial f}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}}$$

$$\boxed{\mathbf{E} = -\nabla V, \ \text{V/m}}$$

$$\nabla V \equiv \text{grad} V$$

$$\text{V}= \int_{\boldsymbol{r}}^{\infty} \boldsymbol{E} d\boldsymbol{l} = \int_{\boldsymbol{r}}^{\infty} \frac{Q}{4\pi\varepsilon_0 l^2} d\boldsymbol{l} = \frac{Q}{4\pi\varepsilon_0 r}$$

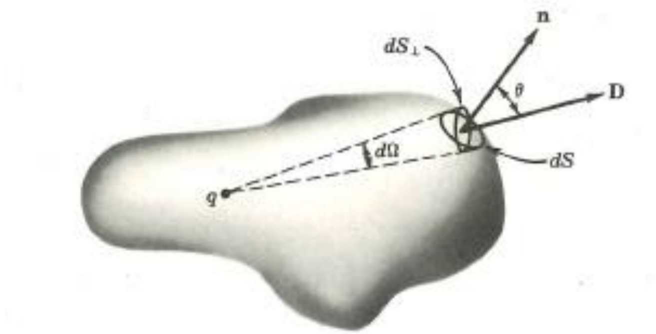
$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}}$$

Gauss' law

Electric flux flowing out of a closed surface = Total charge enclosed divided by the permittivity of the medium

$$\oint E d\boldsymbol{s} = \frac{Q}{\varepsilon}$$

Vector: $\vec{E} = \frac{Q}{4\pi r^2 \varepsilon_0}$ in vacuum, $\vec{D} = \frac{Q}{4\pi r^2}$

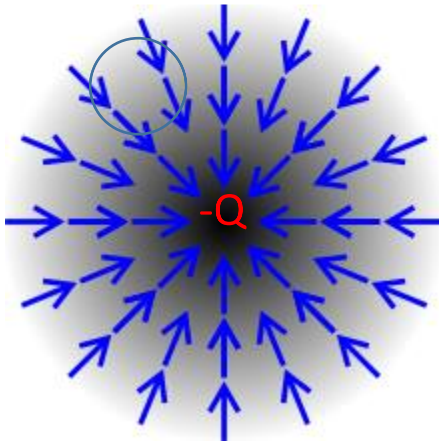


$$\nabla \cdot E = \frac{\rho}{\varepsilon}$$

Electric displacement field D = εE.

D is independent of medium:

$$\epsilon_0 \oint_S \mathbf{E} \cdot d\mathbf{S} = Q_{\text{total i } S}$$



- $\nabla \cdot D = \rho$

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{E} dV.$$

Electric polarization and material permittivity

The influence of electric polarization: $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$

- $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$
- $\boldsymbol{P} = \varepsilon_0 \chi_e \boldsymbol{E}$
- $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \varepsilon_0 \chi_e \boldsymbol{E} = (1 + \chi_e)\varepsilon_0 \boldsymbol{E}$

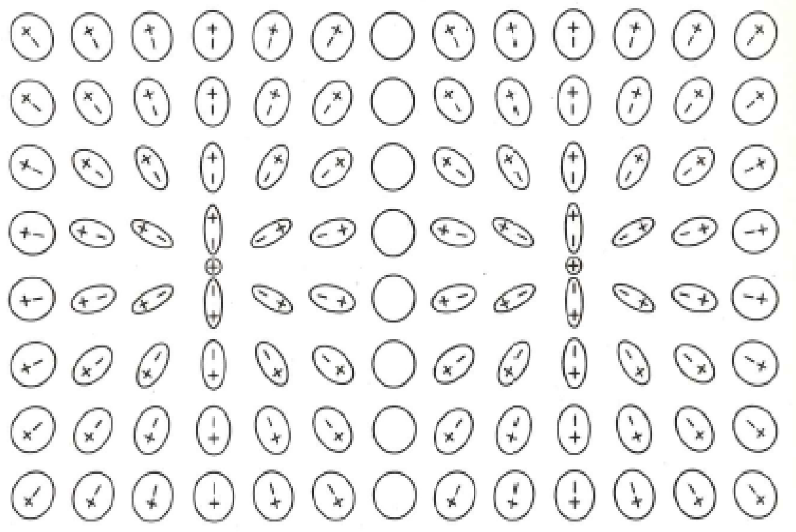


Fig. 1.3c Polarization of the atoms of a dielectric by a pair of equal positive charges.

- $(1 + \chi_e) = \varepsilon_r$ relative permittivity:
- $\varepsilon = \varepsilon_r \varepsilon_0$ electric permittivity .(dielectric material property)

The polarization reduces the electric field E in the dielectric material compared to vacuum. The resultant E becomes less.

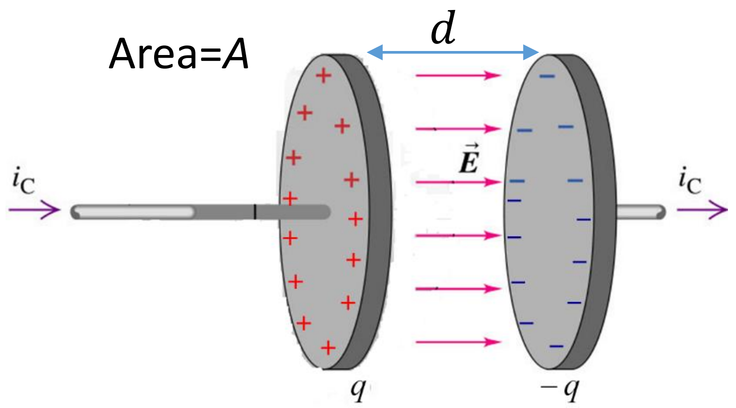
$$\vec{\boldsymbol{E}} = \frac{\boldsymbol{D}}{\boldsymbol{\varepsilon}} = \frac{\boldsymbol{D}}{\boldsymbol{\varepsilon_r \varepsilon_0}}$$

Capacitor and Energy in electric field

$$C = \frac{\varepsilon A}{d} \qquad \text{Volume}=Ad$$

$$W_e = \frac{1}{2}CV^2 = \frac{1}{2}\frac{\varepsilon A}{d}V^2 = \frac{1}{2}\varepsilon Ad\frac{V^2}{d^2} = \frac{1}{2}\varepsilon V_{vol}E^2 = \frac{1}{2}V_{vol}\boldsymbol{DE}$$

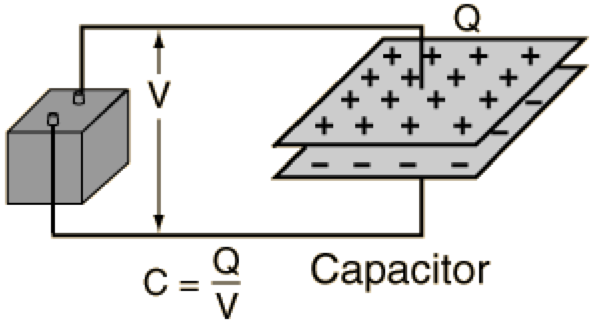
$$\frac{V^2}{d^2}=E^2 \qquad \boldsymbol{D}=\varepsilon\boldsymbol{E}$$



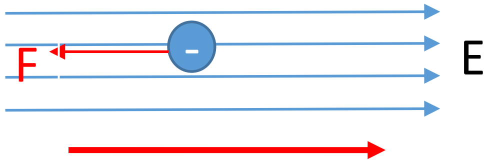
Electric Energy density $\eta_e = \frac{W_e}{V_{vol}} = \frac{1}{2}\varepsilon E^2 = \frac{1}{2}\boldsymbol{DE}$

Electric Energy density

$$\eta_e = \frac{1}{2}ED = \frac{1}{2}\varepsilon E^2$$

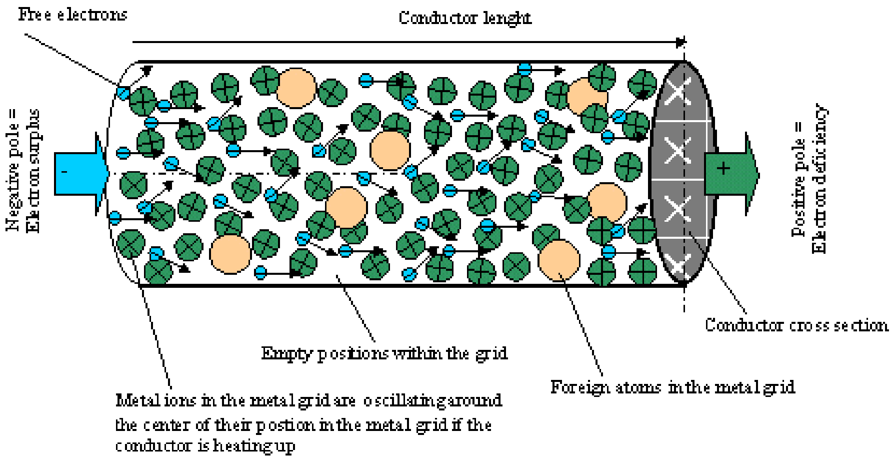


Conductivity and current



$$\mathbf{J} = \sigma \mathbf{E}$$

Vector: $\vec{F} = q\vec{E}$



Current: Free charges move along electric field direction

$$I = \frac{dQ}{dt}$$

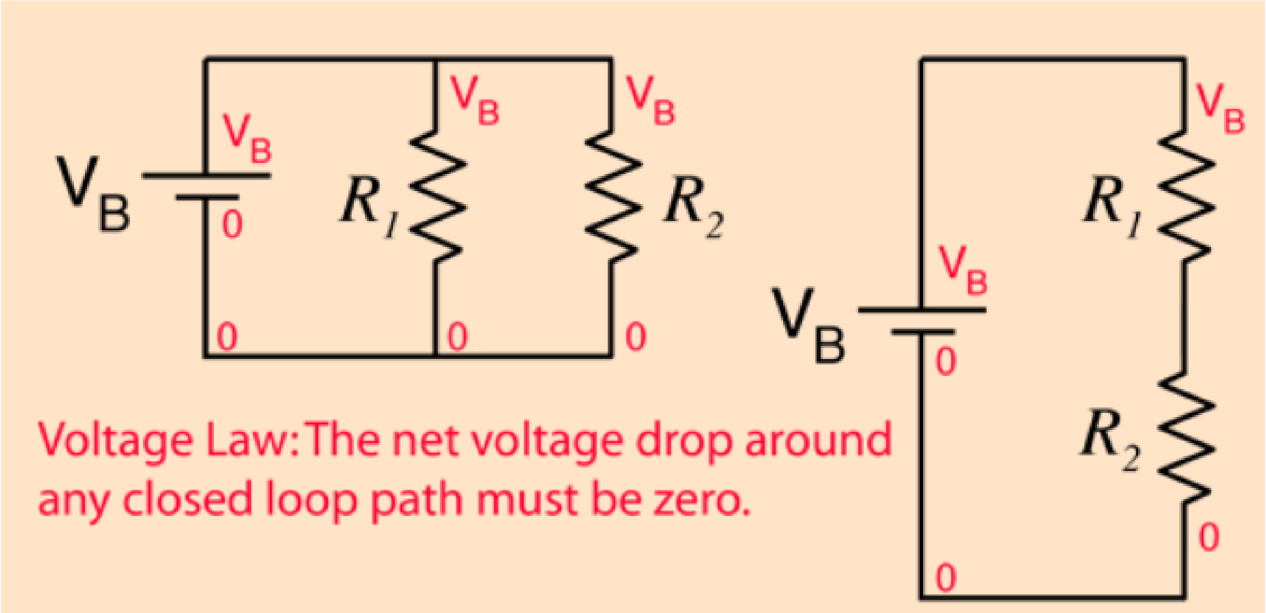
Ohm's law $$I = \frac{V}{R}$$

Stationary electric field

Conservative field:

$\nabla \times E = -\frac{\partial B}{\partial t} = 0$, (No magnetic flux variation)

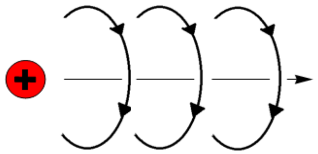
$$V = -\oint_c Edl = 0$$



Charges in motion produce magnetic field

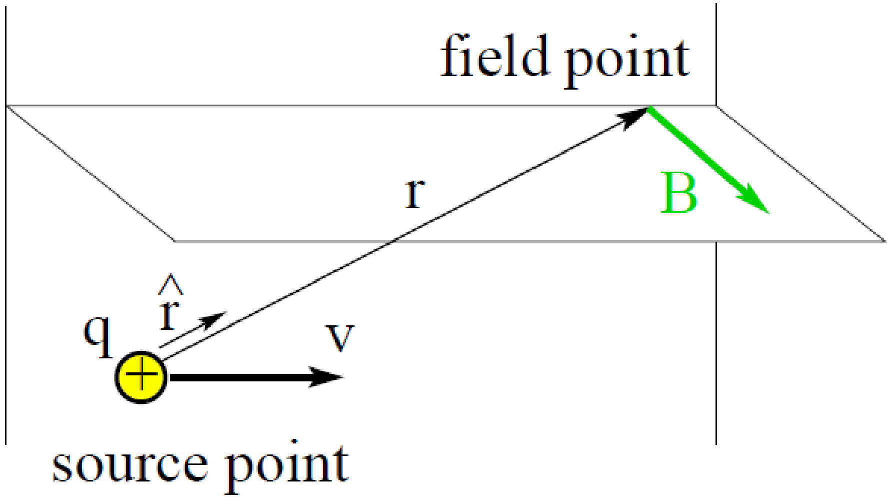
Charges in *motion* (electrical current) produce a magnetic field .

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\frac{q\boldsymbol{v}\times\hat{\boldsymbol{r}}}{r^2} \text{ in vacuum}$$



$\mu_0 = 4\pi \times 10^{-7} H/m$, permeability of free space

$$\boldsymbol{H} = \frac{1}{4\pi}\frac{q\boldsymbol{v}\times\hat{\boldsymbol{r}}}{r^2} = \frac{B}{\mu_0}$$



Magnetic flux and flux contiunity

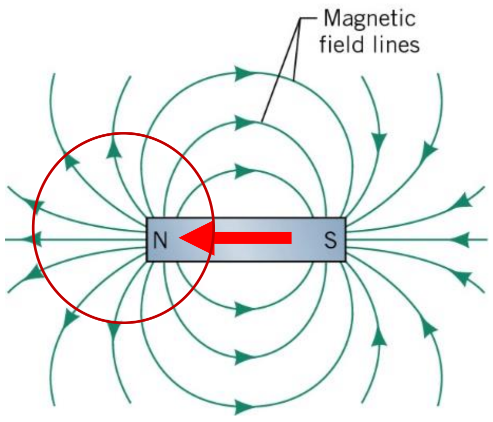
Magnetic flux ϕ is the integral of the flux density accross surface

$$\phi = \int_s \boldsymbol{B}.d\boldsymbol{S},$$

For en enclosed surface, the flux is zero

$$\oint_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0.$$

$$\nabla \cdot B = 0$$



Theres is no magnetic monopole, continuous magnetic field.

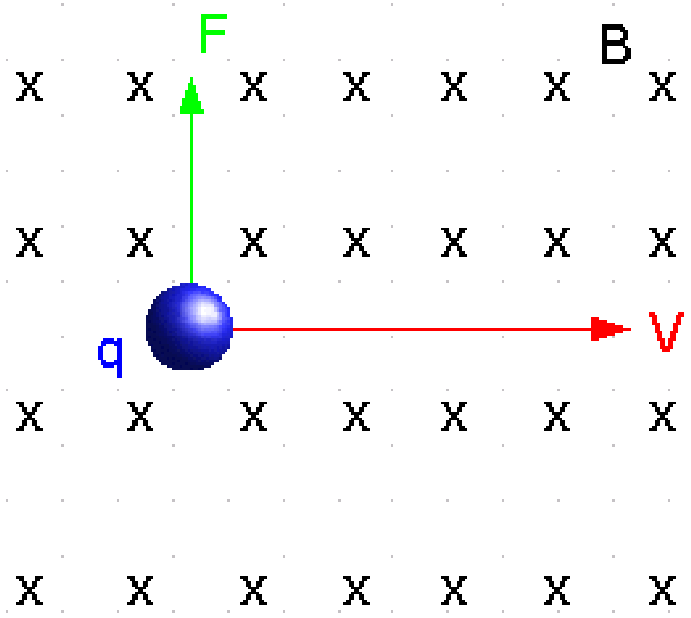
$$\int_V \nabla \cdot \mathbf{B} \mathrm{d}V = \oint_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0,$$

Electro-magnetic force

Force exerted by magnetic field B on a moving point charge Q is:

$$\boldsymbol{F} = Q\,\boldsymbol{v} \times \boldsymbol{B}$$

Magnetic force acting on a moving charge is always pependicular to it's moving directon, so magnetic force does no work on the charges, but changing the charge's moving direction



Lorentz law

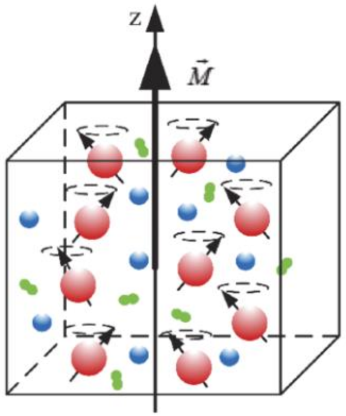
$$\mathbf{F} = q\mathbf{E} + q(\mathbf{v} \times \mathbf{B})$$

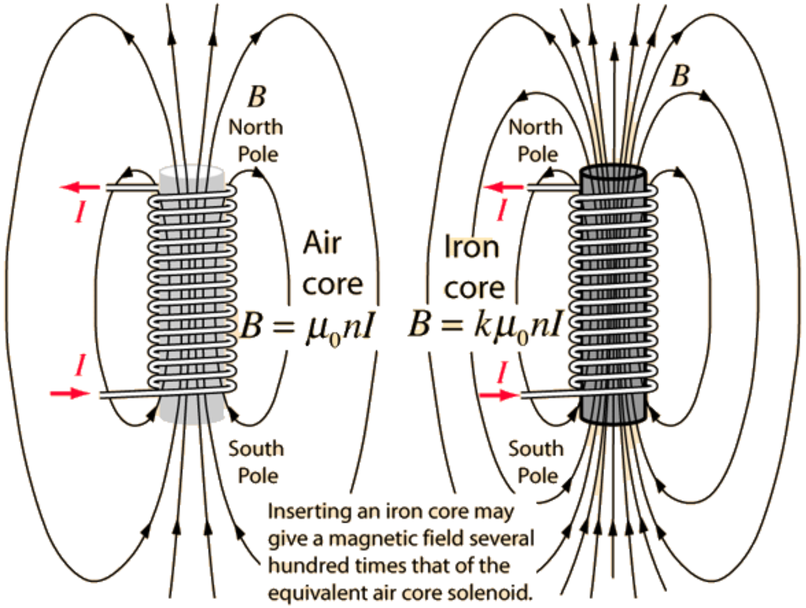
Magnetization and material permeability

The influence of magnetization: $\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M})$

$$\boldsymbol{B} = \mu_0(1 + \chi_m)\boldsymbol{H} = \mu\boldsymbol{H} = \mu_0\mu_r\boldsymbol{H}$$

μ_r: relative permeability



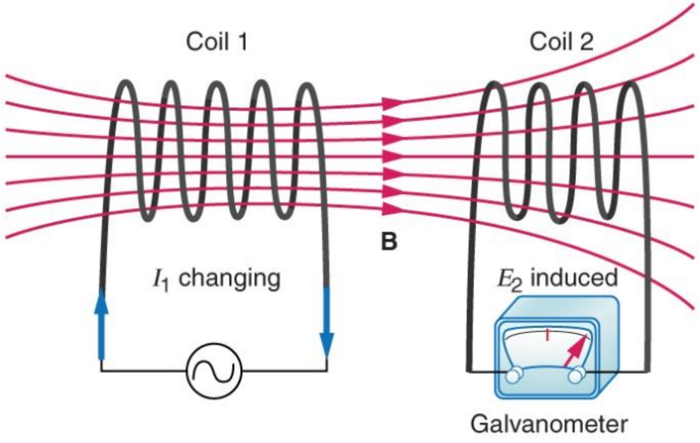


Magnetization increases in the magnetic flux density B in ferro-magnetic materials compared to vacuum.
The resultant B becomes higher.

Faraday's law

$V = -\frac{d\phi}{dt}$ **Faraday's law,** $\phi = \int_s \boldsymbol{B} \cdot \boldsymbol{dS}$ **is the magnetic flux.**

$$\nabla \times \boldsymbol{E} = -\frac{\partial B}{\partial t};$$

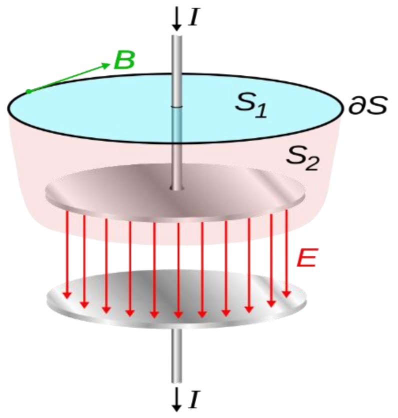


Time varying magnetic field produced electric field

Ampere’s law and Displacement Current:

Displacement current is defined as the rate of change of electric displacement field.

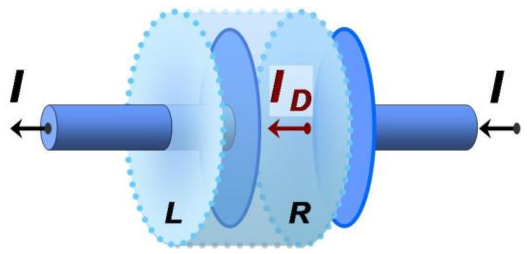
$$J_D = \frac{\partial D}{\partial t}$$



Amperes law: $\nabla \times H = J + \frac{\partial D}{\partial t};$

J $= \sigma E$ is conduction current in materials

$\frac{\partial D}{\partial t}$ is displcement current



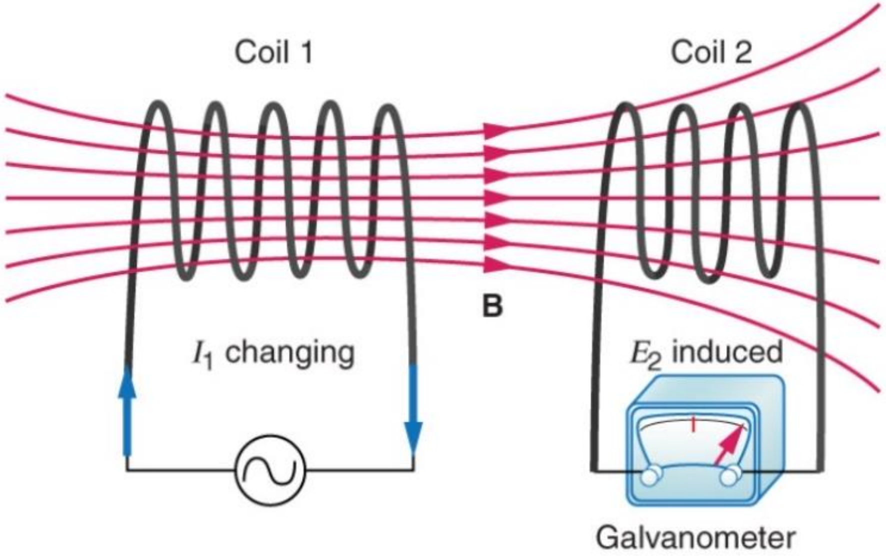
Capacitor

Time varying electric field produce magnetic field

Inductor and magnetic energy

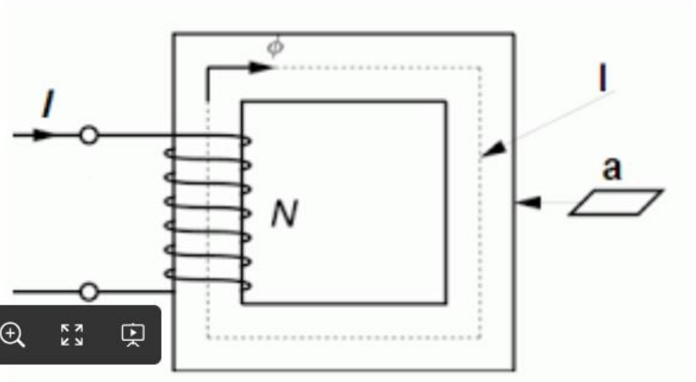
$$L = \frac{\int_S \boldsymbol{B} \cdot d\boldsymbol{S}}{I}$$

$$W = \int_0^I LI \frac{dI}{dt} dt = \int_0^I LI dI = \frac{1}{2} LI^2$$



Magnetic energy density

$$\eta_m = \frac{W}{V_{vol}} = \frac{B^2}{2\mu}$$



$$L = \frac{N^2 a\mu}{l} = \frac{N^2}{R_m}$$

Poynting's theorem and Poynting vector

Poynting's theorem states that the rate of energy transfer per unit volume from a region of space equals the rate of work done on the charge distribution in the region, plus the energy flux leaving that region.

$$-\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon \mathbf{E}^2 + \frac{1}{2}\mu \mathbf{H}^2\right) = \mathbf{J}\cdot\mathbf{E} + \nabla\cdot(\mathbf{E}\times\mathbf{H})$$

$$\mathbf{P} = \mathbf{E}\times\mathbf{H}$$

Poynting's vector: Power density

$$\int_v \left(\mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t} + \mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t} + \mathbf{E}\cdot\mathbf{J}\right) dv = -\oint_s (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{s}$$

$$\mathbf{W} = -\oint(\mathbf{E}\times\mathbf{H})\cdot ds$$

- Magnetic energy density ($\mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t}$)
- Electric energy density ($\mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t}$)
- Ohmic power density ($\mathbf{E}\cdot\mathbf{J}$)

Electromagnetism

Maxwell equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

Material related

$$\mathbf{B} = \mu \mathbf{H}$$

↑

$$\mathbf{D} = \varepsilon \mathbf{E}$$

$$\mathbf{J} = \sigma \mathbf{E}$$

Energy density and force

$$\frac{1}{2} \varepsilon \mathbf{E}^2$$

$$\frac{1}{2} \mu \mathbf{H}^2$$

$$\mathbf{F} = q\mathbf{E} + q(\mathbf{v} \times \mathbf{B})$$