Lecture 5: Electro-dynamics

- Faraday’s law
- Conductive current and displacement current
- Inductance and Lenz’ law
- Energy store in inductance

Changing magnetic field: Faraday's law

$\boldsymbol{\varepsilon} = -\frac{d\phi}{dt}$ **Faraday's law,** $\phi = \int_s \boldsymbol{B} \cdot d\boldsymbol{S}$ **is the magnetic flux.**

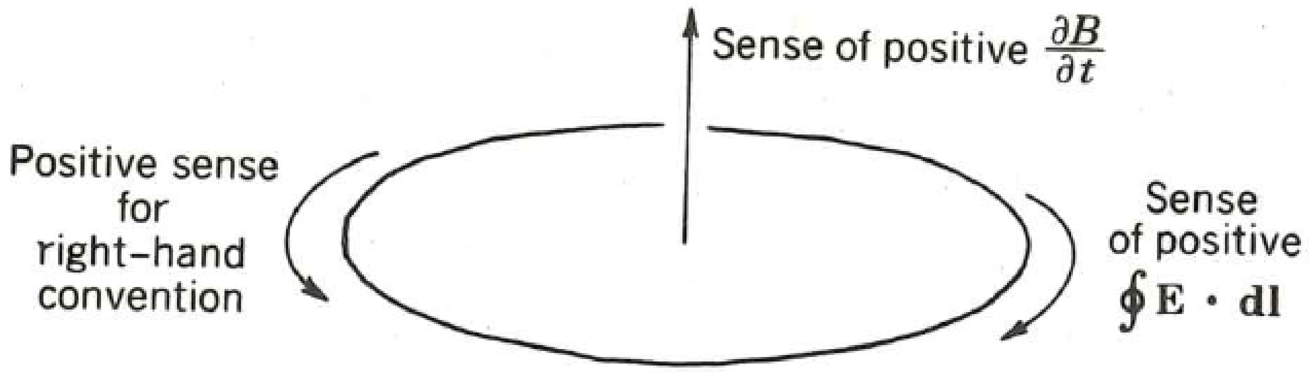
In static-electric field, $\nabla \times \boldsymbol{E} = 0$; and $\oint_c \boldsymbol{E} \cdot d\boldsymbol{l} = 0$, conservative vector field,

When $\frac{\partial B}{\partial t} \neq 0$, $\boldsymbol{\nabla} \times \boldsymbol{E} \neq \boldsymbol{0}$

$$\oint_c \boldsymbol{E} \cdot d\boldsymbol{l} = -\int_s \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{S} = -\frac{d}{dt}\left(\int_s \boldsymbol{B} \cdot d\boldsymbol{S}\right) = -\frac{d\phi}{dt}$$

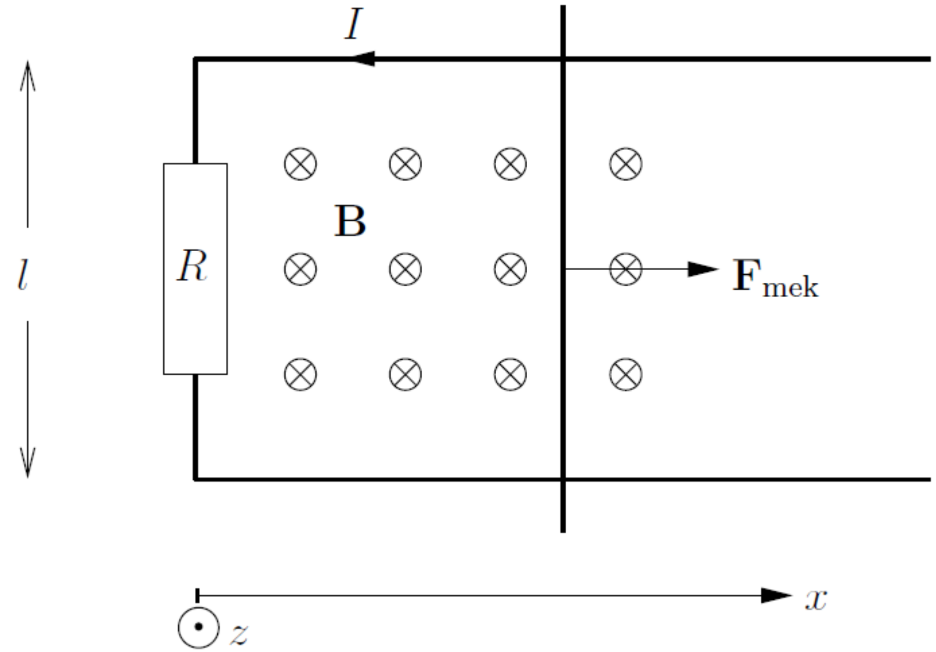
$$\oint_c \boldsymbol{E}\, d\boldsymbol{l} = \int_s (\nabla \times \boldsymbol{E}) \cdot dS$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial B}{\partial t}$$



Electro-motive force (EMF)

Example:



$$\mathbf{S} = (S_0 + lvt)\hat{\mathbf{z}}.$$

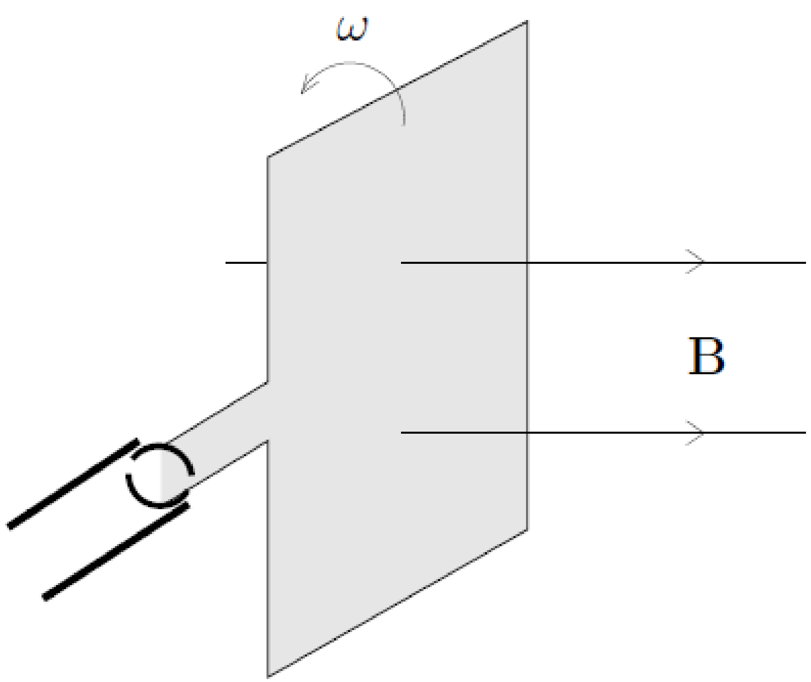
$$e = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_S \mathbf{B}\cdot\mathrm{d}\mathbf{S} = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{B}\cdot\int_S \mathrm{d}\mathbf{S}\right) = -\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{B}\cdot\mathbf{S} = \frac{\mathrm{d}}{\mathrm{d}t}BS = Blv$$

$$e = BLv$$

$$I = \frac{e}{R} = \frac{Blv}{R}$$

Faraday's law for a moving system: generator

The angle between B and surface normal direction is: $\varphi = \omega t.$

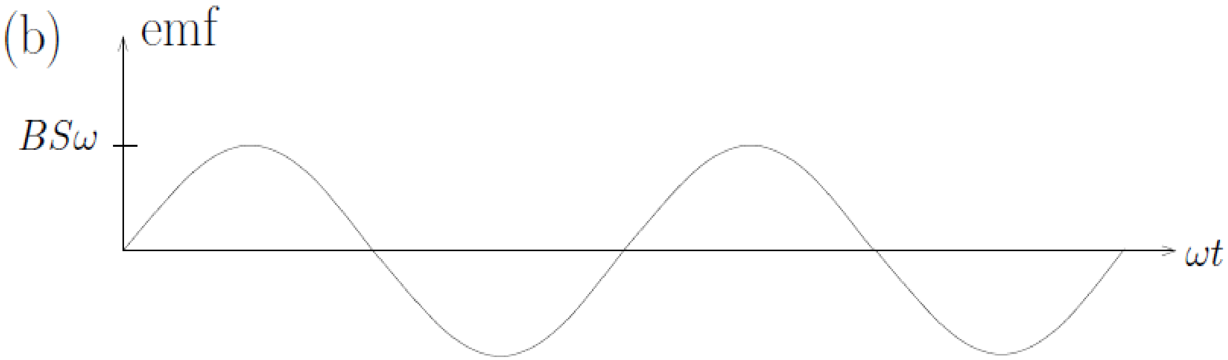


What is the EMF and its waveform with time.

$$\Phi = \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = \mathbf{B} \cdot \mathbf{S} = BS\cos\omega t.$$

$$e = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = BS\omega\sin\omega t$$

(b)



Displacement current

$$\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}; \quad (\mathbf{J} = \sigma\mathbf{E} \text{ is conduction current in materials})$$

$$\nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t}.$$

Displacement current is defined in terms of the rate of change of electric displacement field.

$$\mathbf{J}_D = \frac{\partial\mathbf{D}}{\partial t}$$

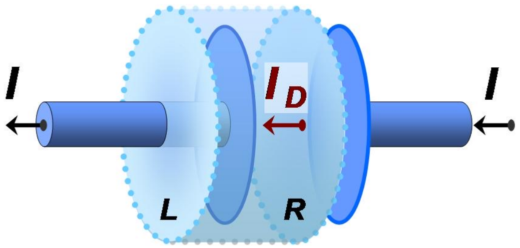
This additional term solves some problems:

1) Charge conservation argument:

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} \Rightarrow \nabla\cdot(\nabla\times\mathbf{B}) = 0, \Rightarrow \nabla\cdot\mathbf{J} = 0.$$

2) Electro-magnetic wave propagates in vacuum, where there is no current J.
3) Current in capacitor

When charging/discharging capacitor, there is current in the cable, but no current between the two plates (assume vacuum medium), but electric field changing.



Capacitor

Capacitor and displacement current

Capacitor is charging with current I:

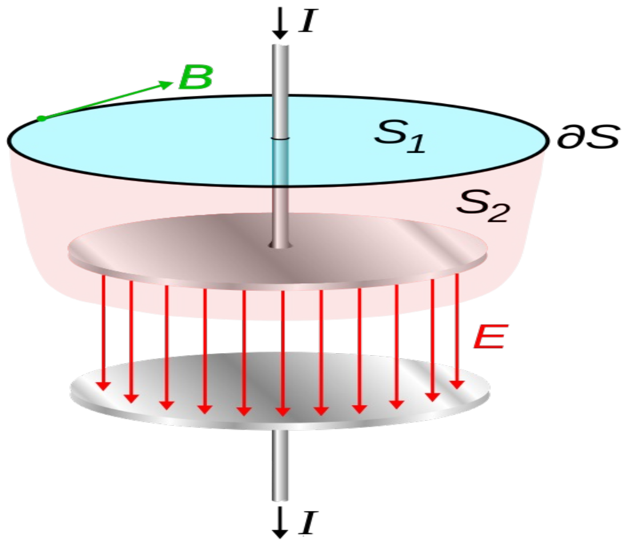
Capacitance : $C = \frac{\varepsilon A}{l}$, charge current is $I = C\frac{dV}{dt}$

$$\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}$$

Calculating the integral over surface S1,

$$\int_s \nabla\times\mathbf{H}\cdot d\mathbf{S}_1 = \int_s \left(\mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}\right)\cdot d\mathbf{S}_1 = \int_s \mathbf{J}\cdot d\mathbf{S}_1 = I \;\left(I = C\frac{dV}{dt} = \frac{\varepsilon A}{l}\frac{dV}{dt}\right)$$

$\frac{\partial D}{\partial t}$=0, out of capacitor, the surface integration is zero



Calculating the integral over surface S2,

$$\int_s \nabla\times\boldsymbol{H}\cdot d\boldsymbol{S_2} = \int_s \left(\boldsymbol{J} + \frac{\partial\boldsymbol{D}}{\partial t}\right)\cdot d\boldsymbol{S_2} = \int_s \frac{\partial\boldsymbol{D}}{\partial t}\cdot d\boldsymbol{S_2} = \int_s \frac{\varepsilon}{l}\frac{dV}{dt}d\boldsymbol{S_2} = \frac{\varepsilon A}{l}\frac{dV}{dt} = I$$

J = 0 at surface S2,

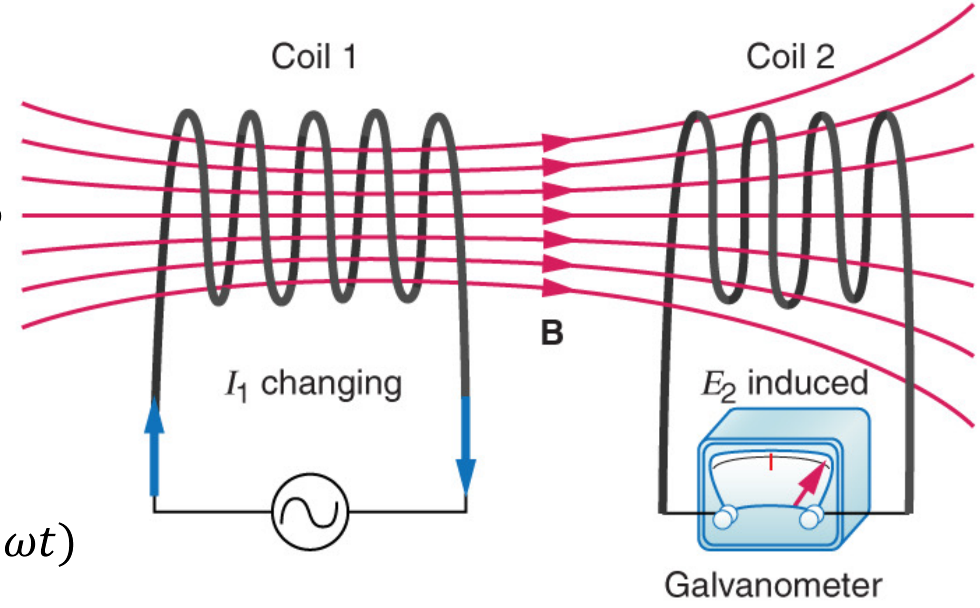
$\boldsymbol{D} = \varepsilon\boldsymbol{E} = \varepsilon\frac{\boldsymbol{V}}{l}$, only inside capacitor.

Maxwell's equations

Name	Integral equations (SI convention)	Differential equations (SI convention)
Gauss's law	$\oiint_{\partial\Omega} \mathbf{D} \cdot \mathrm{d}\mathbf{S} = \iiint_{\Omega} \rho_{\mathrm{f}} \, \mathrm{d}V$	$\nabla \cdot \mathbf{D} = \rho_{\mathrm{f}}$
Ampère's circuital law (with Maxwell's addition)	$\oint_{\partial\Sigma} \mathbf{H} \cdot \mathrm{d}\boldsymbol{\ell} = \iint_{\Sigma} \mathbf{J}_{\mathrm{f}} \cdot \mathrm{d}\mathbf{S} + \frac{d}{dt} \iint_{\Sigma} \mathbf{D} \cdot \mathrm{d}\mathbf{S}$	$\nabla \times \mathbf{H} = \mathbf{J}_{\mathrm{f}} + \frac{\partial \mathbf{D}}{\partial t}$
Gauss's law for magnetism	$\oiint_{\partial\Omega} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0$	$\nabla \cdot \mathbf{B} = 0$
Maxwell–Faraday equation (Faraday's law of induction)	$\oint_{\partial\Sigma} \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} = -\frac{d}{dt} \iint_{\Sigma} \mathbf{B} \cdot \mathrm{d}\mathbf{S}$	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$

Example:

Coil 1 : I(t) = $\hat{I}\sin(\omega t)$, coil crosss−section area 1 = area 2.
The flux produced from 1 going through coil 2: B(t) = $\hat{B}\sin(\omega t)$
1) What is the induced voltage in coil 2?
2) Assuming resistance in coil 2 is R, calculating the current in coil 2?



$\phi = N\int_s \ Bds$ (flux linkage between coil 1 and coil 2)

$$V_2(t) = -\frac{d\phi}{dt} = -\frac{N\int_s \ \hat{B}\sin(\omega t)\,ds}{dt} = -\frac{NA\hat{B}\sin(\omega t)}{dt} = -N\omega A\hat{B}\cos(\omega t)$$

$$I_2(t) = \frac{V_2(t)}{R} = -\frac{N\omega A\hat{B}\cos(\omega t)}{R}$$

Inductance and Lenz's law

Inductance definition : $\boldsymbol{L = \frac{\phi}{I}}$

Inductance definition : $L = \frac{N \int_s \; B\cdot dS}{I}$

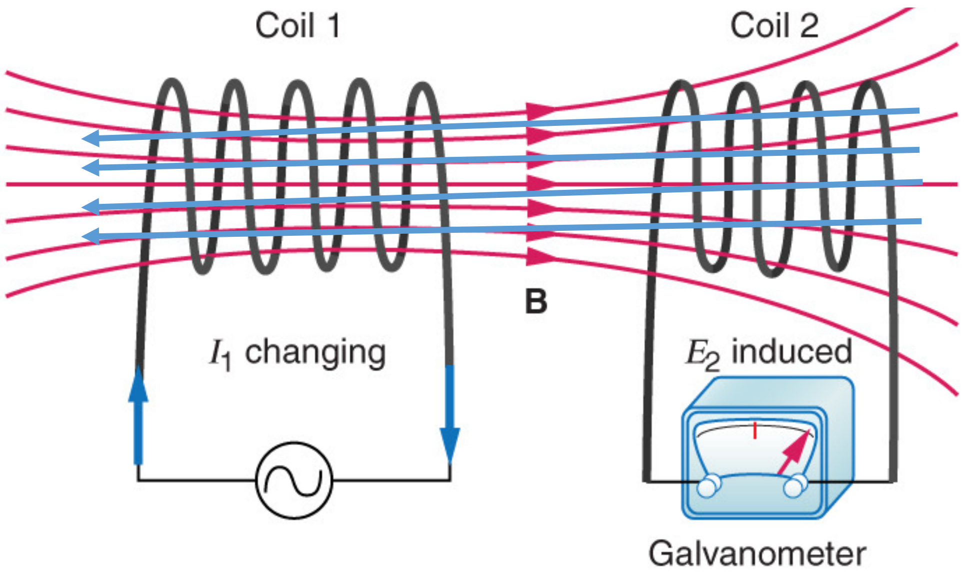
Self-inductance: $L_1 = \frac{\phi_1}{I_1} = \frac{N_1 \int_{s_1} B_1 ds}{I_1}$

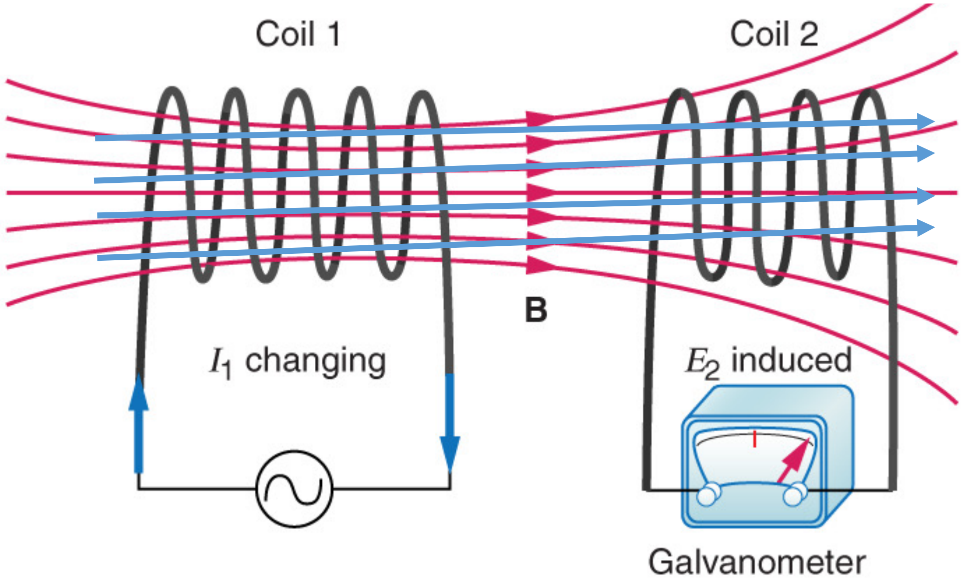
Mutual inductance: $L_{12} = \frac{\phi_{12}}{I_1} = \frac{N_2 \int_{s_2} B_2 dS}{I_1}$

$I_2(t) = \frac{V_2(t)}{R} = -\frac{N\omega A\hat{B}\cos(\omega t)}{R}$ $\quad L_2 = \frac{\phi_2}{I_2} = \frac{N \int_{s_2} B_2 dS_2}{I_2}$

Lenz's Law:
The magnetic field created by the induced current opposes changes in the initial magnetic field.

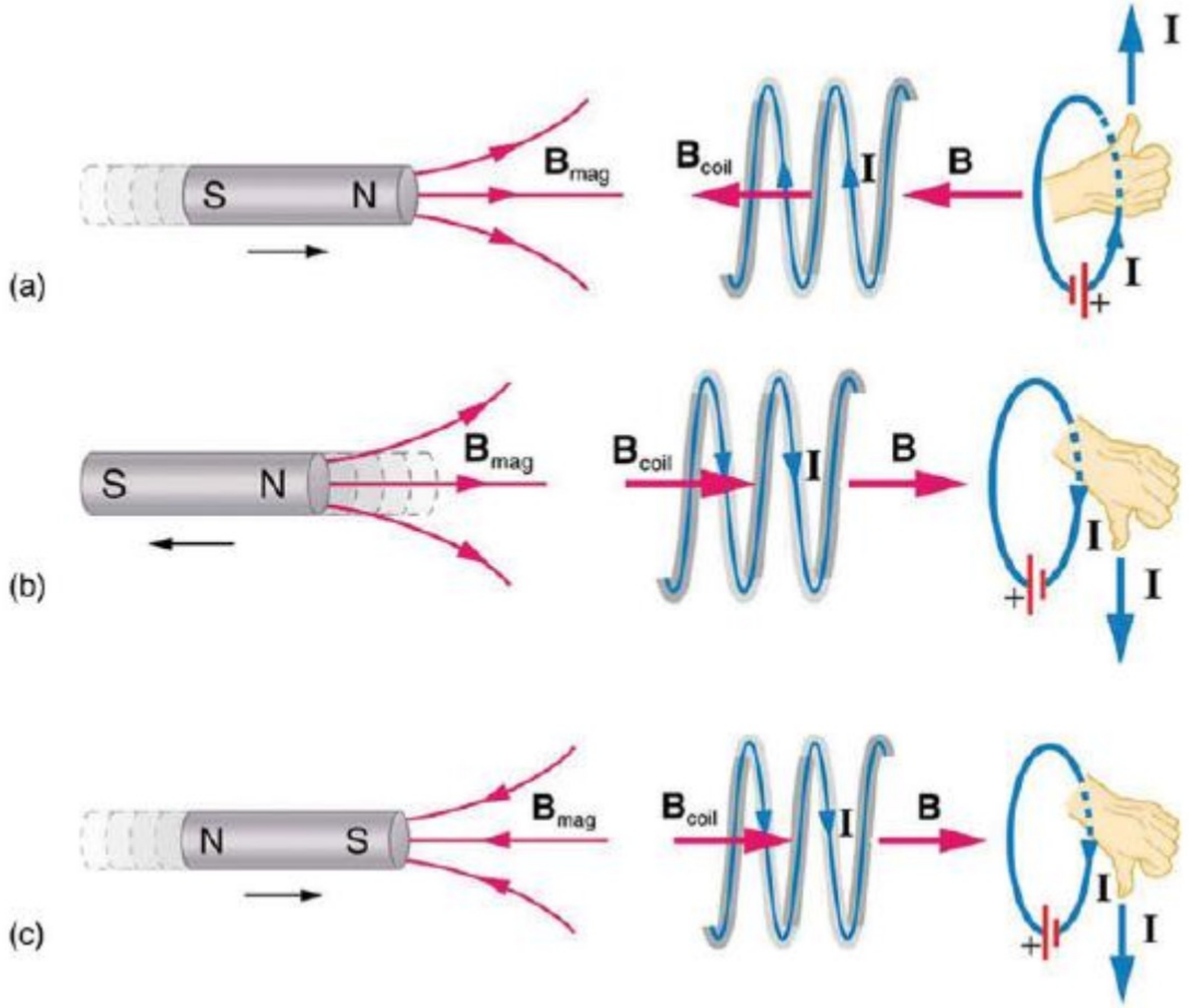
When B1 increase, the direction of B2 is opposite. Against the change





When B1 decreases, the direction of B2 is the same as B1. Against the change

Lenz’s law



Example : calculating inductance

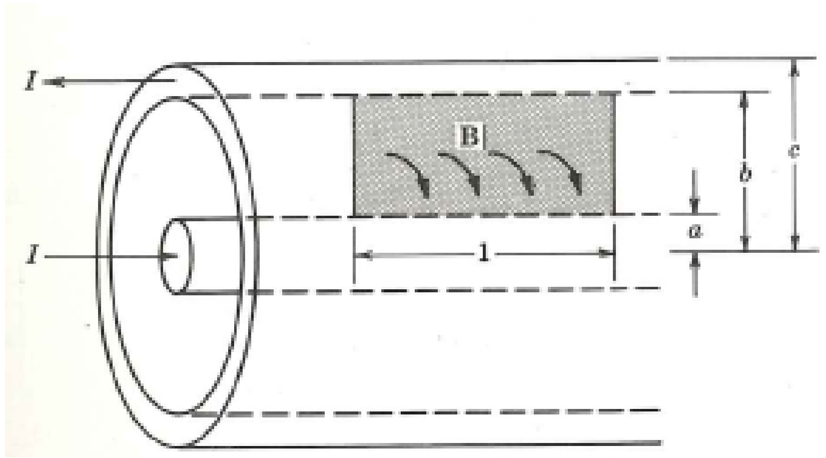
A coaxial line carrying curent I on the inner conductor and –I on the outer .
Calculate the magnetic field H at r distance, (Current evenly distributed in the two conductors)
The

Calculating the external inductance of the coaxial line in unit length.

$$H_\phi = \frac{I}{2\pi r}$$

$$\int_S \mathbf{B} \cdot d\mathbf{S} = \int_a^b \mu\left(\frac{I}{2\pi r}\right) dr = \frac{\mu I}{2\pi} \ln \frac{b}{a}$$

$$L = \frac{\mu}{2\pi} \ln \frac{b}{a} \quad \text{H/m}$$



Inductance calculation:

Assume the total length is l and the cross-section area is a

$$L = \frac{\phi}{I} = \frac{N \int_S \boldsymbol{B} d\boldsymbol{s}}{I}$$

MMF

$$NI = HL = \Phi R$$

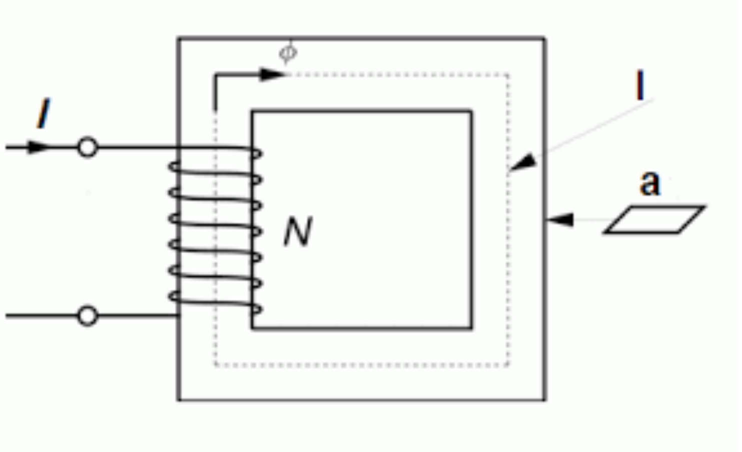
Magnetic reluctance

$$R_m = \frac{l}{\mu a}$$

$$R_m \phi = NI$$

Flux density

$$B = \frac{\phi}{a} = \frac{NI}{aR_m} = \frac{NI\mu}{l}$$



$$L = \frac{N^2 a\mu}{l} = \frac{N^2}{R_m}$$

$$P = \frac{1}{2} L I^2 = \frac{1}{2}\frac{N^2 \mu a}{l} I^2 = \frac{1}{2}\frac{N^2 I^2 \mu^2}{l^2}\frac{la}{\mu} = \frac{B^2}{2\mu} V_{vol}$$

$$\eta_m = \frac{P}{V_{vol}} = \frac{B^2}{2\mu}$$

Energy stored in inductance

$$L = \frac{\int_s \ \boldsymbol{B} \cdot d\boldsymbol{S}}{I}$$

$$V(t) = -\frac{d}{dt}\left(\int_s \ \boldsymbol{B} \cdot d\boldsymbol{S}\right) = -\frac{d\phi}{dt}$$

$$V(t) = \frac{d}{dt}(LI) = L\frac{dI}{dt} \quad \Rightarrow \quad dp = VI = LI\frac{dI}{dt}$$

$$P = \int_0^I LI\frac{dI}{dt}dt = \int_0^I LIdI = \frac{1}{2}LI^2$$

Energy stored in magnetic field

$$P = \frac{1}{2}LI^2 = \frac{1}{2}\frac{N^2\mu a}{l}I^2 = \frac{1}{2}\frac{N^2I^2\mu^2}{l^2}\frac{la}{\mu} = \frac{B^2}{2\mu}V_{vol}$$

$$L = \frac{N^2 a\mu}{l}$$

$$B = \frac{\phi}{a} = \frac{NI}{aR_m} = \frac{NI\mu}{l}$$

$$\eta_m = \frac{p}{V_{vol}} = \frac{B^2}{2\mu}$$

Example

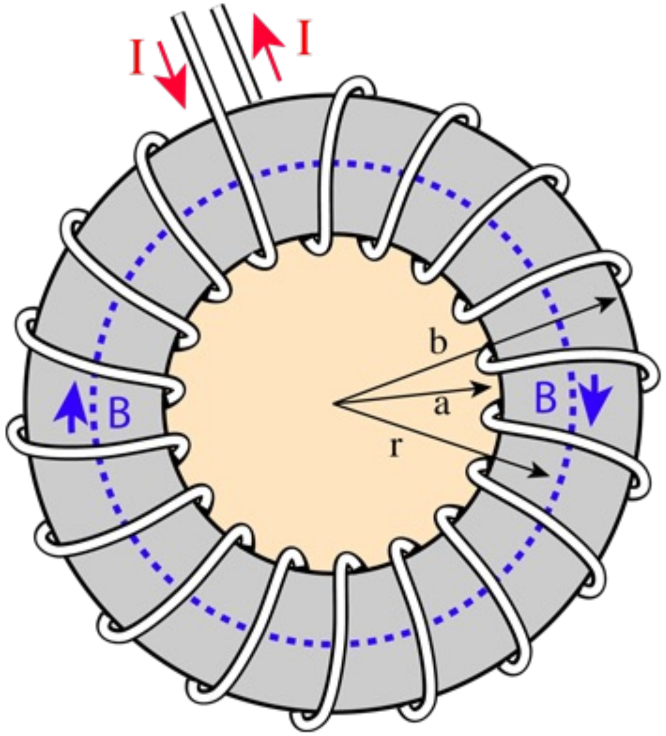
Given a tightly wound toroid with radius a, and N number of turns that is conducting a constant direct current I. Use a = 10cm, N = 1000 and I = 1mA. Find the magnetic field B everywhere (both inside and outside the toroid) assuming the core consists of

a) vacuum
b) an iron core with μ_r = 5000.

$$NI = HL = \Phi R$$

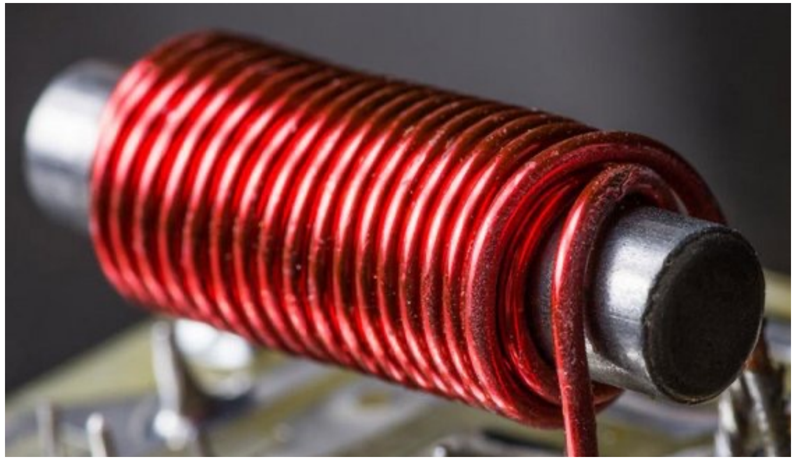
$$B 2\pi r = \mu N I$$

$$B = \frac{\mu N I}{2\pi r}$$



Example

a) Find the self inductance L of a long, tightly wound solenoid.
b) If the number of turns is doubled (and everything else remains the same), what will happen with the self inductance?
c) Assume that the current is decaying from I_0 to 0 during the time T. Find the induced voltage as a function of I_0, T and L.



$$\mathbf{H} = H\hat{\mathbf{z}} = \frac{NI}{l}\hat{\mathbf{z}}. \qquad B = \mu\frac{NI}{l}$$

$$\Phi_{\mathrm{cs}} = \int_S \mathbf{B}\cdot \mathrm{d}\mathbf{S} = \int_S B\mathrm{d}S = B\int_S \mathrm{d}S = \mu\frac{NI}{l}\cdot \pi a^2.$$

$$L = \frac{\Phi}{I} = \frac{N\Phi_{\mathrm{cs}}}{I} = \frac{\mu\pi a^2 N^2}{l}.$$

$$e(t) = -L\frac{\mathrm{d}I(t)}{\mathrm{d}t},$$

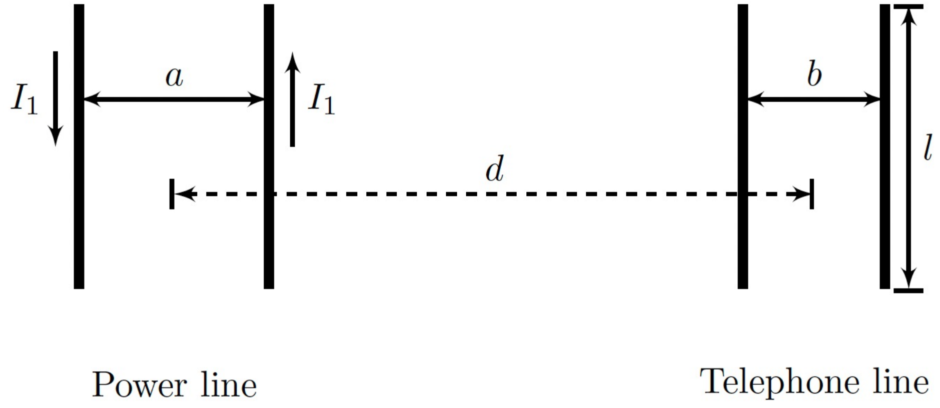
$$I(t) = I_0\left(1-\frac{t}{\tau}\right), \text{ for } 0 \le t \le \tau.$$

$$e(t) = -L\frac{\mathrm{d}}{\mathrm{d}t}I_0\left(1-\frac{t}{\tau}\right) = \frac{LI_0}{\tau}.$$

Example

Problem 3

A telephone line and a power line is running parallel with each other. Both the power line and the telephone line consists of two thin, parallel conductors as shown in the figure below. The power line is assumed to be infinitely long, and the telephone line is assumed to be a closed, rectangular loop with length l and width b. Assume that the thickness of the conductor is negligible compared to the distances a, b, d, and l.



a) Find the mutual inductance between the to lines.

Hint: Use $L_{12} = \Phi_{12}/I_1$.

b) Find the amplitude of the induced electromotive force in the telephone line when there is a harmonic alternating current with an amplitude I_0 and a frequency f in the power line. As a numerical example, we say that $f = 50\text{Hz}$, $I_0 = 100\text{A}$, $l = 500\text{m}$, $d = 10\text{m}$, $a = 50\text{cm}$, and $b = 10\text{cm}$.

(Answer: 1.57mV.)

Solution: a)

We have decided z = 0 at the left conductor of the telephone line

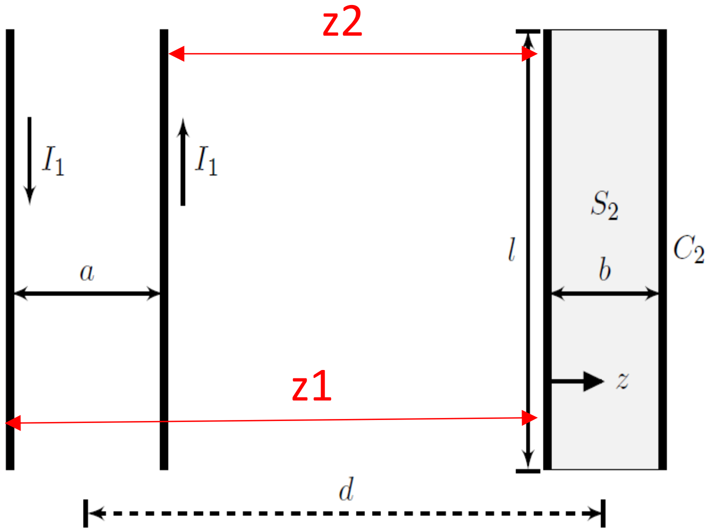
the magnetic flux within C_2 is given by

$$z_1 = -d - \tfrac{a}{2} + \tfrac{b}{2}$$

$$z_2 = -d + \tfrac{a}{2} + \tfrac{b}{2}$$

$$\begin{aligned} B(z) &= B_1(z) + B_2(z) \\ &= \frac{\mu_0 I_1}{2\pi r_1} - \frac{\mu_0 I_1}{2\pi r_2} \\ &= \frac{\mu_0 I_1}{2\pi}\left(\frac{1}{z - z_1} - \frac{1}{z - z_2}\right) \end{aligned}$$

$$\begin{aligned} \Phi_{12} &= \int_{S_2} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = l \int_0^b B(z)\mathrm{d}z \\ &= \frac{\mu_0 I_1 l}{2\pi} \int_0^b \left(\frac{1}{z - z_1} - \frac{1}{z - z_2}\right) \\ &= \frac{\mu_0 I_1 l}{2\pi}\left[\ln\left(\frac{b - z_1}{-z_1}\right) - \ln\left(\frac{b - z_2}{-z_2}\right)\right] \end{aligned}$$



$$L_{12} = \frac{\Phi_{12}}{I_1} = \frac{\mu_0 l}{2\pi}\left[\ln\left(\frac{b - z_1}{-z_1}\right) - \ln\left(\frac{b - z_2}{-z_2}\right)\right]$$

Solution: b)

Faraday's law $$e = -\frac{\mathrm{d}\Phi_{12}}{\mathrm{d}t}$$

$$I_1(t) = I_0 \cos(2\pi f t)$$

$$e = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mu_0 I_0 \cos(2\pi f t) l}{2\pi}\left[\ln\left(\frac{b - z_1}{-z_1}\right) - \ln\left(\frac{b - z_2}{-z_2}\right)\right]\right)$$
$$= \mu_0 I_0 f \sin(2\pi f t) l \left[\ln\left(\frac{b - z_1}{-z_1}\right) - \ln\left(\frac{b - z_2}{-z_2}\right)\right].$$

The amplitude e_0 of the induced electromotive force:

$$e_0 = \left|\mu_0 I_0 f l \left[\ln\left(\frac{b - z_1}{-z_1}\right) - \ln\left(\frac{b - z_2}{-z_2}\right)\right]\right|$$

$$e_0 = \left|4\pi \cdot 10^{-7} \cdot 100 \cdot 50 \cdot 500 \left[\ln\left(\frac{10.3}{10.2}\right) - \ln\left(\frac{9.8}{9.7}\right)\right]\right| \mathrm{V} = 1.57\mathrm{mV}.$$