Lecture 5: Electro-dynamics

- Faraday’s law
- Displacement current
- Inductance and Lenz’s law
- Energy stored in inductor/magnetic field

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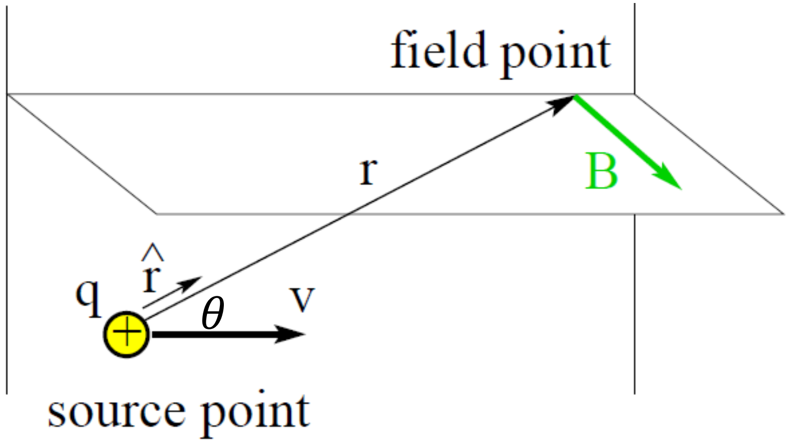
Magnetic field

Charges in motion generate magnetic field



Magnetic field H in vacuum generated by moving charge q: $$\boldsymbol{H} = \frac{1}{4\pi}\frac{q\boldsymbol{v}\times\hat{\boldsymbol{r}}}{r^2}$$

Magnetic flux density B in vacuum generated by moving charge q: $$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{q\boldsymbol{v}\times\hat{\boldsymbol{r}}}{r^2}$$

$$\mathbf{B} = \mu_0\,\mathbf{H}$$

$$\boldsymbol{v}\times\hat{\boldsymbol{r}} = |v|\sin(\theta)\,\boldsymbol{n}$$

n is perpendicular to the plane containing ***v*** and ***r***, in the direction given by **the right-hand rule**

In free space is $\mu_0 = 4\pi\times10^{-7}H/m$.

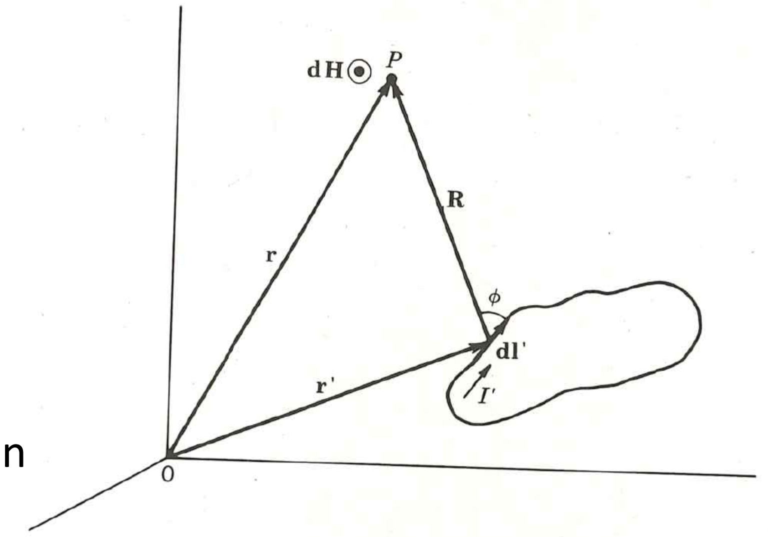
A *steady current* *I* generates magnetic field, Biot–Savart law

$$\boldsymbol{H}(\boldsymbol{r}) = \int_c \frac{I'(r)d\boldsymbol{l}'\times\widehat{\boldsymbol{R}}}{4\pi R^2}$$

$$\boldsymbol{B}(\boldsymbol{r}) = \int_c \frac{\mu_0 I'(r)d\boldsymbol{l}'\times\widehat{\boldsymbol{R}}}{4\pi R^2}$$

$$I'd\boldsymbol{l}'\times\widehat{\boldsymbol{R}} = |I'|dl'\sin(\phi)\,\boldsymbol{n}$$

n is perpendicular to the plane containing $\boldsymbol{l}'$ and $\widehat{\boldsymbol{R}}$, in the direction given by **the right-hand rule**

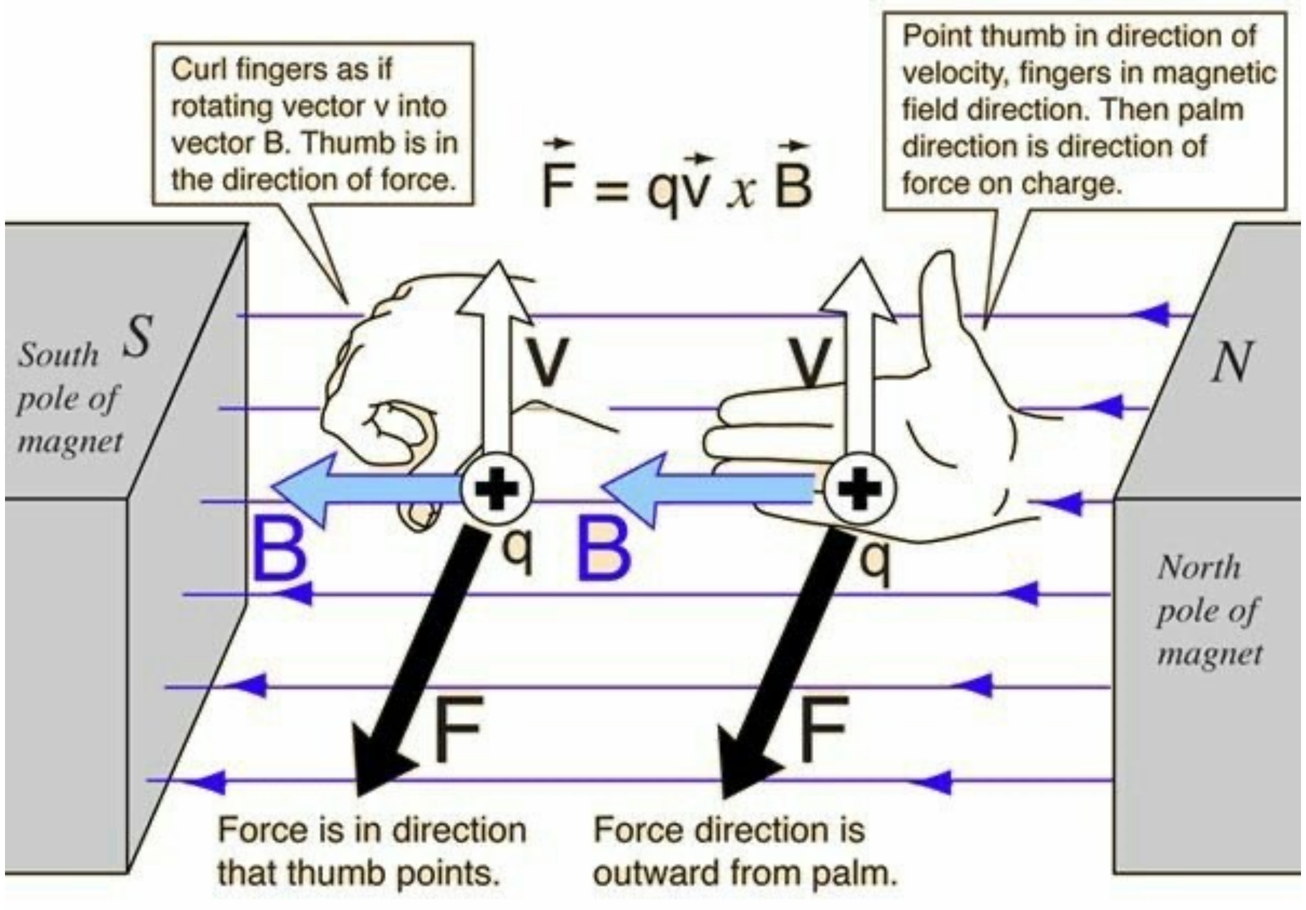


Lorentz law

Force exerted by magnetic field **B** on a moving point charge Q is:

$$\boldsymbol{F} = Q\,\boldsymbol{v}\times\boldsymbol{B}$$

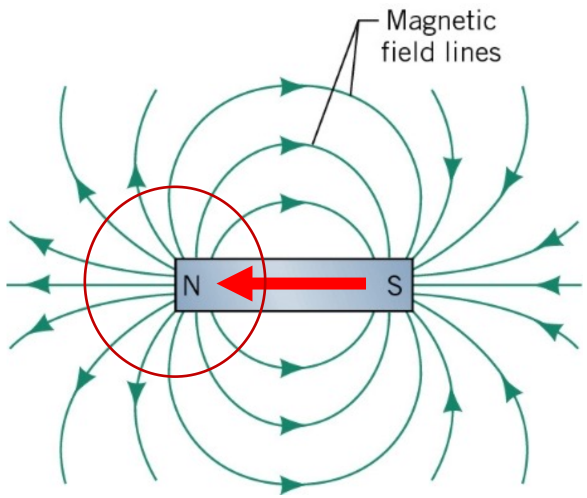
The direction is given by the right-hand rule



Lorentz law

$$\mathbf{F} = q\mathbf{E} + q(\mathbf{v} \times \mathbf{B})$$

Magnetic flux and Gauss's law



Magnetic flux ϕ is the integral of the flux density across surface

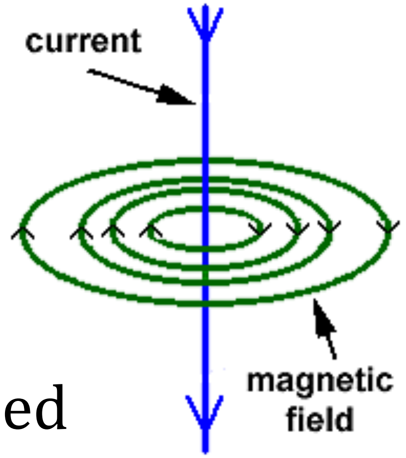
$$\phi = \int_s \mathbf{B} \cdot d\mathbf{S}$$

For an enclosed surface, the flux is zero

$$\oint_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0.$$

Gauss's law $\nabla \cdot \mathbf{B} = 0$

Ampere's law



Ampere's law states that the line integral of the magnetic field around closed loop C is equal to the electric current enclosed in the loop.

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int_s \mathbf{J} \cdot d\mathbf{s} = I$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J},$$

Currents generate magnetic field

Magnetic field in material

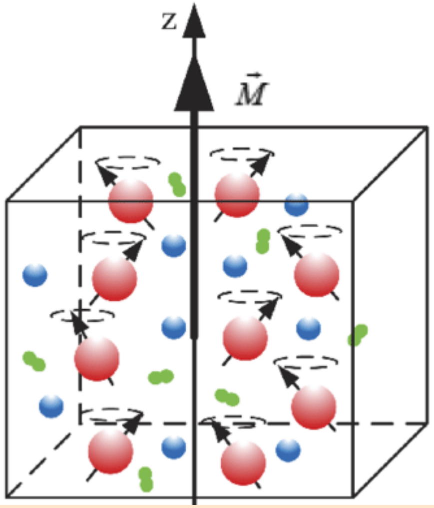
Once there is magnetic field applied to medium, magnetization, **M**, occurs.

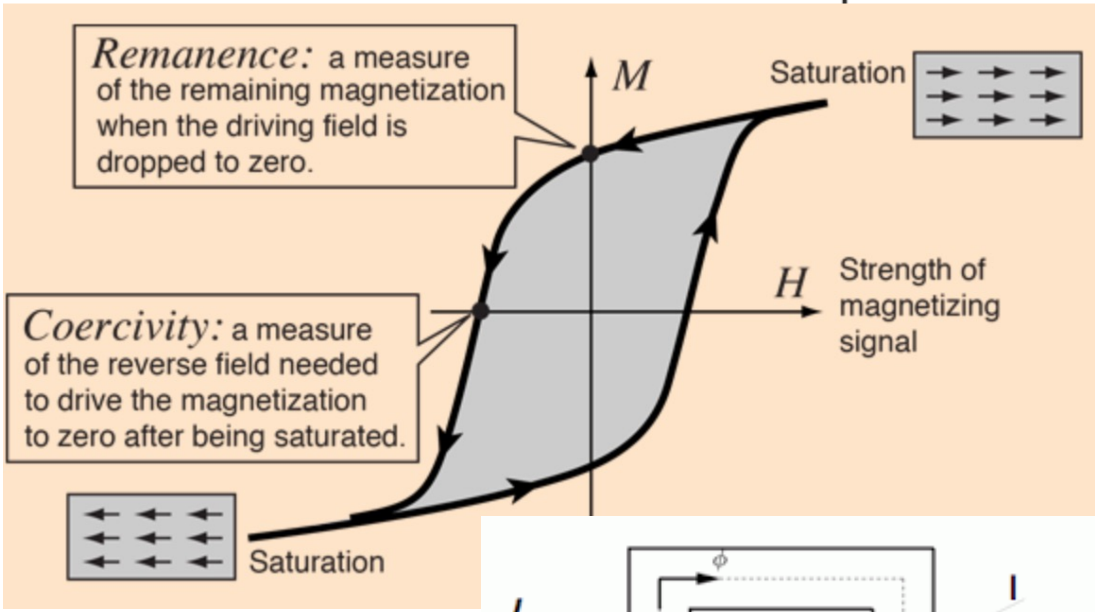
$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$$

$$\mathbf{M} = \chi_m \mathbf{H}$$

$$\mathbf{B} = \mu_0(1 + \chi_m)\boldsymbol{H} = \mu_0 \mu_r \mathbf{H} = \mu \mathbf{H}$$

χ_m is magnetic susceptibility, used to quantify the additional field **M**.
μ_r *relative permeability*.



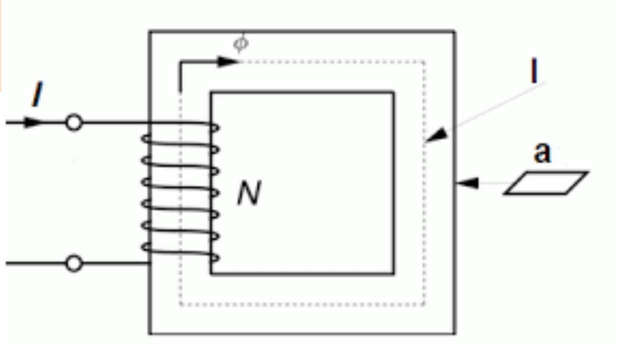


Magnetic circuit

Magnetomotive force (MMF): $F = NI = HL = \Phi R$

Φ, magnetic flux

$R = \frac{l}{\mu S}$, magnetic reluctance



Faraday's law

$\boldsymbol{\varepsilon} = -\frac{d\phi}{dt}$ **Faraday's law,** $\phi = \int_s \mathbf{B} \cdot d\mathbf{s}$ **is the magnetic flux.**

In static-electric field, $\nabla \times \mathbf{E} = 0$; and $\oint_c \mathbf{E} \cdot d\mathbf{l} = 0$, conservative vector field.

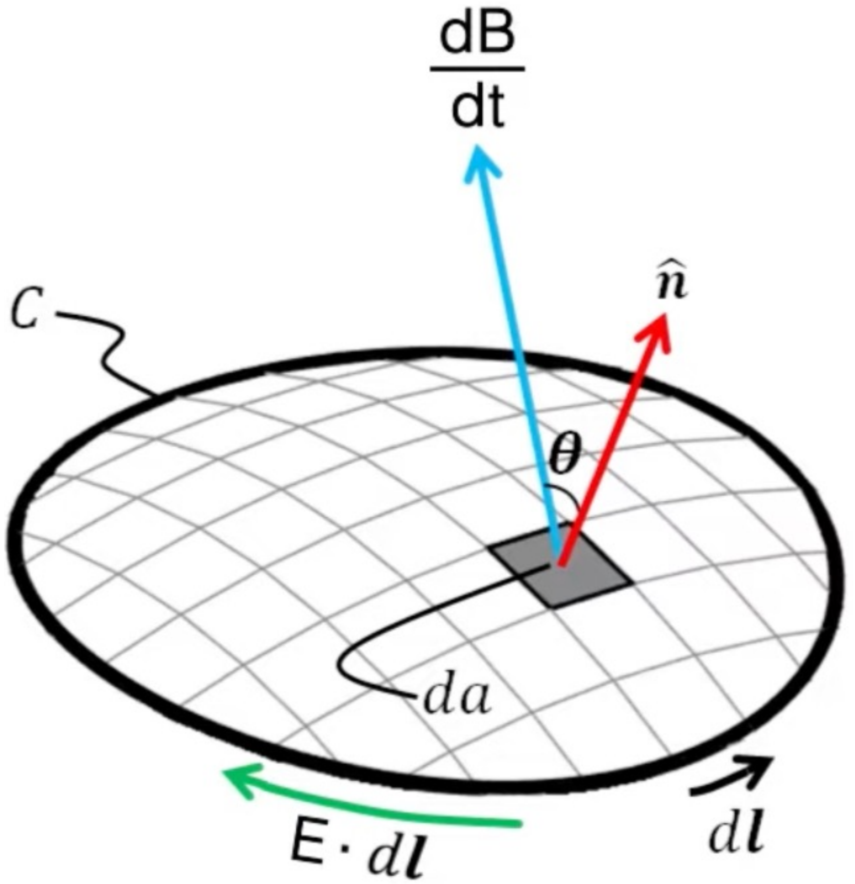
When $\frac{\partial \mathbf{B}}{\partial t} \neq 0$, $\boldsymbol{\nabla} \times \mathbf{E} \neq \mathbf{0}$

$$\oint_c \mathbf{E} \cdot d\mathbf{l} = -\frac{d\phi}{dt} = -\frac{d}{dt}\left(\int_s \mathbf{B} \cdot d\mathbf{s}\right) = -\int_s \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

$$\oint_c \mathbf{E} \cdot d\mathbf{l} = \int_s (\nabla \times \mathbf{E}) \cdot d\mathbf{s}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Time varying magnetic field generates electric field.



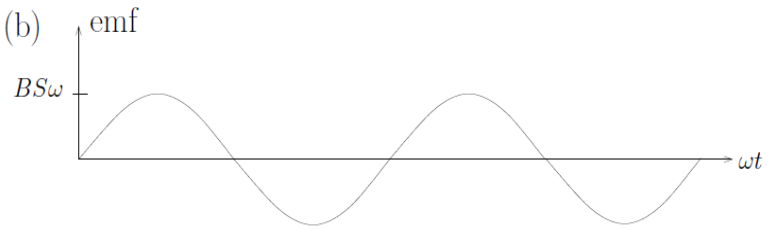
Faraday’s law for generator

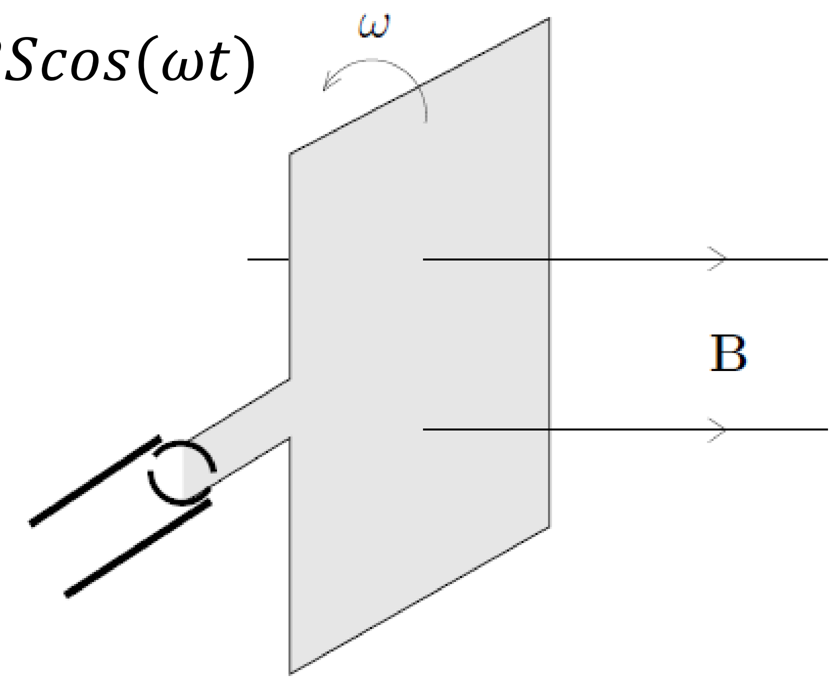
The angle between B and surface normal direction is: $\varphi = \omega t.$

What is the EMF and its waveform with time?

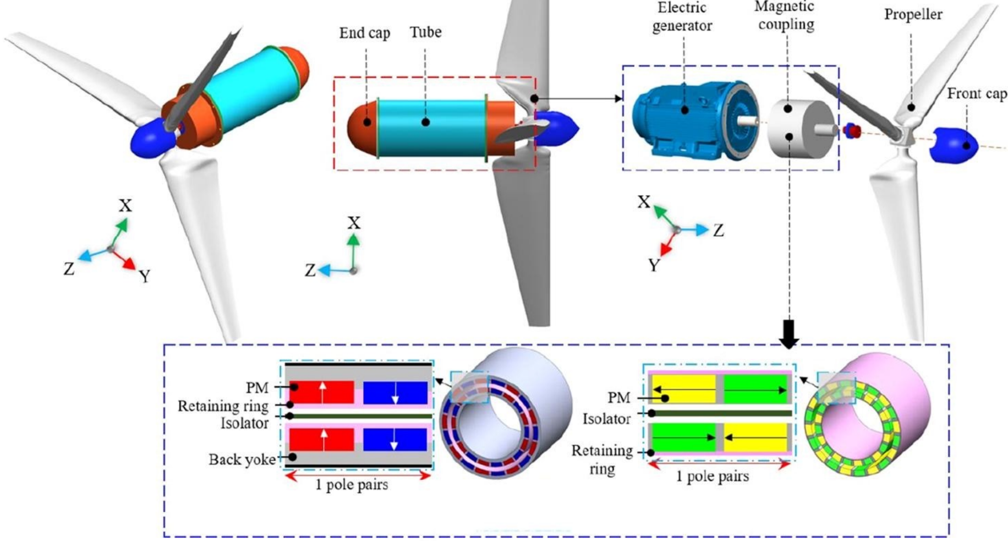
$$\phi = \int_s \mathbf{B} \cdot d\mathbf{s} = \mathbf{B} \cdot \mathbf{S} = BScos(\omega t)$$

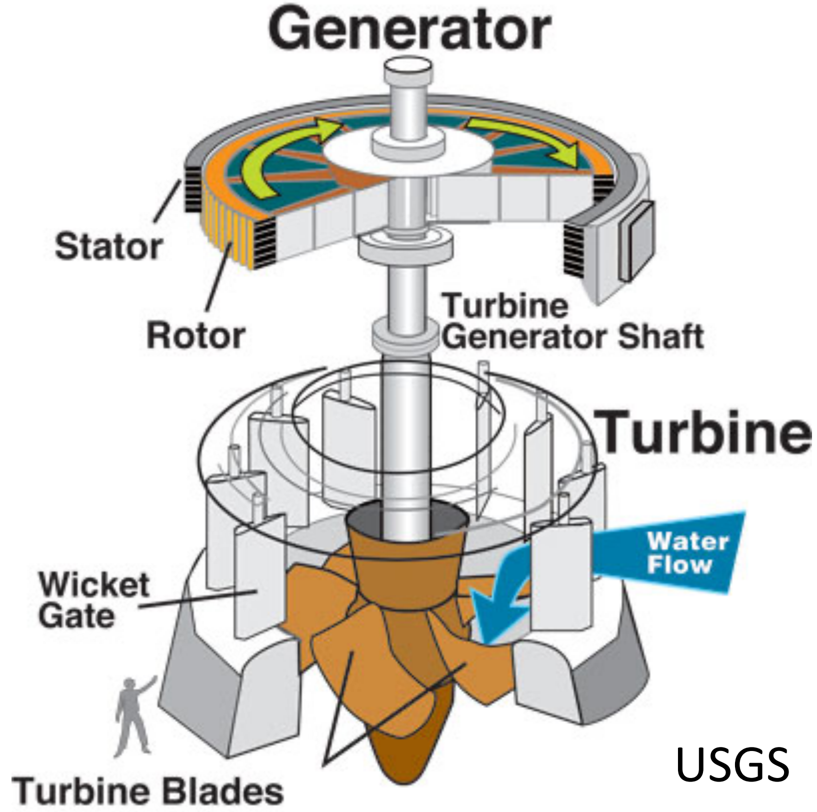
$$e = -\frac{d\phi}{dt} = BS\omega sin(\omega t)$$





Park et al. 2020





USGS

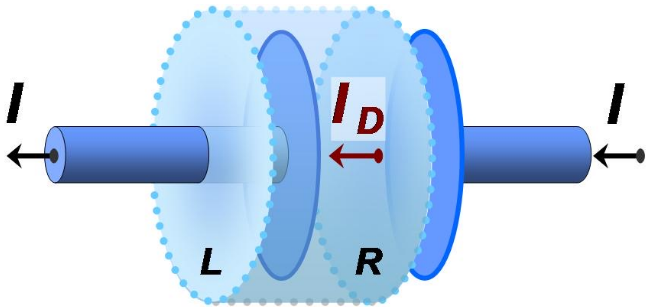
Displacement current

Displacement current is defined as the rate of change of electric displacement field.

$$\mathbf{J}_D = \frac{\partial \mathbf{D}}{\partial t}$$

Ampere's law

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}; \quad (\,\mathbf{J} = \sigma \mathbf{E}\ \text{is conduction current in materials})$$



Capacitor

Displacement current solves some puzzles:

1) Electro-magnetic wave propagates in vacuum, where there is no current **J**
2) Current in capacitor

When charging/discharging a capacitor, displacement current exists between the two plates (vacuum medium), and electric field changes.

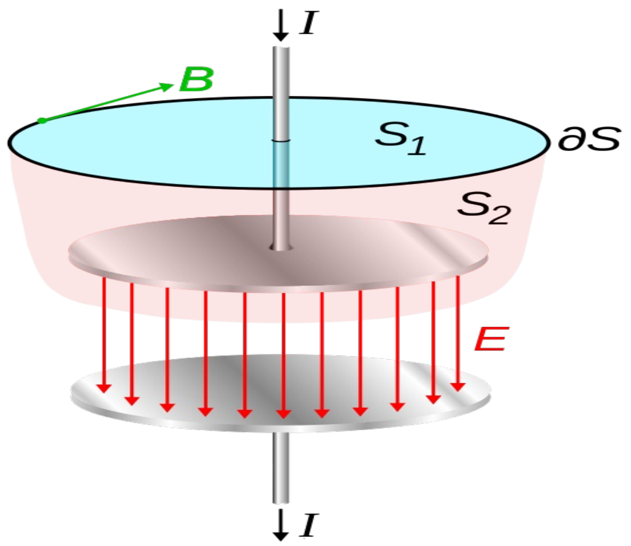
Time varying electric field generates magnetic field.

Capacitor and displacement current

Capacitor is charging with current I

Capacitance is $C = \frac{\varepsilon A}{l}$ charge current is $I = C\frac{dV}{dt}$

$$\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}$$



Calculating the integral over surface S1,

$$\int_s \nabla\times\mathbf{H}\cdot d\mathbf{S}_1 = \int_s \left(\mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}\right)\cdot d\mathbf{S}_1 = \int_s \mathbf{J}\cdot d\mathbf{S}_1 = I$$

$\frac{\partial D}{\partial t}$ = 0 out of capacitor

Calculating the integral over surface S2,

$$\int_s \nabla\times\mathbf{H}\cdot d\mathbf{S}_2 = \int_s \left(\mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}\right)\cdot d\mathbf{S}_2 = \int_s \frac{\partial\mathbf{D}}{\partial t}\cdot d\mathbf{S}_2 = \int_s \frac{\varepsilon}{l}\frac{dV}{dt}dS_2 = \frac{\varepsilon A}{l}\frac{dV}{dt} = I$$

J = 0 at surface S_2

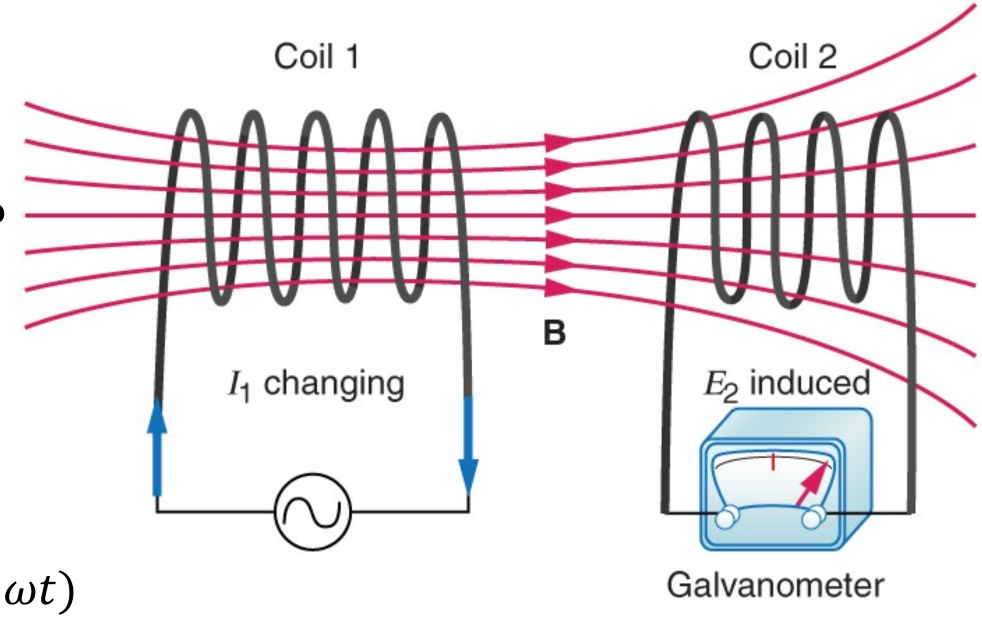
$\boldsymbol{D} = \varepsilon\boldsymbol{E} = \varepsilon\frac{\boldsymbol{V}}{l}$, only inside capacitor.

Maxwell's equations

Differential equations (SI convention)	
$\nabla \cdot \mathbf{D} = \rho_\mathrm{f}$	Gauss's law for electric field
$\nabla \times \mathbf{H} = \mathbf{J}_\mathrm{f} + \frac{\partial \mathbf{D}}{\partial t}$	Ampere's law
$\nabla \cdot \mathbf{B} = 0$	Gauss's law for magnetic field
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	Faraday's law

Example

In coil 1: I(t) = $I\sin(\omega t)$, coil cross−section area 1 = area 2.
The flux produced from coil 1 going through coil 2: B(t) = $B\sin(\omega t)$
1) What is the induced voltage in coil 2?
2) Assuming resistance in coil 2 is R, calculating the current in coil 2?



$\phi = N \int_s \mathbf{B} \cdot d\mathbf{s}$ (flux linkage between coil 1 and coil 2)

$$V_2(t) = -\frac{d\phi}{dt} = -\frac{N\int_s B\sin(\omega t)\,ds}{dt} = -\frac{NAB\sin(\omega t)}{dt} = -N\omega AB\cos(\omega t)$$

$$I_2(t) = \frac{V_2(t)}{R} = -\frac{N\omega A\hat{B}\cos(\omega t)}{R}$$

Inductance and Lenz's law

Inductance definition:

$$L = \frac{\phi}{I} = \frac{N \int_S \mathbf{B} \cdot d\mathbf{S}}{I}$$

Self inductance: $L_1 = \frac{\phi_1}{I_1} = \frac{N_1 \int_{S_1} B_1 ds_1}{I_1}$

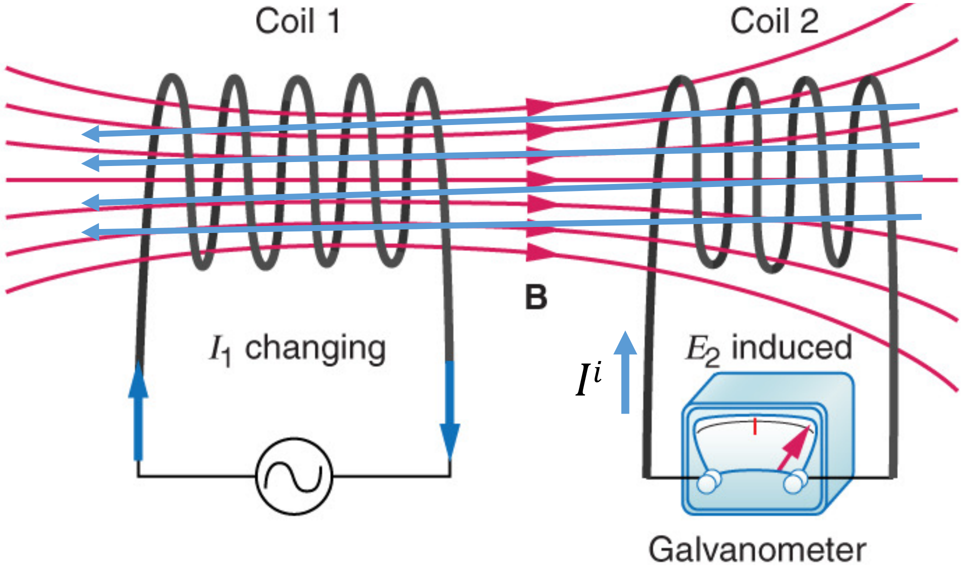
Mutual inductance: $M_{12} = \frac{\phi_{12}}{I_1} = \frac{N_2 \int_{S_2} B_1 ds_2}{I_1}$

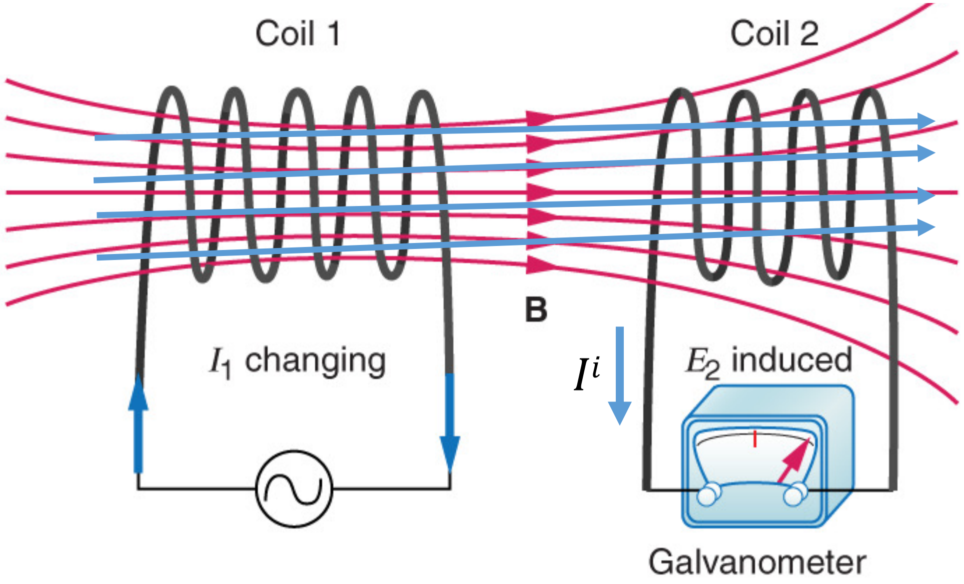
$$L_2 = \frac{\phi_2}{I_2} = \frac{N_2 \int_{S_2} B_2 dS_2}{I_2} \qquad M_{21} = \frac{\phi_{21}}{I_2} = \frac{N_1 \int_{S_1} B_2 ds_1}{I_2}$$

Lenz's Law

The magnetic field created by the induced current opposes changes in the initial magnetic field.

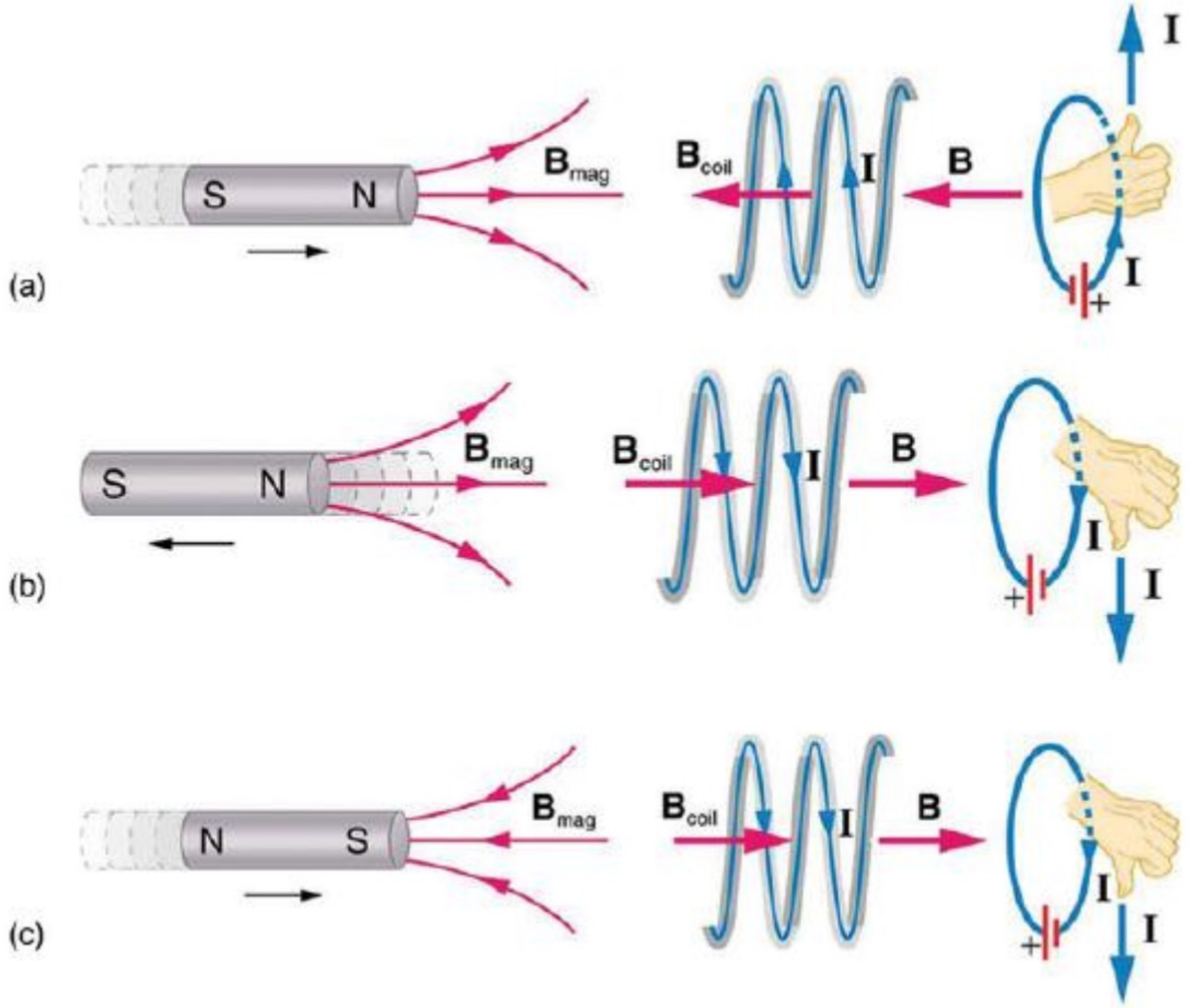
When B1 increase, the direction of B2 is opposite. Against the change





When B1 decreases, the direction of B2 is the same as B1. Against the change

Lenz’s law



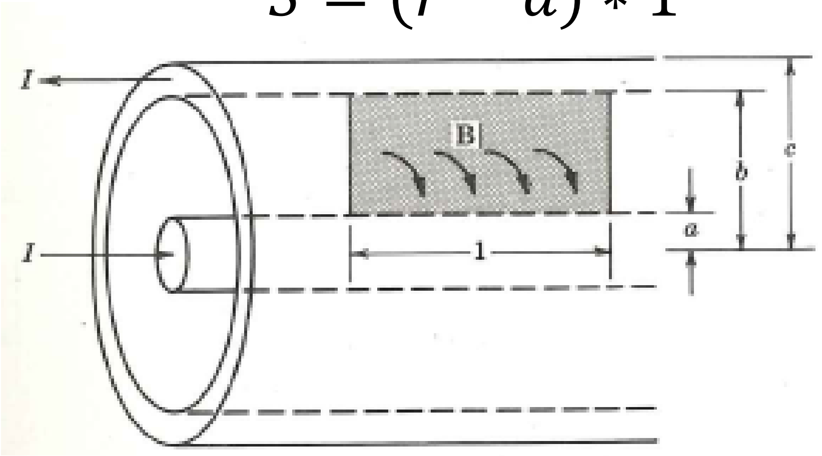
Example: calculating inductance

A coaxial line carrying current I on the inner conductor and –I on the outer.
Calculate the magnetic field H at r distance (current evenly distributed in the two conductors)

Calculate the external inductance of the coaxial line in unit length.

$$S = (r - a) * 1$$

$$H_\phi = \frac{I}{2\pi r}$$

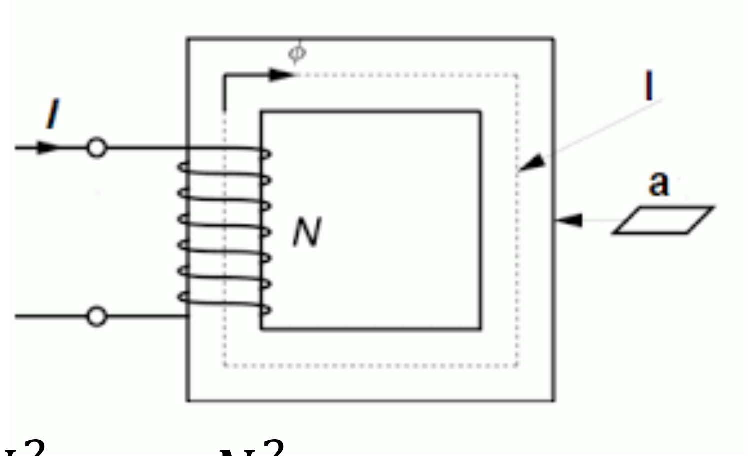


$$\int_S \mathbf{B}_\phi \cdot d\mathbf{S} = \int_a^b \mu \frac{I}{2\pi r} dr = \frac{\mu I}{2\pi} \ln \frac{b}{a}$$

$$L = \frac{\mu}{2\pi} \ln \frac{b}{a}$$

Inductance

Assume the total length of an inductor is l and the cross-section area is a



$$L = \frac{\phi}{I} = \frac{N\int_S \mathbf{B}\cdot d\mathbf{s}}{I}$$

MMF $NI = HL = \Phi R$

Magnetic reluctance $R_m = \frac{l}{\mu a}$

$$R_m\phi = NI$$

Flux density $B = \frac{\phi}{a} = \frac{NI}{aR_m} = \frac{NI\mu}{l}$

$$L = \frac{N^2 a\mu}{l} = \frac{N^2}{R_m}$$

Energy stored in inductor

$$L = \frac{\int_s \boldsymbol{B} \cdot d\boldsymbol{S}}{I}$$

$$V(t) = -\frac{d}{dt}\left(\int_s \boldsymbol{B} \cdot d\boldsymbol{S}\right) = -\frac{d\phi}{dt}$$

$$V(t) = \frac{d}{dt}(LI) = L\frac{dI}{dt} \quad \Rightarrow \quad dp = VI = LI\frac{dI}{dt}$$

$$W = \int_0^I LI\frac{dI}{dt}dt = \int_0^I LI dI = \frac{1}{2}LI^2$$

Energy in capacitor

$$W_e = \frac{1}{2}CV^2$$

Energy stored in magnetic field

$$L = \frac{N^2 a\mu}{l}$$

$$B = \frac{\phi}{a} = \frac{NI}{aR_m} = \frac{NI\mu}{l}$$

$$W = \frac{1}{2}LI^2 = \frac{1}{2}\frac{N^2\mu a}{l}I^2 = \frac{1}{2}\frac{N^2 I^2 \mu^2}{l^2}\frac{la}{\mu} = \frac{B^2}{2\mu}V_{vol}$$

Energy in electric field

$$W_e = \frac{1}{2}CV^2 = \frac{1}{2}V_{vol}DE$$

Magnetic energy density

$$\eta_m = \frac{W}{V_{vol}} = \frac{B^2}{2\mu}$$

Electric energy density

$$\eta_e = \frac{W_e}{V_{vol}} = \frac{1}{2}ED = \frac{1}{2}\varepsilon E^2$$

Example

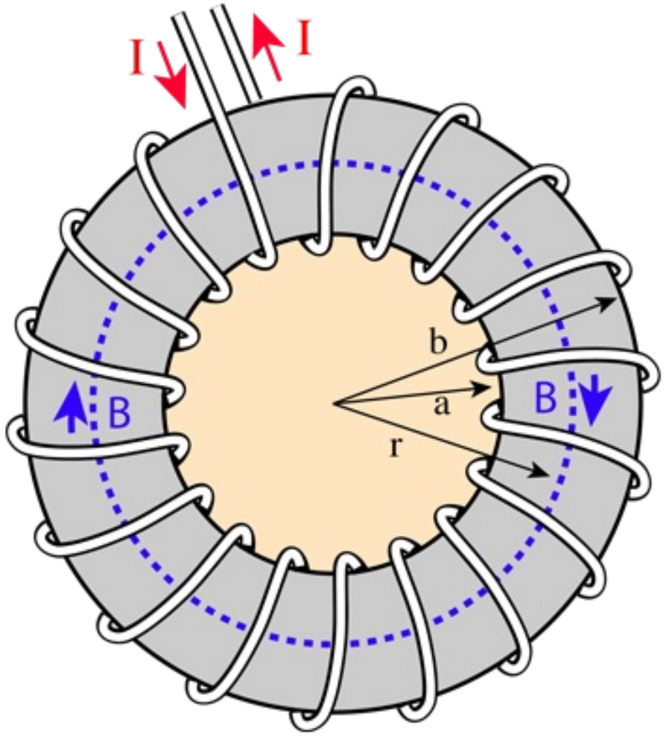
Given a tightly wound toroid with radius a, and N number of turns that conducts a constant direct current I. Use a = 10cm, N = 1000 and I = 1mA. Find the magnetic field **B** everywhere assuming the core consists of

a) vacuum
b) an iron core with μ_r = 5000.

$$NI = Hl = \Phi R$$

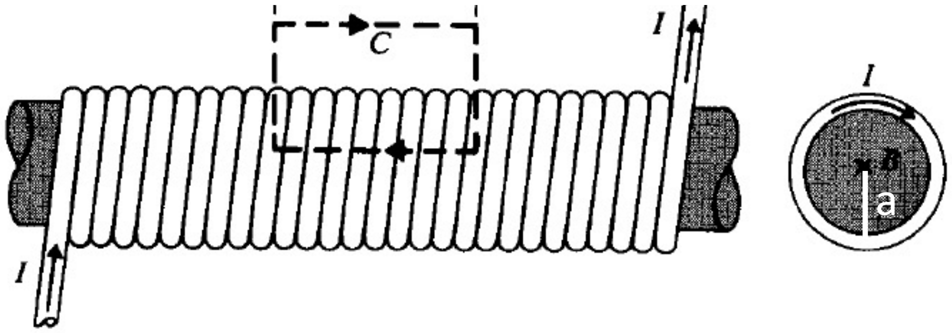
$$H2\pi r = NI$$

$$B = \frac{NI\mu}{2\pi r}$$



Example

a) Find the self inductance L of a long, tightly wound solenoid.
b) If the number of turns is doubled (and everything else remains the same), what will happen with the self inductance?
c) Assume that the current is decaying from I_0 to 0 during the time T. Find the induced voltage as a function of I_0, T and L.



$$\mathbf{H} = H\hat{\mathbf{z}} = \frac{NI}{l}\hat{\mathbf{z}}. \qquad B = \mu\frac{NI}{l}$$

$$\Phi_{\text{cs}} = \int_S \mathbf{B}\cdot \mathrm{d}\mathbf{S} = \int_S B\mathrm{d}S = B\int_S \mathrm{d}S = \mu\frac{NI}{l}\cdot \pi a^2.$$

$$L = \frac{\Phi}{I} = \frac{N\Phi_{\text{cs}}}{I} = \frac{\mu\pi a^2 N^2}{l}.$$

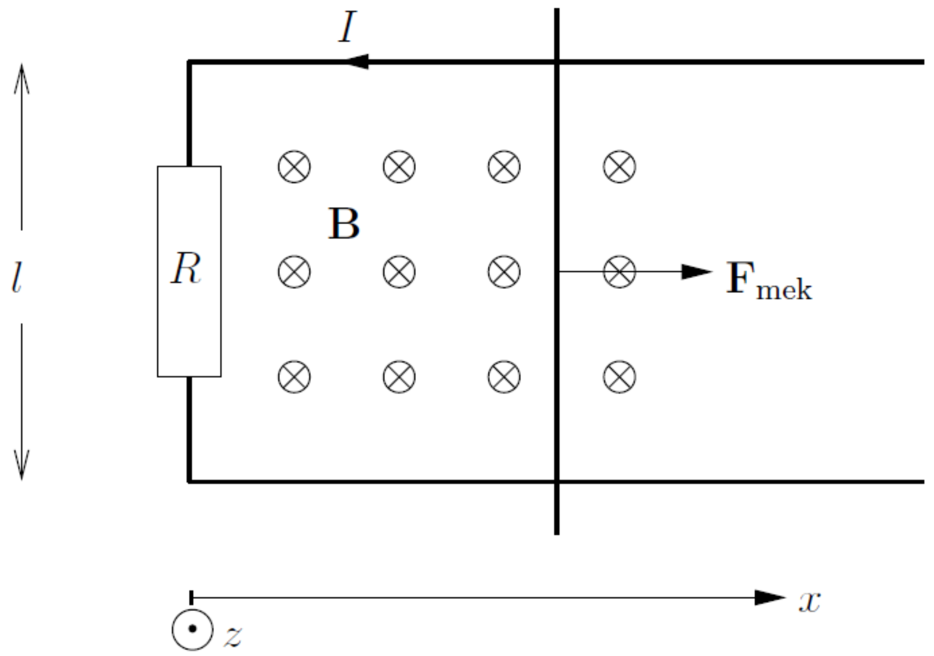
$$e(t) = -L\frac{\mathrm{d}I(t)}{\mathrm{d}t},$$

$$I(t) = I_0\left(1-\frac{t}{\tau}\right), \quad \text{for } 0 \le t \le \tau.$$

$$e(t) = -L\frac{\mathrm{d}}{\mathrm{d}t}I_0\left(1-\frac{t}{\tau}\right) = \frac{LI_0}{\tau}.$$

Example

Given a circuit, the right boundary can move. When it moves at the speed of v, what is current in the circuit? S is the area of enclosed by the circuit.



$$\mathbf{S} = (S_0 + lvt)\hat{\mathbf{z}}.$$

$$e = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_S \mathbf{B}\cdot\mathrm{d}\mathbf{S} = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{B}\cdot\int_S \mathrm{d}\mathbf{S}\right) = -\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{B}\cdot\mathbf{S} = \frac{\mathrm{d}}{\mathrm{d}t}BS = Blv$$

$$e = BLv$$

$$I = \frac{e}{R} = \frac{Blv}{R}$$