Lecture 4: Stationary magnetic field

- Magnetic field
- Gauss's law
- Ampere's law
- Magnetic field in material

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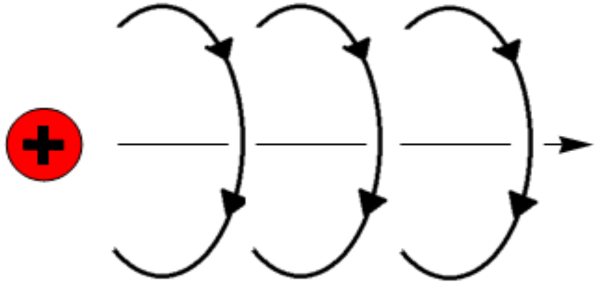
Magnetic field

Charges in motion (currents) produce magnetic field



Magnetic field in vacuum generated by moving charge q: $$\boldsymbol{H} = \frac{1}{4\pi}\frac{q\boldsymbol{v}\times\hat{\boldsymbol{r}}}{r^2}$$ Independent of material property

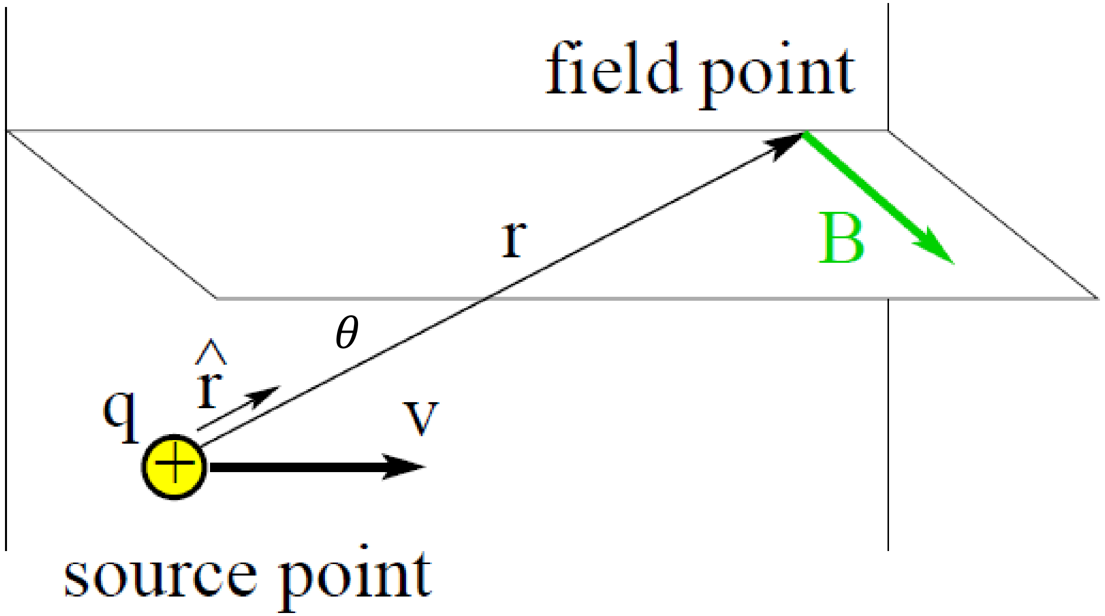
Magnetic flux density B in vacuum generated by moving charge q: $$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{q\boldsymbol{v}\times\hat{\boldsymbol{r}}}{r^2}$$

$$\boldsymbol{v}\times\hat{\boldsymbol{r}} = |v|\sin(\theta)\,\boldsymbol{n}$$

n is perpendicular to the plane containing ***v*** and ***r***, in the direction given by the right-hand rule

$$\mathbf{B} = \mu_0\,\mathbf{H}$$

μ is called permeability and material dependent, in free space is $\mu_0 = 4\pi\times10^{-7}H/m$.



Magnetic (Lorentz) force

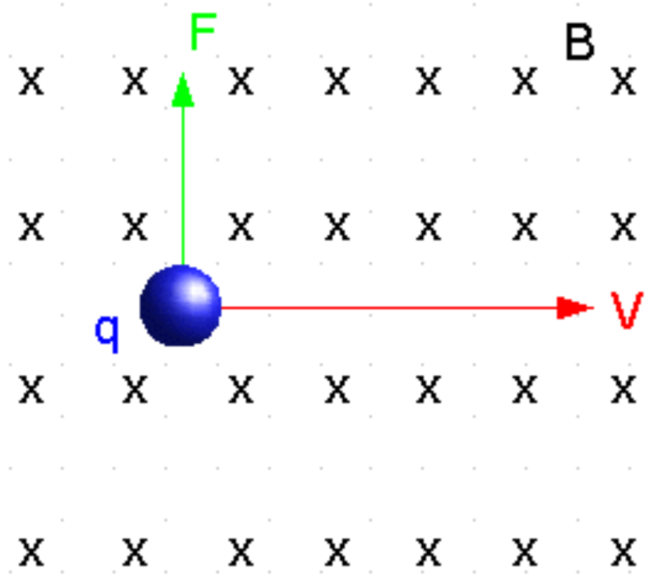
Force exerted by magnetic field **B** on a moving point charge Q is:

$$\boldsymbol{F} = Q\ \boldsymbol{v}\times\boldsymbol{B}$$

Lorentz law

Compare to $\boldsymbol{F}_c$

Coulomb's law



Magnetic force acting on a moving charge is always perpendicular to it's moving direction, so magnetic force only changes the charges' moving direction.

$$W = \boldsymbol{F}\cdot\boldsymbol{L} = \int \boldsymbol{F}\cdot d\boldsymbol{l} = \int \boldsymbol{F}\cdot\boldsymbol{v}dt$$

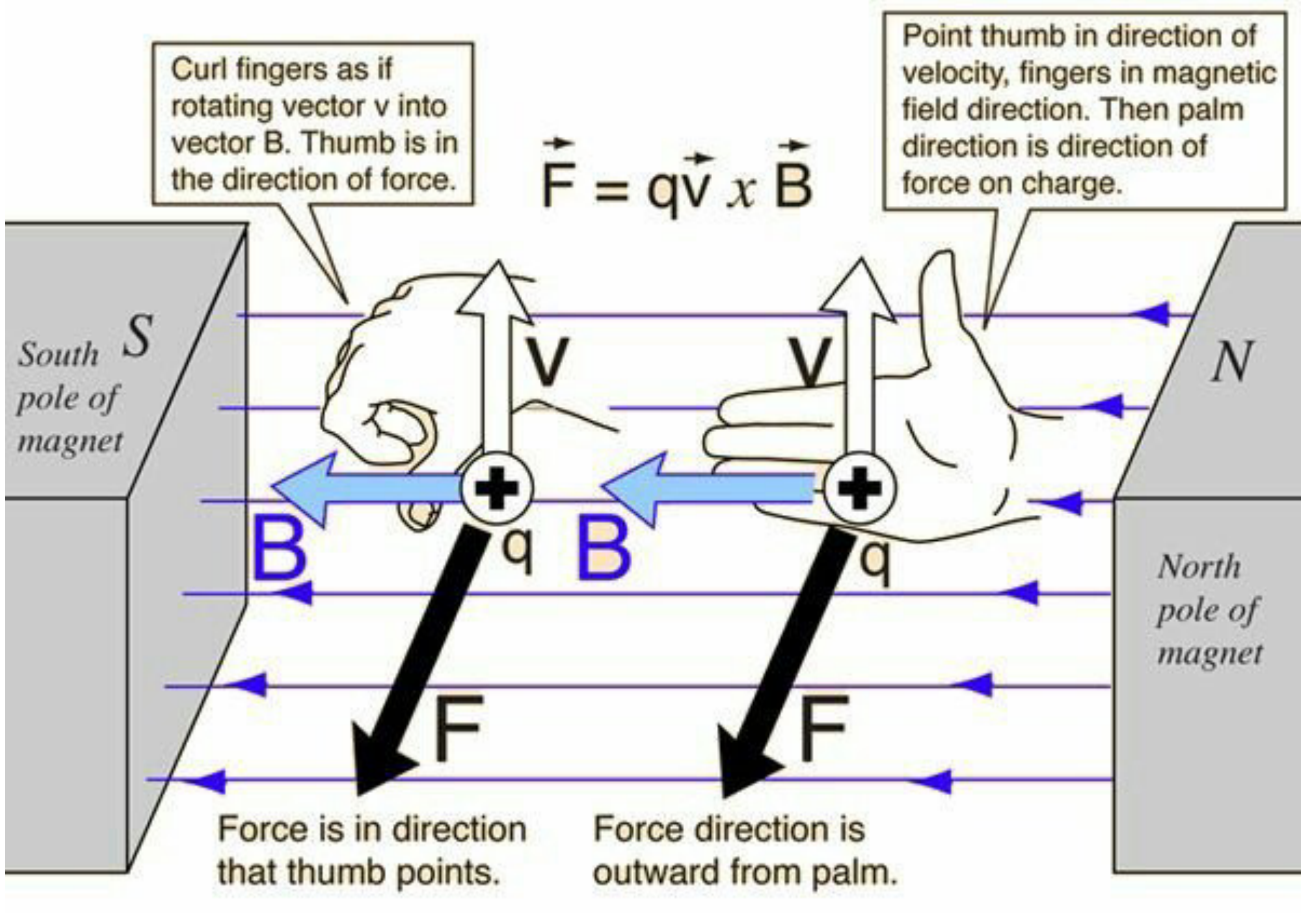
Work is application of force, ***F****, to move an object over a distance,* ***L****,* in the direction that the force is applied.

Lorentz law

Force exerted by magnetic field **B** on a moving point charge Q is:

$$\boldsymbol{F} = Q\, \boldsymbol{v} \times \boldsymbol{B}$$

The direction given by the right-hand rule

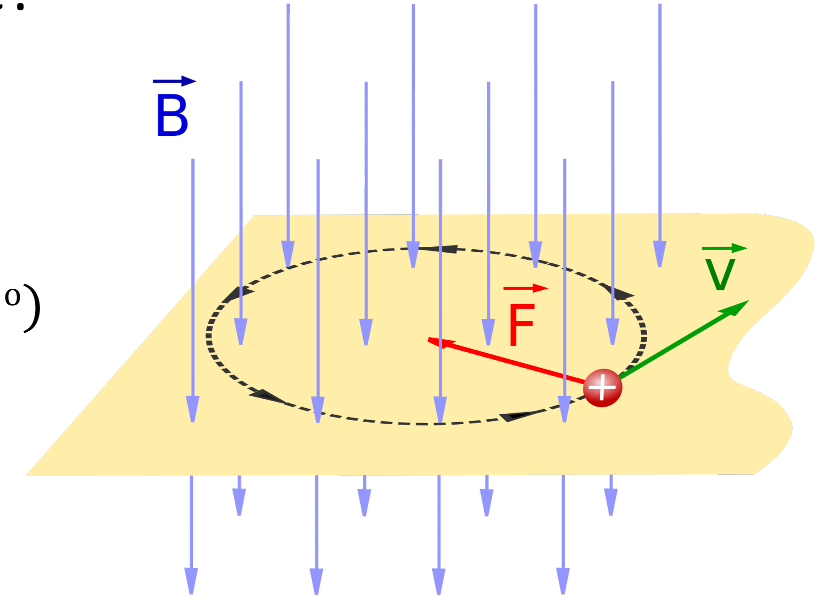


Lorentz law

$$\mathbf{F} = q\mathbf{E} + q(\mathbf{v} \times \mathbf{B})$$

Example: Point charge's movement in constant uniform magnetic filed

A constant uniform magnetic field **B**, a charge q with mass m is shot perpendicularly to the magnetic field with speed **V**(0), what is the radius of charge?



Centrifugal force

$$F = \frac{mv^2}{r} = qvBsin\theta$$ (θ is the angle between **v** and **B**, 90°)

$$r = \frac{mv^2}{qvB} = \frac{mv}{qB}$$

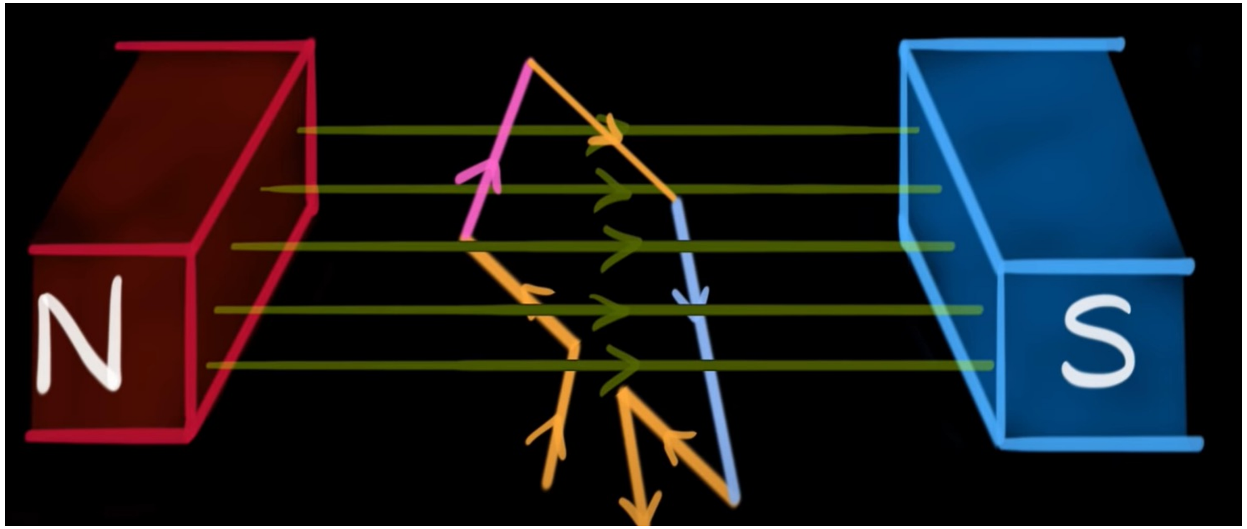
https://www.youtube.com/watch?v=orsMYomjwIw

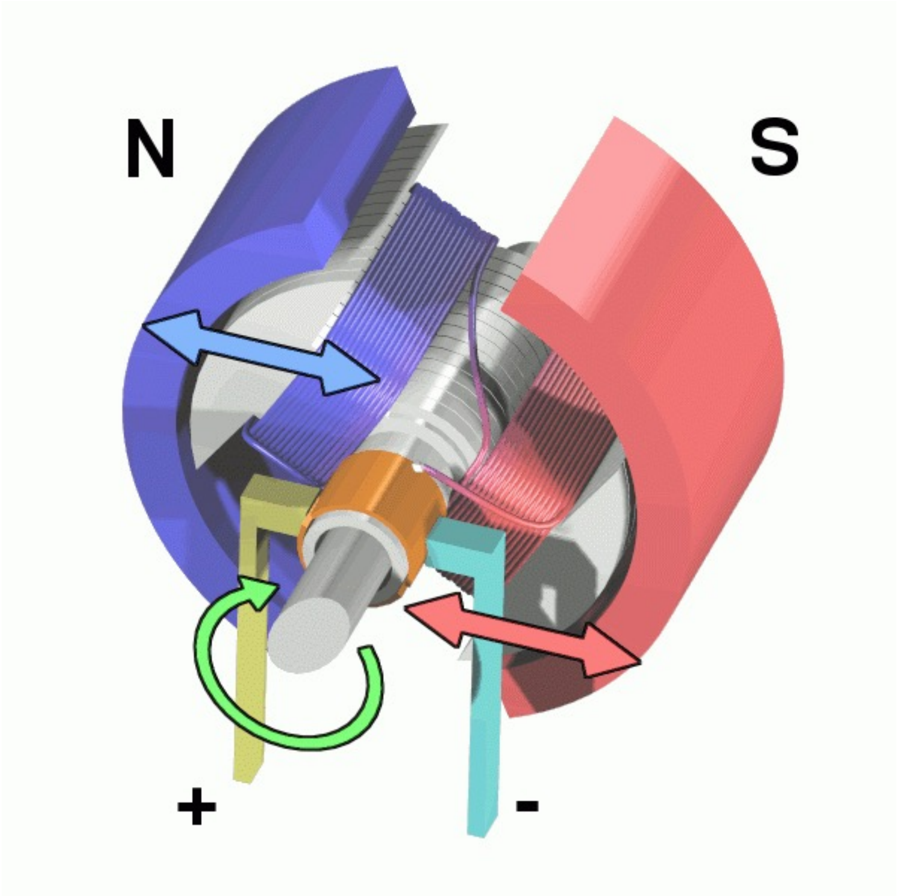
What happens if θ is not 90°?

Electric motor

It produces mechanical energy by **current and magnets**

Rotating windings: electric current (moving charge) flows





Biot-Savart law: Magnetic field generated by current

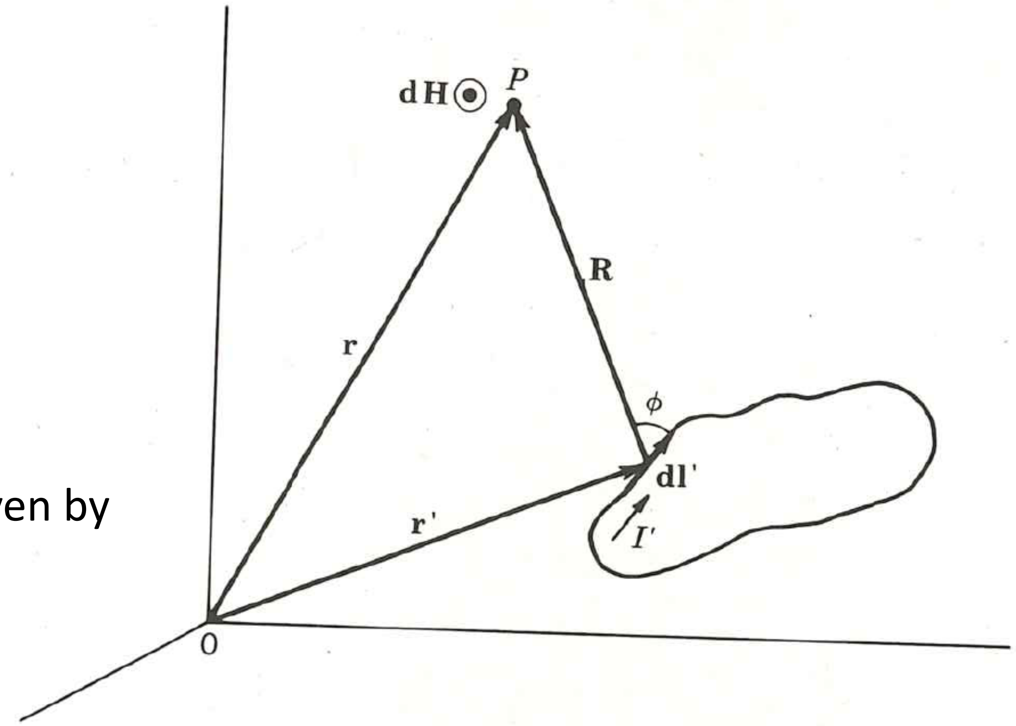
The Biot–Savart law is used for computing the resultant magnetic field **B** at position **r** in 3D-space generated by a *steady current I.*

$$\boldsymbol{H}(\boldsymbol{r}) = \int_c \frac{I'(r)d\boldsymbol{l}'\times\widehat{\boldsymbol{R}}}{4\pi R^2}$$

$$\boldsymbol{B}(\boldsymbol{r}) = \int_c \frac{\mu_0 I'(r)d\boldsymbol{l}'\times\widehat{\boldsymbol{R}}}{4\pi R^2}$$

$$I'd\boldsymbol{l}'\times\widehat{\boldsymbol{R}} = |I'|dl'\sin(\phi)\,\boldsymbol{n}$$

n is perpendicular to the plane containing $\boldsymbol{l}'$ and $\widehat{\boldsymbol{R}}$, in the direction given by **the right-hand rule**



Scalar calculation

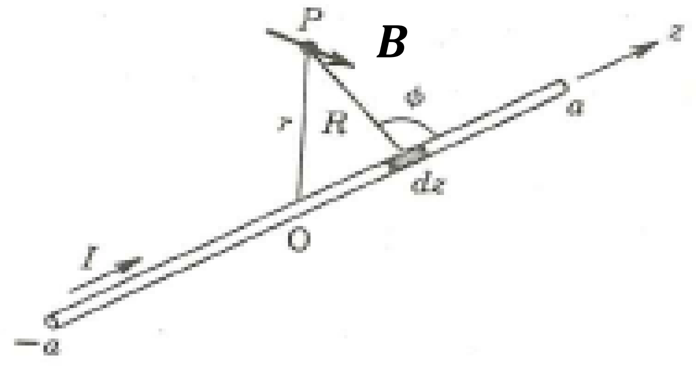
$$H(r) = \frac{1}{4\pi}\int_c \frac{I'(r)dl'sin\phi}{R^2}$$

$$B(r) = \frac{\mu_0}{4\pi}\int_c \frac{I'(r)dl'sin\phi}{R^2}$$

Example: Magnetic flux density

Find the magnetic field B at a point **P** at perpendicular distance r from the center of a finite length of current I, the total current length is 2a.

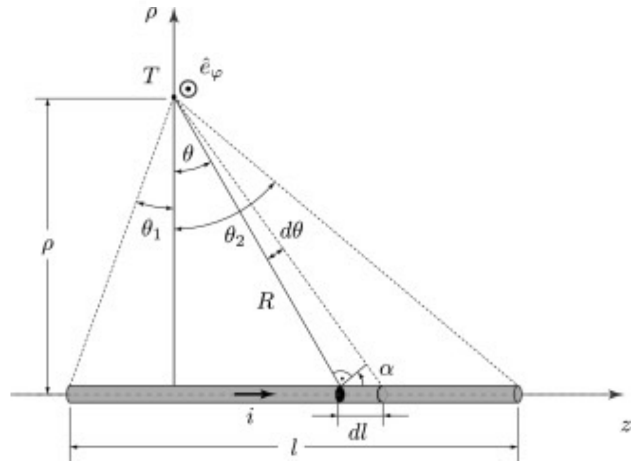
$$d\boldsymbol{B} = \frac{\mu_0}{4\pi}\frac{Id\boldsymbol{l}\times\widehat{\boldsymbol{R}}}{R^2}$$



$$\boldsymbol{B} = \int_{-a}^{a}\frac{\mu_0}{4\pi}\frac{Id\boldsymbol{z}}{R^2}\hat{R} = \int_{-a}^{a}\frac{\mu_0 \boldsymbol{I}dz}{4\pi R^2}\hat{R} = \int_{-a}^{a}\frac{\mu_0 I\sin\phi dz}{4\pi R^2}$$

$$\sin(\phi) = \frac{r}{\sqrt{r^2+z^2}} \qquad R^2 = r^2 + z^2$$

$$B = \int_{-a}^{a}\frac{\mu_0 I\sin\phi dz}{4\pi R^2} = \frac{\mu_0 Ir}{4\pi}\int_{-a}^{a}\frac{dz}{(r^2+z^2)^{3/2}} = \frac{\mu_0 I}{2\pi r}\frac{a}{(r^2+a^2)^{1/2}}$$



Assuming a>>r, what is B?

$$B = \frac{\mu_0 I}{2\pi ra}$$

Example: Magnetic flux density

$$\frac{z}{r} = \tan(u)$$

$$dz = \frac{r\,du}{\cos^2(u)}$$

$$A = \int \frac{dz}{(r^2+z^2)^{3/2}} = \int \frac{dz}{(r^2+z^2)^{3/2}} = \int \frac{dz}{(r^2+r^2\tan^2(u))^{3/2}} = \int \frac{1}{(r^2+r^2\tan^2(u))^{3/2}} \frac{r\,du}{\cos^2(u)} =$$
$$\int \frac{1}{r^3(1+\tan^2(u))^{3/2}} \frac{r\,du}{\cos^2(u)} = \int \frac{1}{r^2\left(\frac{\cos^2(u)}{\cos^2(u)}+\frac{\sin^2(u)}{\cos^2(u)}\right)^{3/2}} \frac{du}{\cos^2(u)} = \int \frac{\cos(u)}{r^2(1)^{3/2}} du = \int \frac{\cos(u)}{r^2} du =$$
$$\frac{\sin(u)}{r^2} + c = \frac{z}{r^2(r^2+z^2)^{1/2}} + c$$

$$B = \frac{\mu_0 I r}{4\pi} A = \frac{\mu_0 I r}{4\pi} \int_{-a}^{a} \frac{dz}{(r^2+z^2)^{3/2}} = \frac{\mu_0 I}{2\pi r} \frac{a}{(r^2+a^2)^{1/2}}$$

Example: Field on axis of circular loop

A ring with radius *a* and current I, calculate B at point on the z axis.

$$\mathrm{d}\mathbf{B}\cdot\hat{\mathbf{z}} = \mathrm{d}B\cos\beta = \mathrm{d}B\sin\alpha,$$

B and d**B** directions are different

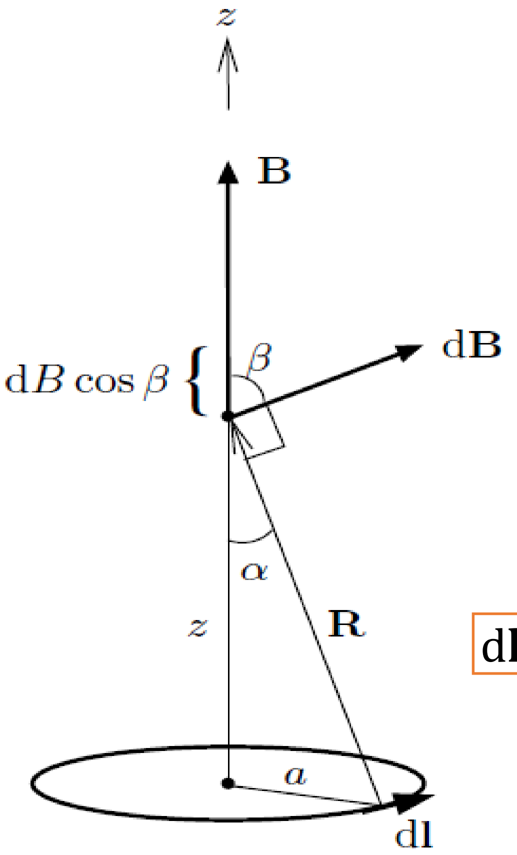
$$\mathrm{d}\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I\mathrm{d}\mathbf{l}\times\hat{\mathbf{R}}}{R^2} \qquad \mathrm{d}B = \frac{\mu_0}{4\pi}\frac{I\mathrm{d}l}{R^2}.$$

$$B = \oint_{\text{ring}} \mathrm{d}B\sin\alpha = \frac{\mu_0 I\sin\alpha}{4\pi R^2}\oint \mathrm{d}l = \frac{\mu_0 I\sin\alpha(2\pi a)}{4\pi R^2} = \frac{\mu_0 I a^2}{2R^3}$$

$$R = \sqrt{z^2 + a^2}$$

$$\sin\alpha = \frac{a}{R}$$

$$\mathbf{B} = \frac{\mu_0 I a^2}{2R^3}\hat{\mathbf{z}} = \frac{\mu_0 I a^2}{2(z^2 + a^2)^{3/2}}\hat{\mathbf{z}}$$

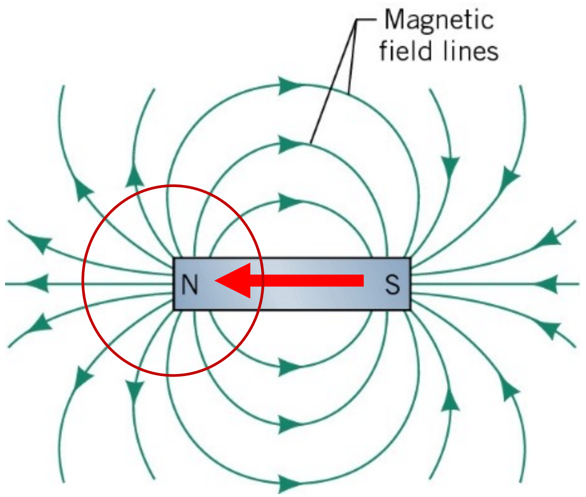


$$\mathbf{dl}\times\hat{\boldsymbol{R}} = \mathrm{d}l$$

Magnetic flux and Gauss's law

Magnetic flux ϕ is the integral of the flux density across surface

$$\phi = \int_s \boldsymbol{B} \cdot d\boldsymbol{S}$$



For an enclosed surface, the flux is zero

$$\oint_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0.$$

Gauss's law $\nabla \cdot B = 0$

There is no magnetic monopole.

$$\oint_S \mathbf{A} \cdot \mathrm{d}\mathbf{S} = \int_v \nabla \cdot \mathbf{A} \mathrm{d}v = 0$$

Ampere's law

Ampere's Law states that for any closed loop C, the line integral of the magnetic field around closed loop C is equal to the electric current enclosed in the loop.

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int_s \mathbf{J} \cdot d\mathbf{s} = I$$

Current generates magnetic field

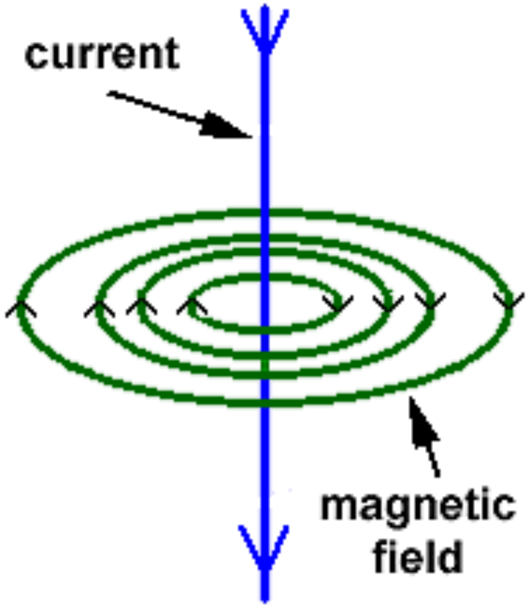
$$\oint_C \mathrm{B} \cdot \mathrm{dl} = \int_S \nabla \times \mathrm{B} \cdot \mathrm{dS} = \mu_0 \int_S \mathbf{J} \cdot \mathrm{dS}.$$

Stokes' theorem

$$\nabla \times \mathrm{B} = \mu_0 \mathbf{J},$$

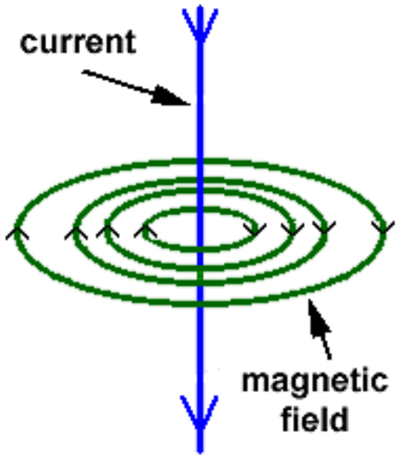
$$\frac{1}{\mu_0} \nabla \times \mathbf{B} = \nabla \times \mathbf{H} = \boldsymbol{J}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$$



In magnetostatic, such as constant DC current, $\frac{\partial \boldsymbol{D}}{\partial \boldsymbol{t}} = \mathbf{0}$

Ampere's law: Describing the magnetic field around a current



Stokes' theorem: $$\oint_{\partial\Sigma} \mathbf{B} \cdot \mathrm{d}\boldsymbol{l} = \iint_{\Sigma} \nabla \times \mathbf{B} \cdot \mathrm{d}\mathbf{S}$$

↓

$$\int_s \nabla\times\mathbf{B} \cdot d\mathbf{S} = \oint \mathbf{B} \cdot d\mathbf{l} = 2\pi r B(r)$$

$$\int_s \nabla\times\mathbf{B} \cdot d\mathbf{S} = \int_s \mu_0 \mathbf{J} \cdot d\mathbf{S} = \mu_0 I$$

↑

$$\nabla\times\mathbf{B} = \mu_0 \mathbf{J}$$

⇒ $2\pi r B(r) = \mu_0 I$ ⇒ $B(r) = \dfrac{\mu_0 I}{2\pi r}$

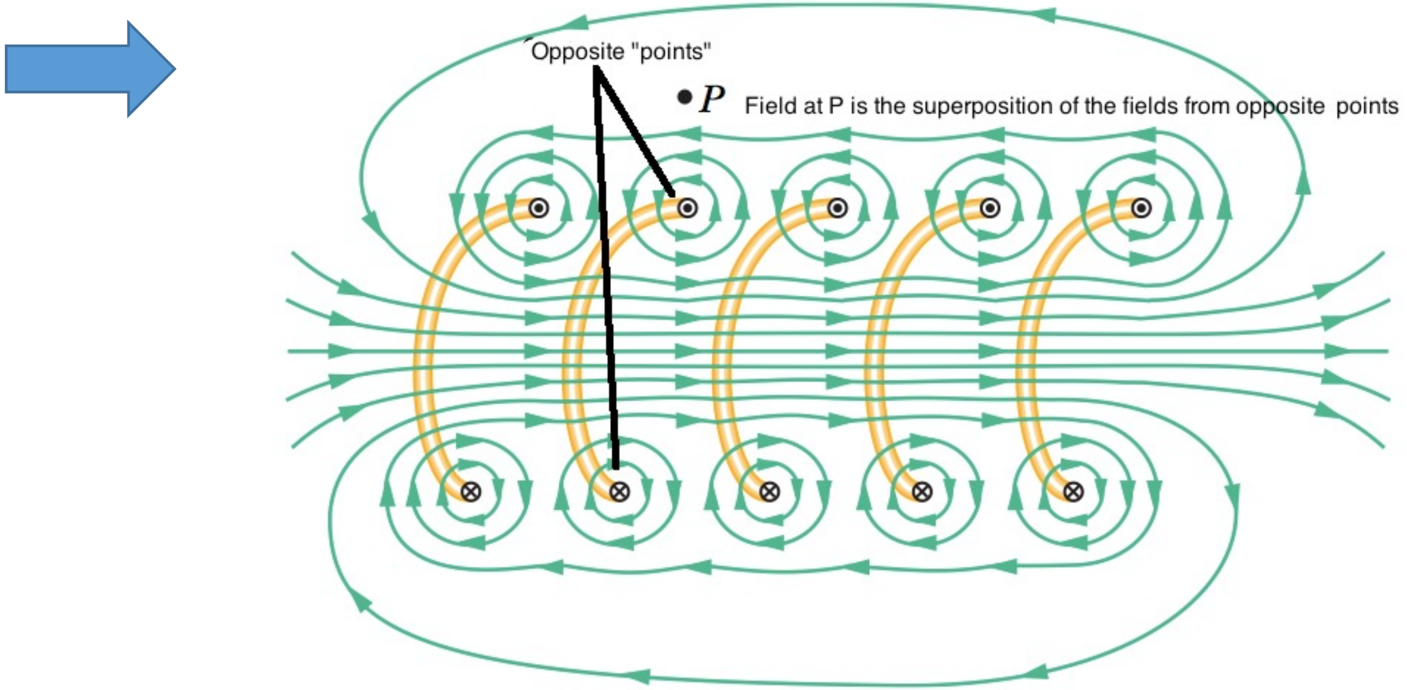
Magnetic field for solenoid

$$\int_s \nabla \times \mathbf{H} \cdot d\mathbf{S} = \oint \mathbf{H} \cdot d\mathbf{l} = Hl$$

$$\int_s \nabla \times \mathbf{H} \cdot d\mathbf{S} = \int_s \mathbf{J} \cdot d\mathbf{S} = NI$$

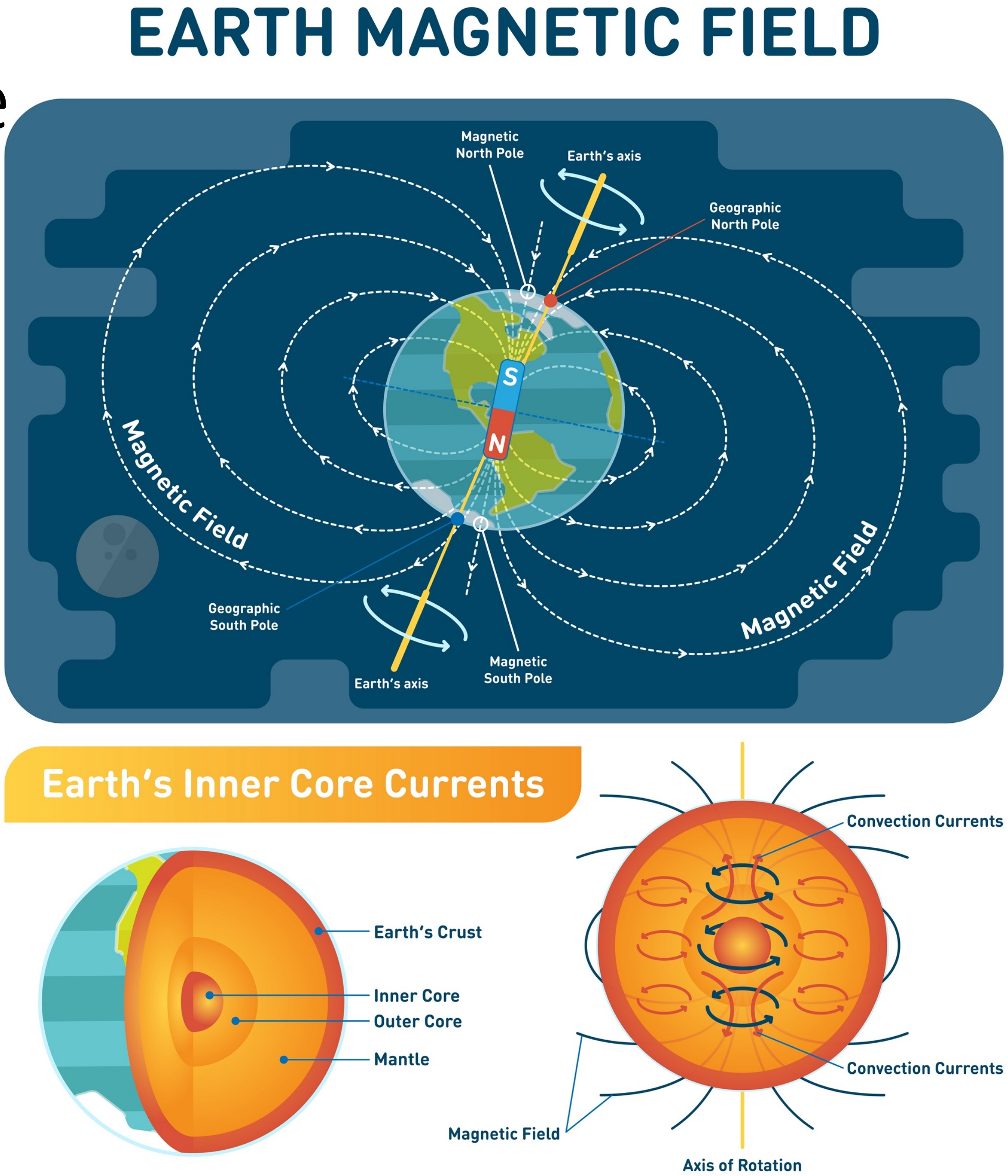
$$\boldsymbol{B} = \mu \boldsymbol{H}$$

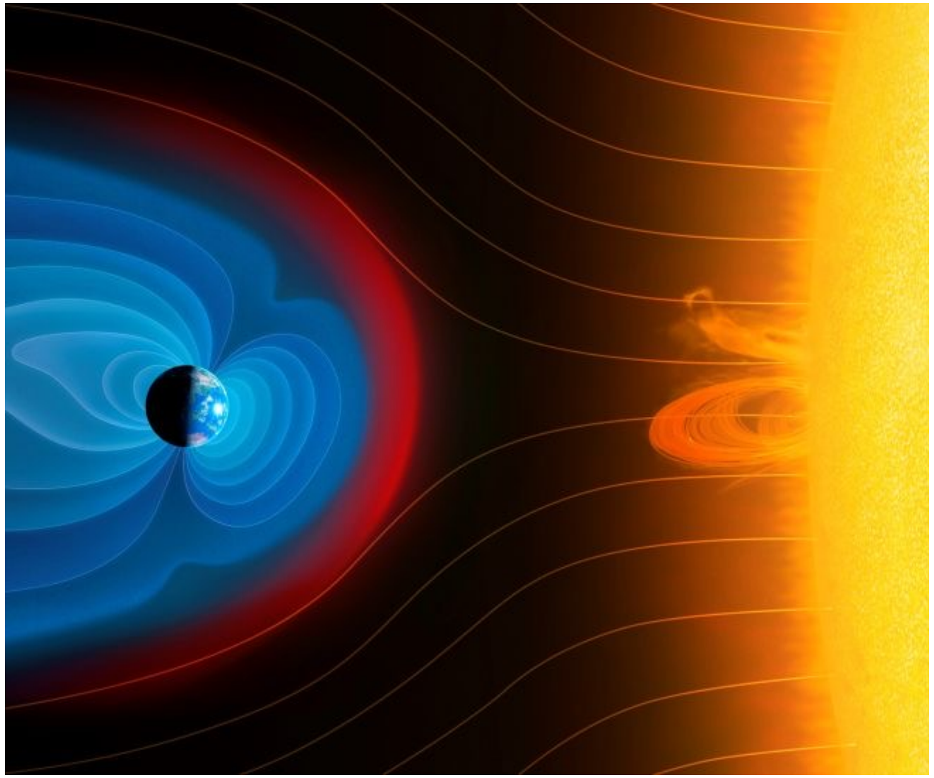
$$H = \frac{NI}{l}$$



If the core is iron instead of air, the flux density **B** is much stronger.

Example





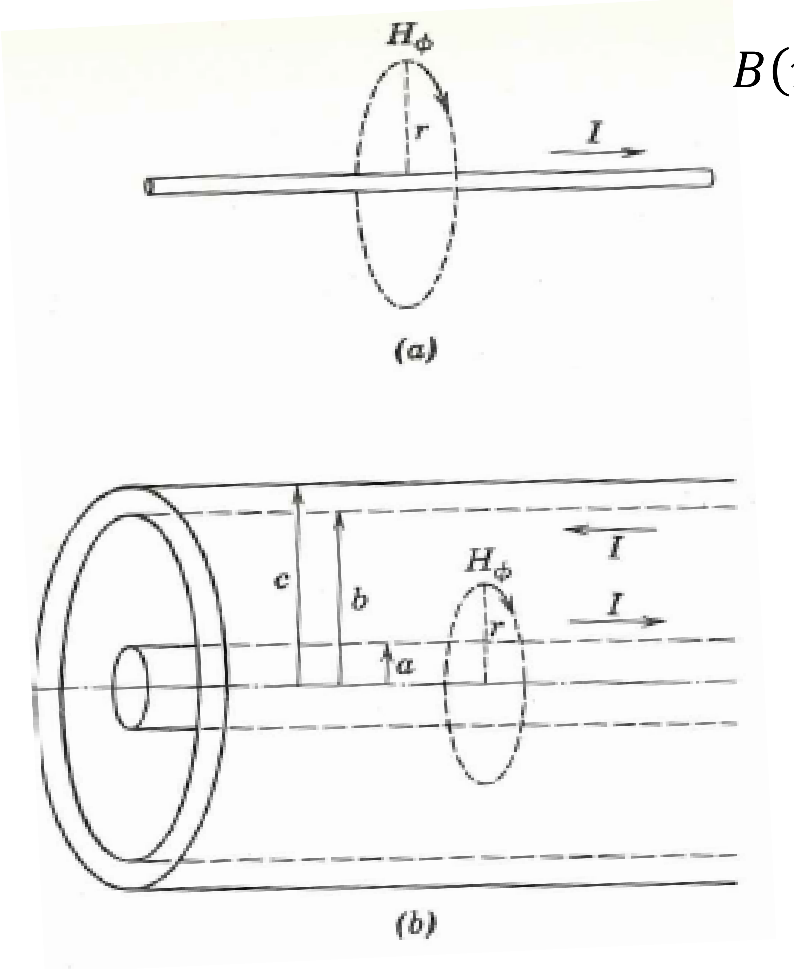
https://www.space.com/earths-magnetic-field-explained

Example: Magnetic field

A coaxial line carrying current I on the inner conductor and –I on the outer. Calculate the magnetic field **H** at r distance, current evenly distributed in the two conductors.

1) 0<r<a,
2) a<r<b

$$B(r) = \frac{\mu_0 I}{2\pi r}$$



$$\int_s \mathbf{J} \cdot d\mathbf{S} = I(r) = J\pi r^2$$

$$\int_S \mathbf{J} \cdot d\mathbf{S} = I = J\pi a^2$$

$$I(r) = I\frac{r^2}{a^2}$$

$$H_\phi(r) = \frac{I(r)}{2\pi r} = \frac{Ir}{2\pi a^2} \qquad (0 < r < a)$$

$$H_\phi = \frac{I}{2\pi r} \qquad (a < r < b)$$

Magnetic field in material

Once there is magnetic field applied to medium, the electronic spin motions in the atoms can be thought of as circulating current that produces a field **M**, magnetization.

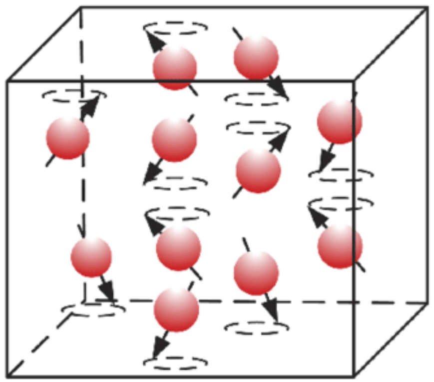
$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$$

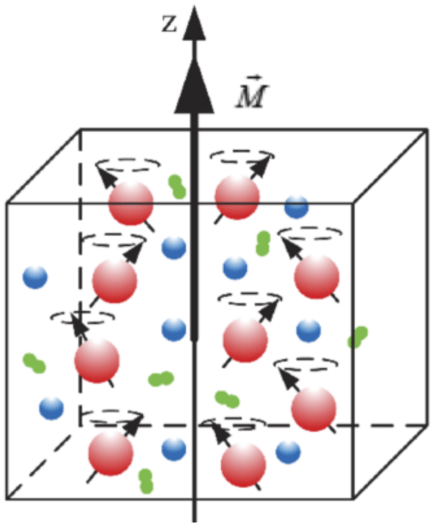
$$\mathbf{M} = \mu_0 \chi_m \boldsymbol{H}$$

$$\mathbf{B} = \mu_0(1 + \chi_m)\boldsymbol{H} = \mu_0 \mu_r \mathbf{H} = \mu \mathbf{H}$$

χ_m is magnetic susceptibility, used to quantify the additional field **M**.

μ_r *relative permeability.*





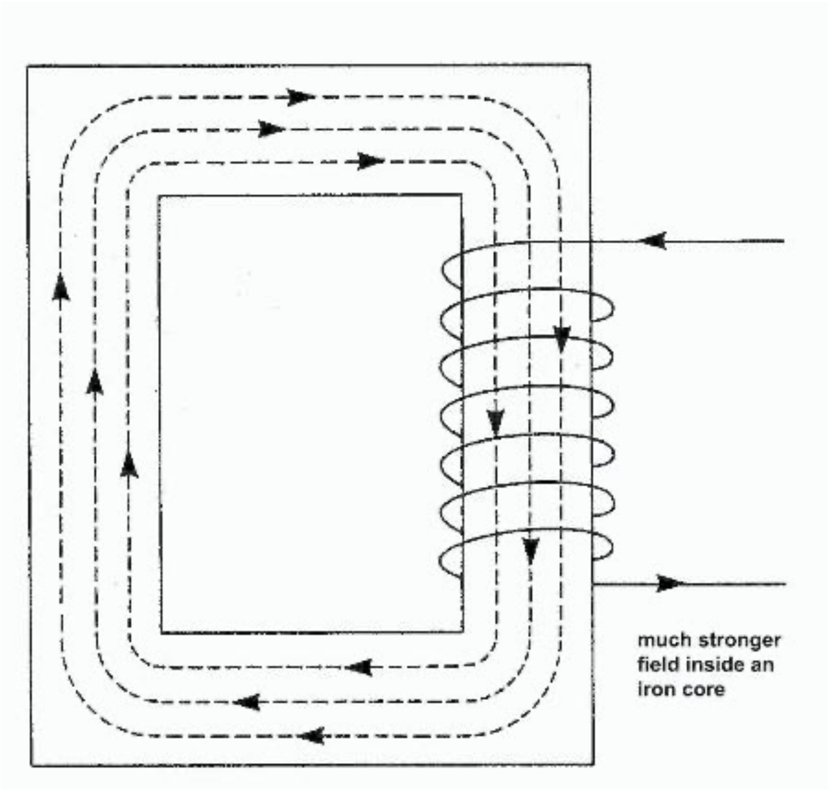
Good magnetic material relative permeability
Iron: ~ 5000

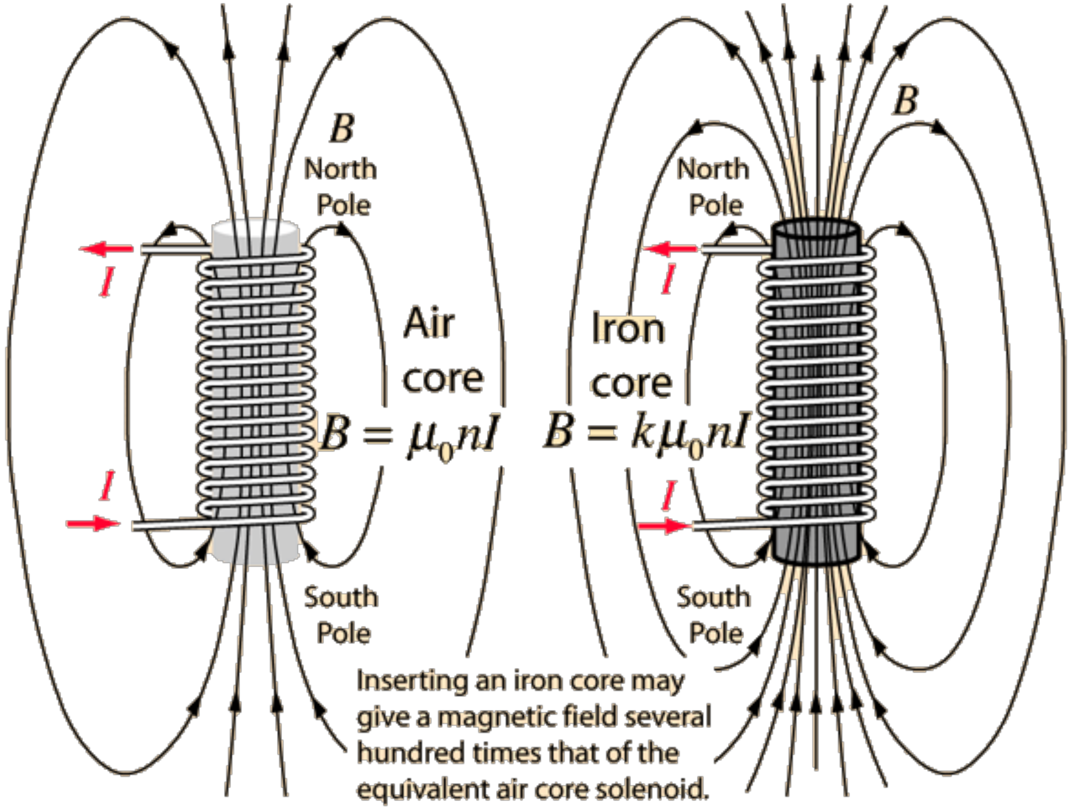
Bad magnetic material relative permeability
Silver:1
Copper : 1
Gold: 1
Aluminium: 1

Magnetization increases the magnetic flux density **B** in ferro-magnetic materials compared to vacuum.

Field in magnetic material

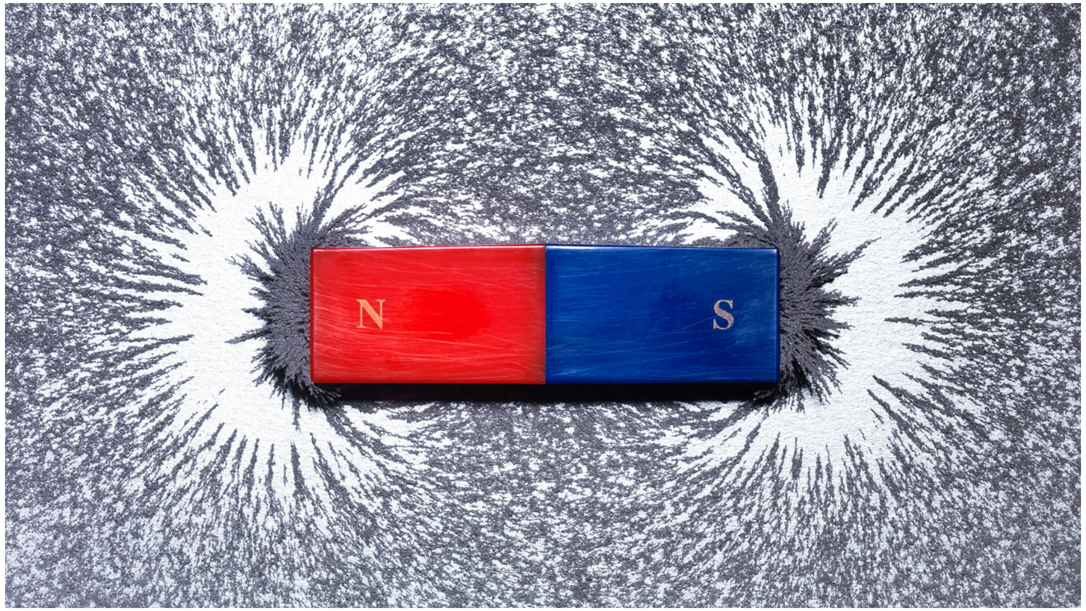
Magnetic material can be used to guide magnetic field path.

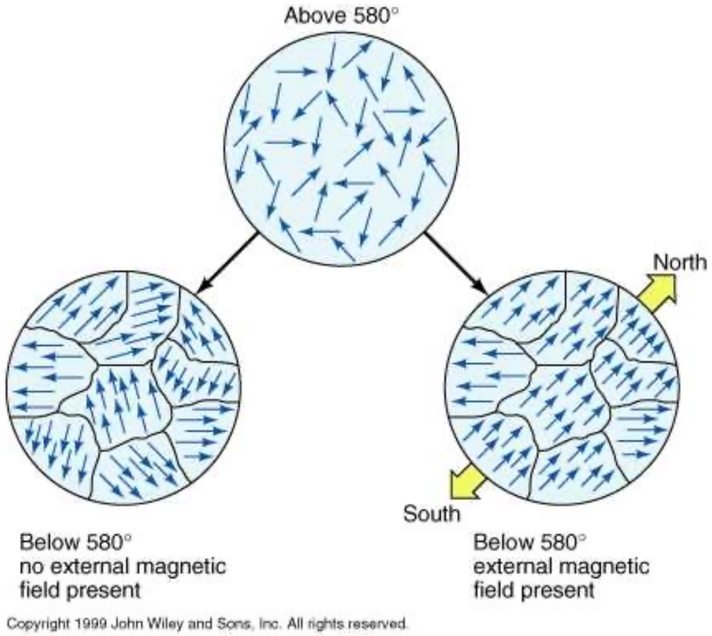




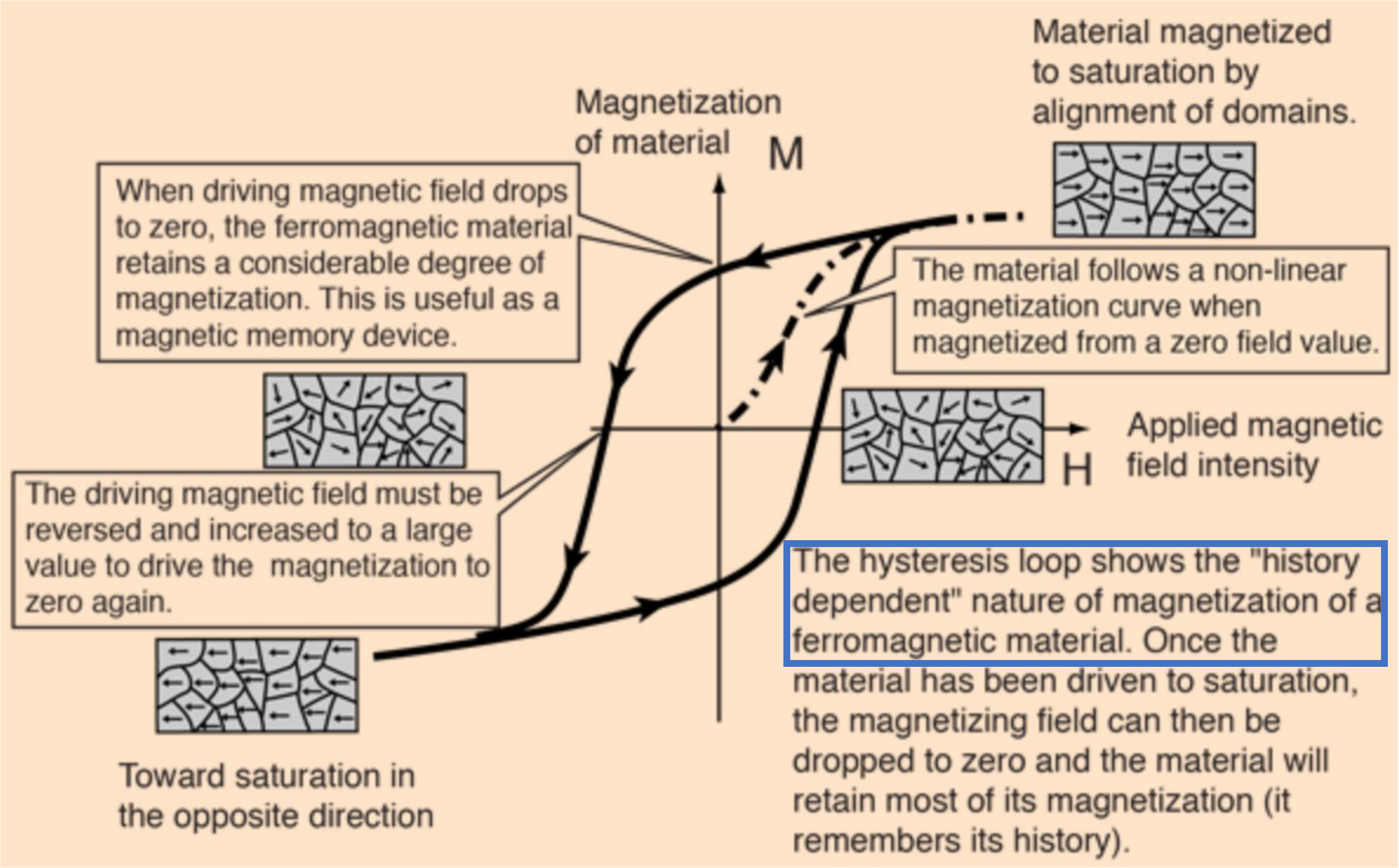
Permanent magnet

For some materials, after magnetization the electron movement can remain when external magnetic field disappears. The electron movement can be distorted at high temperatures, strong opposite external field or strong shock.

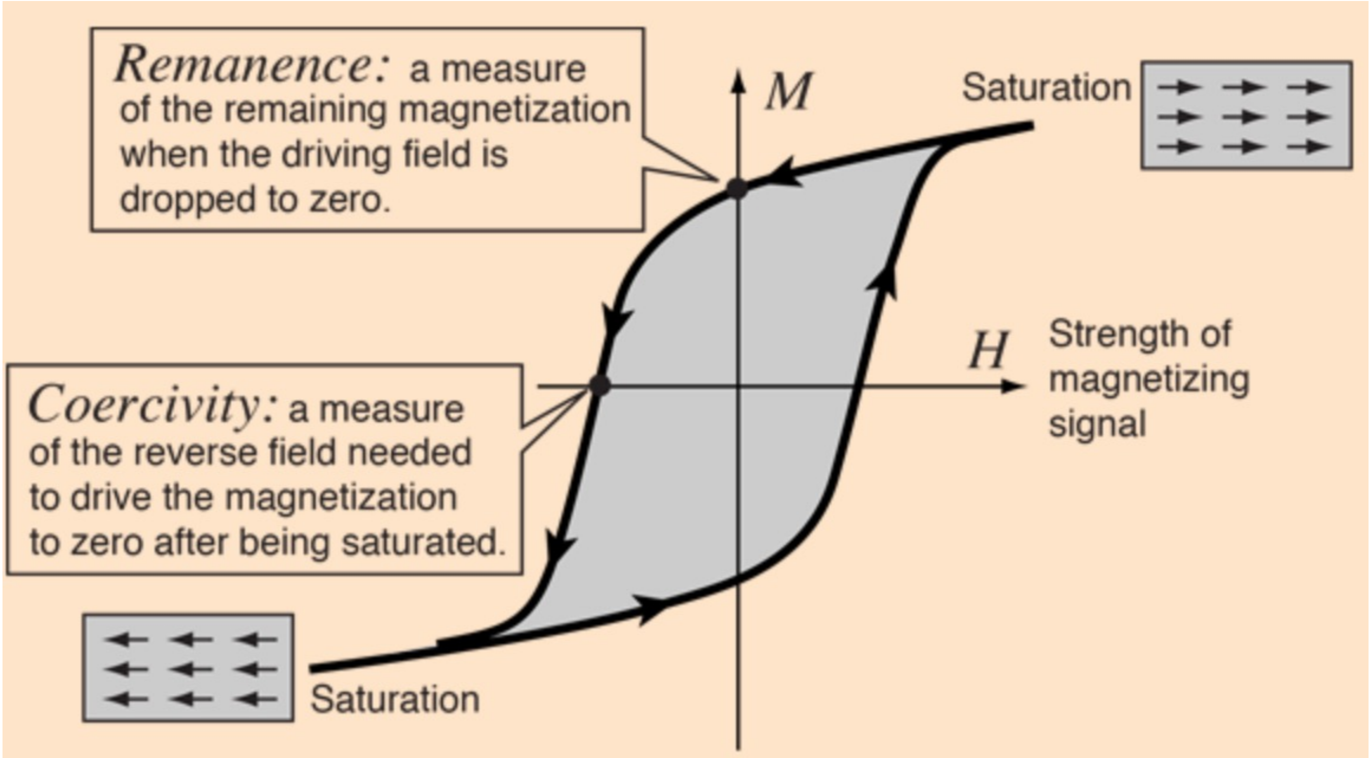


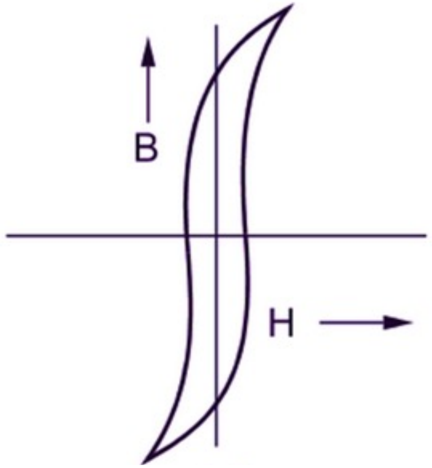


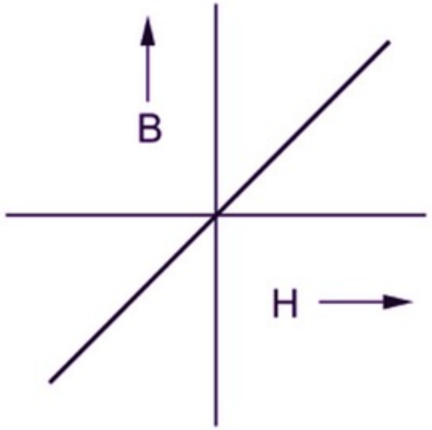
Hysteresis loop



Hysteresis loop





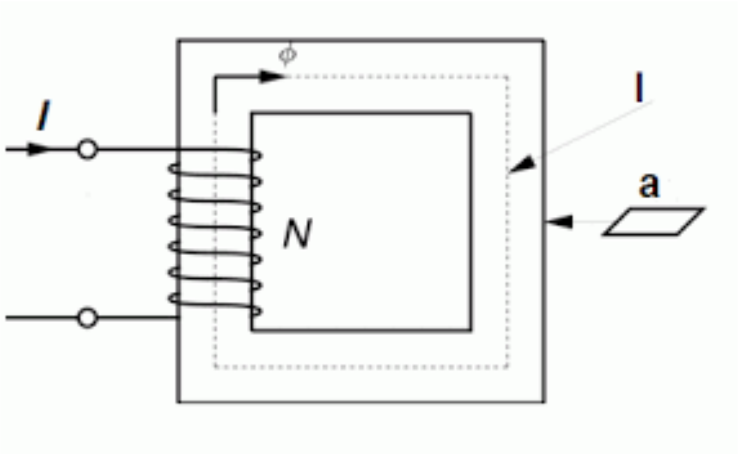


a. For permanent magnet, the material should have a large HL to gain high remanence and coercive force.

b. For electro-magnet, high permeability and low coercivity are required.

Magnetic circuit

Magnetomotive force (MMF): $F = NI = HL = \Phi R$



Φ, magnetic flux

$R = \frac{l}{\mu S}$, magnetic reluctance

- The magnet (*HL*) or current (*NI*) possesses a magneto-motive force (MMF).
- The MMF generates a magnetic flux.
- The enclosed flux path is called a magnetic circuit.
- A stronger MMF produces more flux.
- The lower the reluctance, the more the flux.

Magnetic circuit and electric circuit

Magnetic Circuit	***Electrical Circuit***
1. The closed path for magnetic flux is called a magnetic circuit.	1. The closed path for electric current is called an electric circuit.
2. Flux , ø = $\frac{mmf}{Reluctance}$	2. Current, $I = \frac{emf}{resistance}$
3. mmf (Ampere – turns)	3. emf (Volts)
4. Reluctance, $S = \frac{l}{a\mu_0\mu_r}$	4. Resistance, $R = \rho \frac{l}{a}$
5. Flux density, $B = \frac{\emptyset}{a}$ Wb/m^2	5. Current density $\delta = \frac{I}{a}$ A/m^2
6. mmf drop = $\emptyset\ S$	6. Voltage drop = $I\ R$
7. Magnetic Intensity, $H = \frac{N\ I}{l}$	7. Electric intensity, $E = \frac{V}{d}$

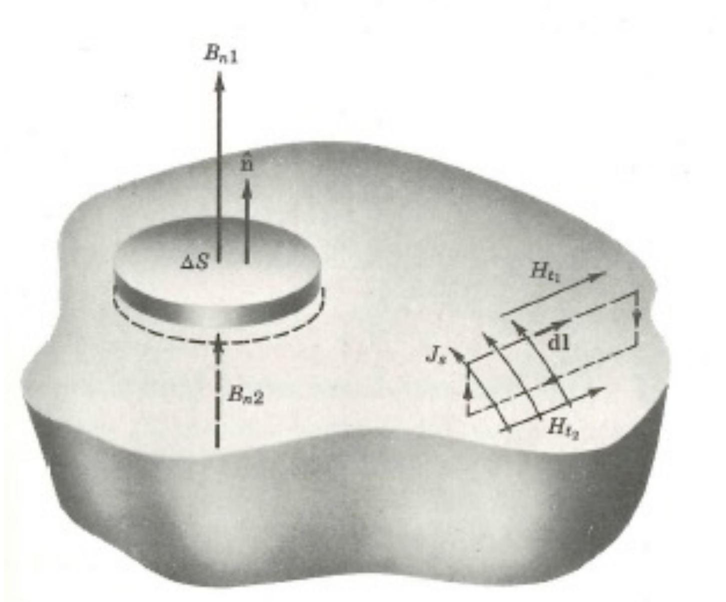
Boundary condition for static magnetic field

Normal component

$$\oint \mathbf{B} \cdot d\mathbf{S} = 0$$

$$B_{n1}\Delta S - B_{n2}\Delta S = 0$$

$$B_{n1} = B_{n2}$$



Tangential component

$$\oint \mathbf{H} \cdot d\mathbf{l} = H_{t1}\Delta l - H_{t2}\Delta l = J_s\Delta l$$

Line current density

$$H_{t1} - H_{t2} = J_s$$

$$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \int_S \nabla \times \mathbf{B} \cdot \mathrm{d}\mathbf{S} = \mu_0 \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S}.$$