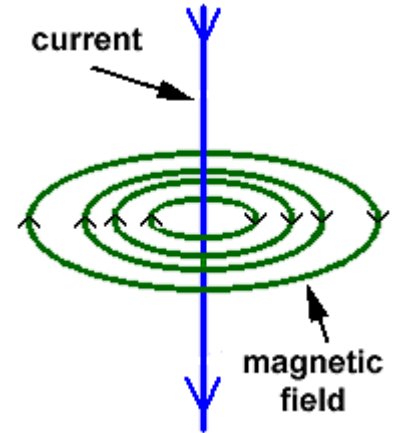
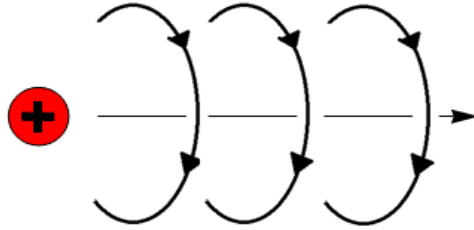


Lecture 4: Stationary magnetic field

- Charge in motion
- Magnetic field
- Gauss's law for magnetic field
- Ampere's Law

Charge in motion

Charges in *motion* (electrical current) produce a magnetic field

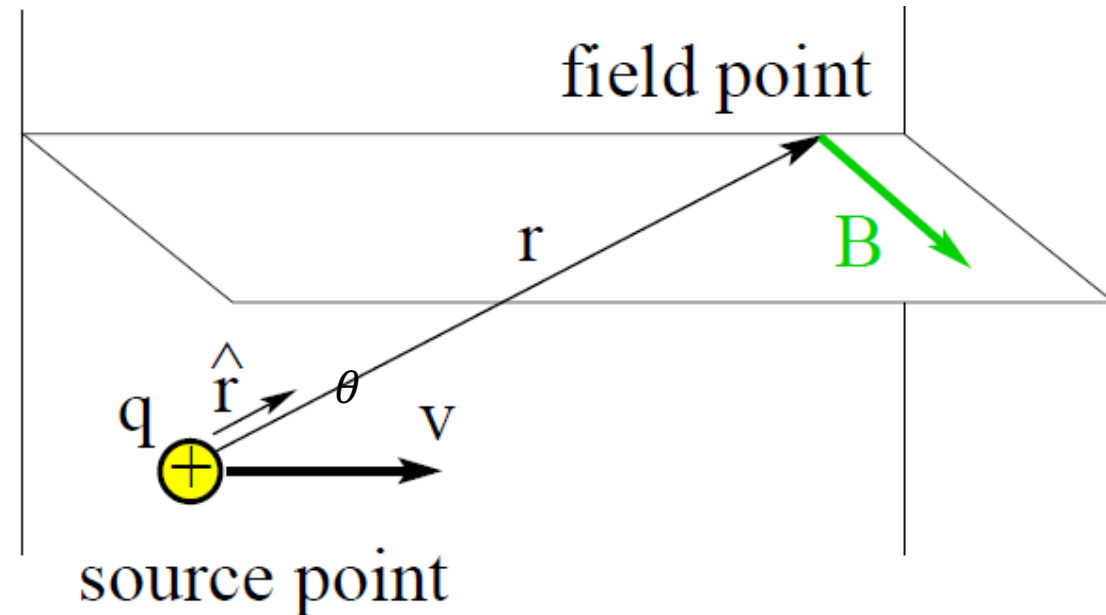


Magnetic flux density \mathbf{B} in vacuum generated by moving charge q :
$$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{q\mathbf{v} \times \hat{\mathbf{r}}}{r^2}$$

$$\mathbf{v} \times \hat{\mathbf{r}} = |\mathbf{v}| \sin(\theta) \mathbf{n}$$

\mathbf{n} is a [unit vector](#) perpendicular to the plane containing \mathbf{v} and \mathbf{r} in the direction given by the right-hand rule

$\mu_0 = 4\pi \times 10^{-7} \text{ H/m}$, permeability of free space



Electromagnetic (Lorentz) force

Force exerted by magnetic field \mathbf{B} on a moving point charge Q is:

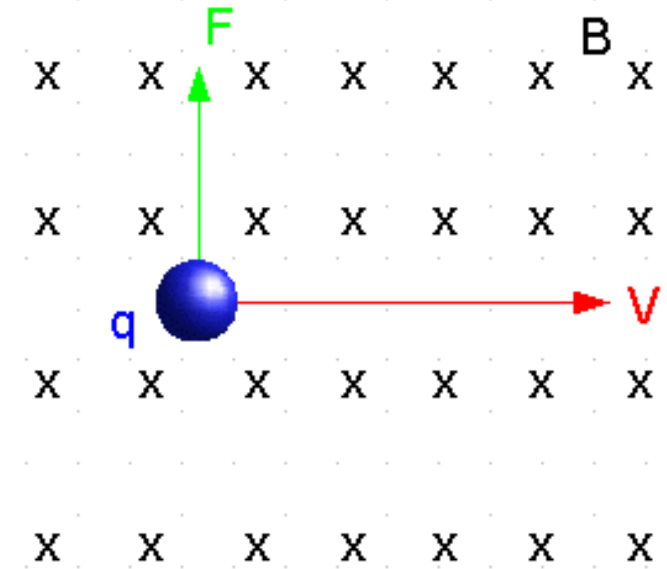
$$\mathbf{F} = Q \mathbf{v} \times \mathbf{B} \quad \text{Compare to } \mathbf{F}_c$$

Lorentz law $\mathbf{F} = q\mathbf{E} + q(\mathbf{v} \times \mathbf{B})$

Magnetic force acting on a moving charge is always perpendicular to it's moving direction, so magnetic force does not work on the charges, but changing the charges' moving direction

$$W = \mathbf{F} \cdot \mathbf{l} = \int \mathbf{F} \cdot d\mathbf{l} = \int \mathbf{F} \cdot \mathbf{v} dt$$

Work is application of force, \mathbf{F} , to move an object over a distance, l , in the direction that the force is applied.



Example: Point charge's movement in constant uniform magnetic field

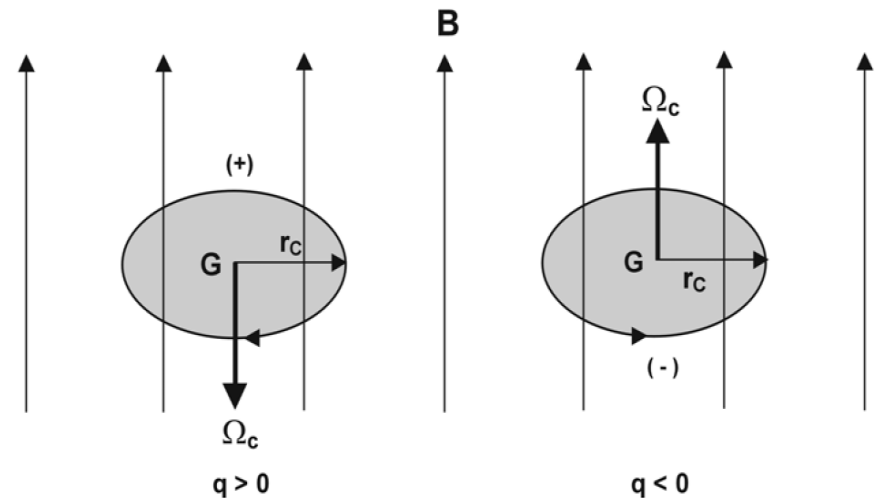
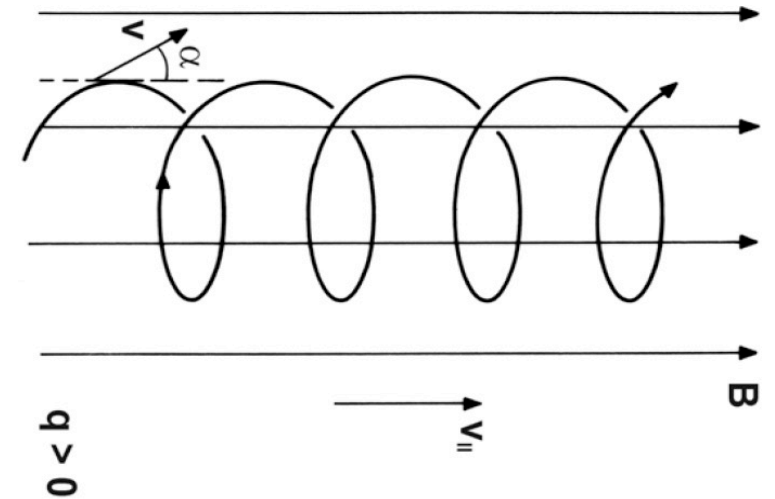
A constant uniform magnetic field \mathbf{B} , a charge q with mass m is shot perpendicularly to the magnetic field with speed $\mathbf{V}(0)$, what is the radius of charge?

Centrifugal force

$$F = \frac{mv^2}{r} = qvB \sin\theta \quad (\theta \text{ is the angle between } \mathbf{v} \text{ and } \mathbf{B}, 90^\circ)$$

$$r = \frac{mv^2}{qvB} = \frac{mv}{qB}$$

<https://www.youtube.com/watch?v=orsMYomjwlw>

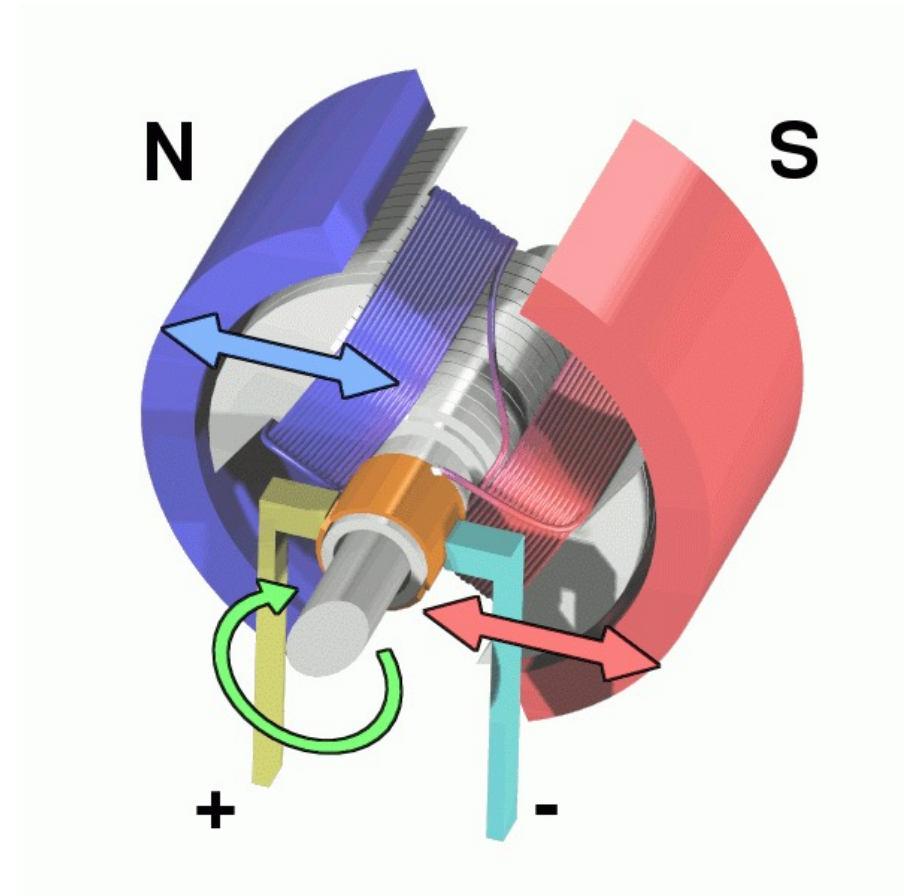
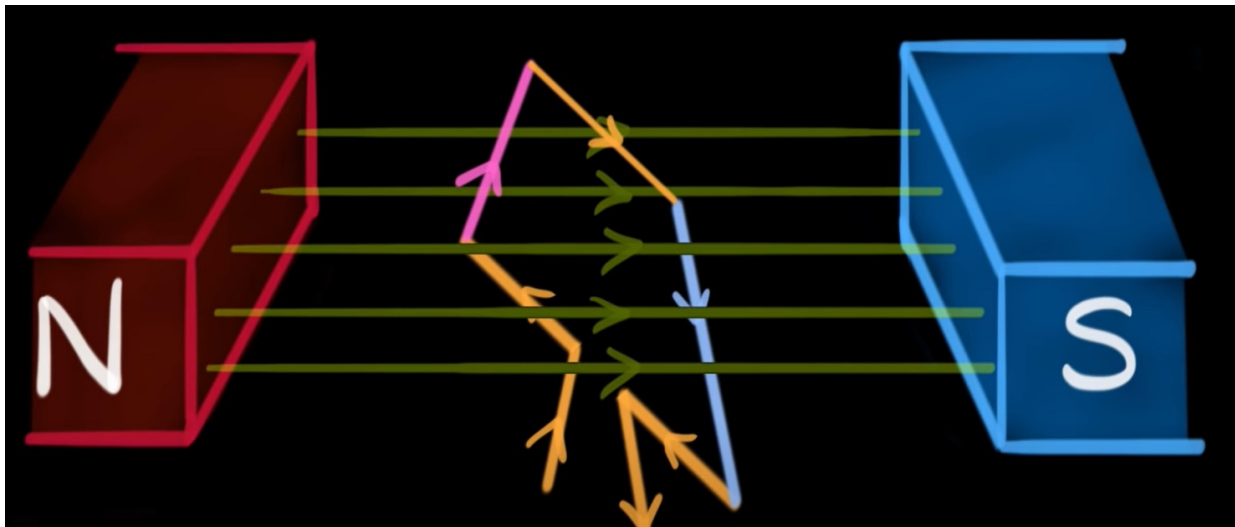


Electric motor

An electrical motor consist of: stator and rotor

One produces mechanical energy by **current and magnets**

Rotating windings: electric current (moving charge) flows



Magnetic field

Magnetic field $\mathbf{H} = \frac{\mathbf{B}}{\mu}$, μ is called permeability and material dependent, the value of μ for free space is $\mu_0 = 4\pi \times 10^{-7} \text{ H/m}$.

$$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{q\mathbf{v} \times \hat{\mathbf{r}}}{r^2}$$

$$\mathbf{H} = \frac{1}{4\pi} \frac{q\mathbf{v} \times \hat{\mathbf{r}}}{r^2}$$

The magnetic field intensity is independent of material property.

Biot-Savart law : Magnetic field generated by current

The Biot–Savart law is used for computing the resultant magnetic field \mathbf{B} at position \mathbf{r} in 3D-space generated by a *steady current* I .

Vector expression:
$$\mathbf{H}(\mathbf{r}) = \int_C \frac{I'(r) d\mathbf{l}' \times \hat{\mathbf{R}}}{4\pi R^2}$$

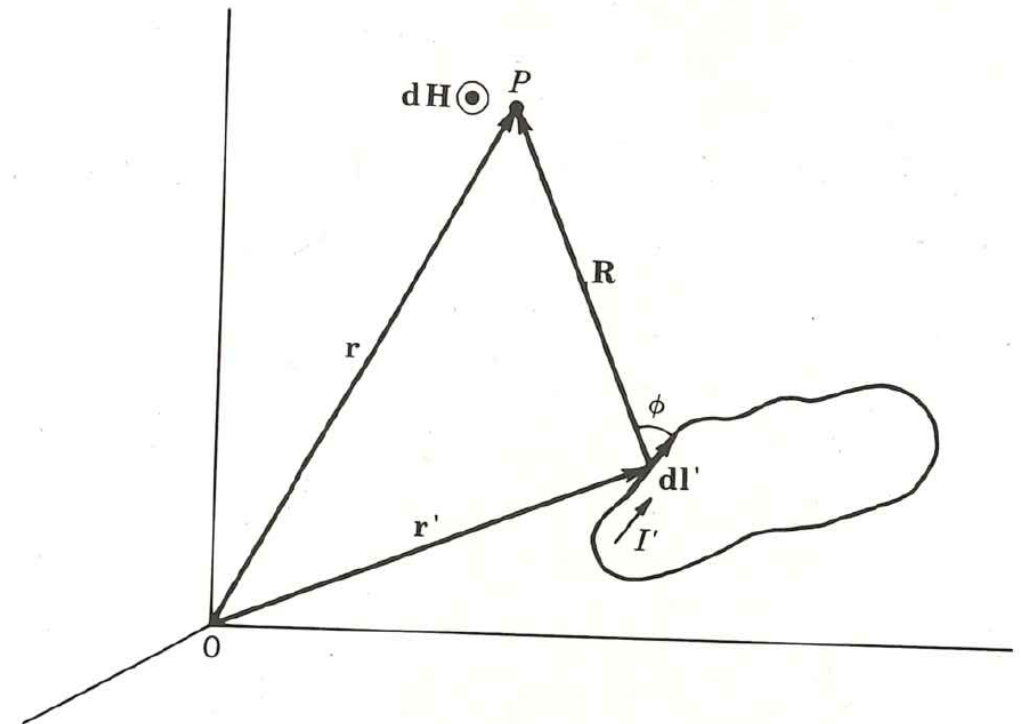
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_C \frac{I'(r) d\mathbf{l}' \times \hat{\mathbf{R}}}{R^2}$$

$$I' d\mathbf{l}' \times \hat{\mathbf{R}} = |I'| dl' \sin(\phi) \mathbf{n}$$

\mathbf{n} is a unit vector perpendicular to the plane containing \mathbf{l} and \mathbf{R} in the direction given by the right-hand rule

Scalar calculation
$$H(r) = \int_C \frac{I'(r) dl' \sin\phi}{4\pi R^2}$$

$$B(r) = \frac{\mu_0}{4\pi} \int_C \frac{I'(r) dl' \sin\phi}{R^2}$$



Example: Magnetic flux density

Find the magnetic field \mathbf{B} at a point \mathbf{P} at perpendicular distance r from the center of a finite length of current I , the total current length is $2a$.

$$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{Id\mathbf{l} \times \hat{\mathbf{R}}}{R^2}$$

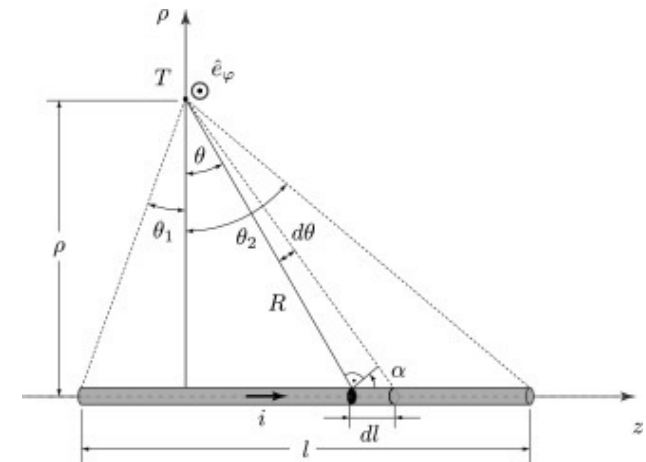
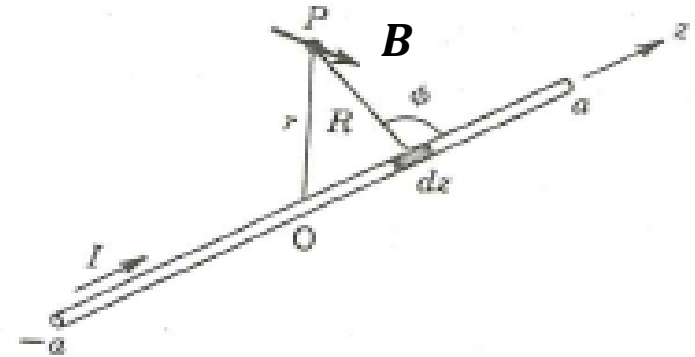
$$\mathbf{B} = \int_{-a}^a \frac{\mu_0}{4\pi} \frac{Idz}{R^2} \hat{\mathbf{R}} = \int_{-a}^a \frac{\mu_0 I \sin \phi dz}{4\pi R^2}$$

$$\sin(\phi) = \frac{r}{\sqrt{r^2 + z^2}}$$

$$R^2 = r^2 + z^2$$

$$B = \int_{-a}^a \frac{\mu_0 I \sin \phi dz}{4\pi R^2} = \frac{\mu_0 I r}{4\pi} \int_{-a}^a \frac{dz}{(r^2 + z^2)^{3/2}} = \frac{\mu_0 I}{2\pi r} \frac{a}{(r^2 + a^2)^{1/2}}$$

Assuming $a \gg r$, what is B ?



Field on axis of circular loop

A ring with radius a and current I , calculate B at point on the z axis.

$$d\mathbf{B} \cdot \hat{\mathbf{z}} = dB \cos \beta = dB \sin \alpha,$$

B and **dB** directions are different

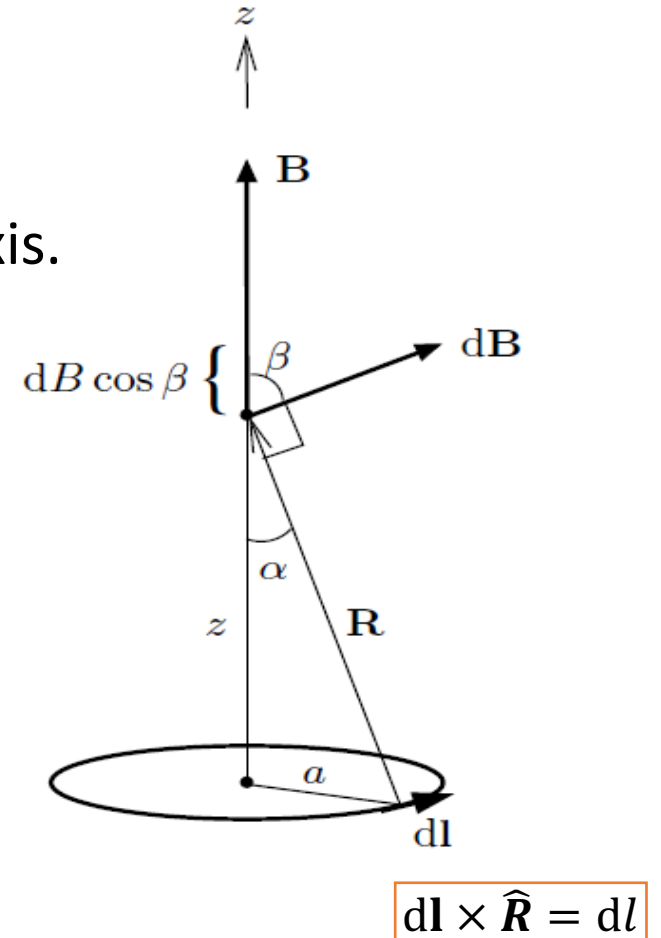
$$d\mathbf{B} = \frac{\mu_0 I d\mathbf{l} \times \hat{\mathbf{R}}}{4\pi R^2} \quad dB = \frac{\mu_0 I dl}{4\pi R^2}.$$

$$B = \oint_{\text{ring}} dB \sin \alpha = \frac{\mu_0 I \sin \alpha}{4\pi R^2} \oint dl = \frac{\mu_0 I \sin \alpha (2\pi a)}{4\pi R^2} = \frac{\mu_0 I a^2}{2R^3}$$

$$R = \sqrt{z^2 + a^2}$$

$$\sin \alpha = \frac{a}{R}$$

$$\mathbf{B} = \frac{\mu_0 I a^2}{2R^3} \hat{\mathbf{z}} = \frac{\mu_0 I a^2}{2(z^2 + a^2)^{3/2}} \hat{\mathbf{z}}$$



Magnetic flux and flux continuity

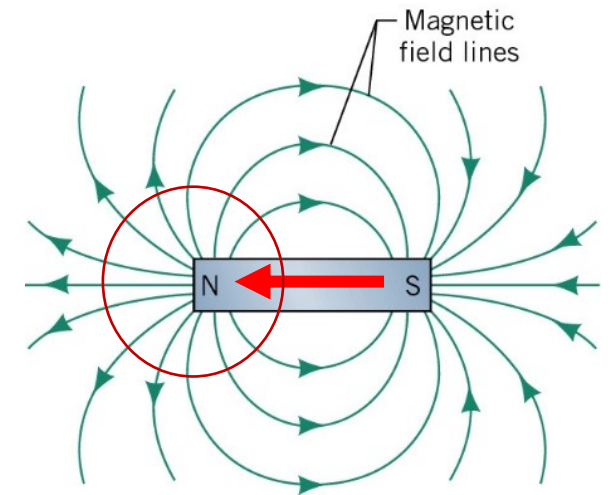
Magnetic flux ϕ is the integral of the flux density across surface

$$\phi = \int_S \mathbf{B} \cdot d\mathbf{S},$$

For an enclosed surface, the flux is zero

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0.$$

$$\nabla \cdot \mathbf{B} = 0$$



There is no magnetic monopole, continuous magnetic field.

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{A} dv = 0$$

Ampere's law

Ampere's Law states that for any closed loop path, the line integral of the magnetic field around closed curve C is equal to the electric current enclosed in the loop.

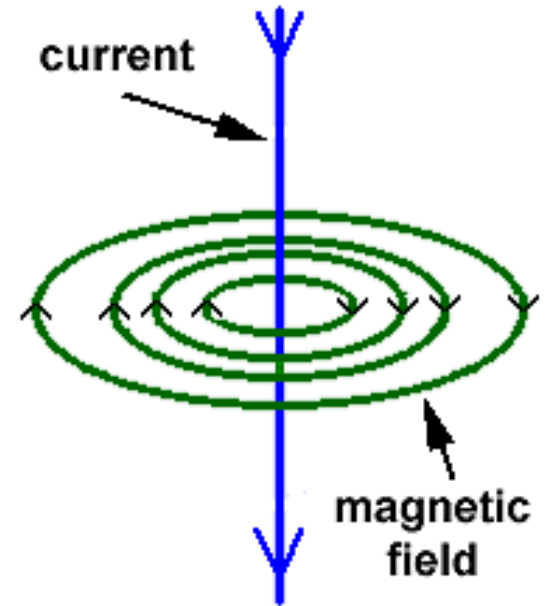
$$\oint \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{s}$$

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{B} \cdot d\mathbf{S} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{S}. \quad \text{Stokes' theorem}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J},$$

$$\frac{1}{\mu_0} \nabla \times \mathbf{B} = \nabla \times \mathbf{H} = \mathbf{J}$$

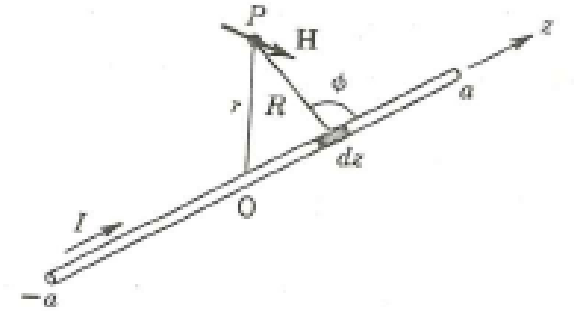
$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$$



In magnetostatic, such as constant DC current, $\frac{\partial \mathbf{D}}{\partial t} = \mathbf{0}$

Ampere's law: describing the magnetic field around a conductor

$$B = \int_{-a}^a \frac{\mu_0 I \sin \phi dz}{4\pi R^2} = \frac{\mu_0 I r}{4\pi} \int_{-a}^a \frac{dz}{(a^2 + z^2)^{3/2}} = \frac{\mu_0 I}{2\pi r} \frac{a}{(r^2 + a^2)^{1/2}}$$



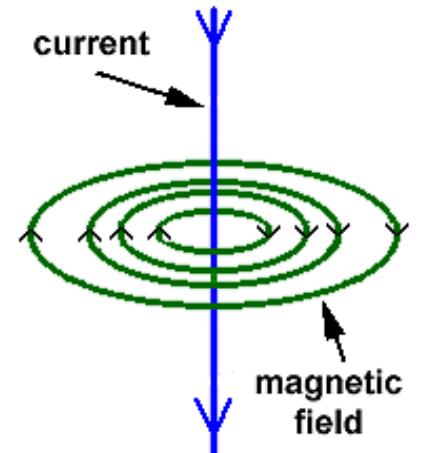
Stokes' theorem: $\oint_{\partial\Sigma} \mathbf{B} \cdot d\mathbf{l} = \iint_{\Sigma} \nabla \times \mathbf{B} \cdot d\mathbf{S}$

$$\int_s \nabla \times \mathbf{B} \cdot d\mathbf{S} = \oint \mathbf{B} \cdot d\mathbf{l} = 2\pi r B(r)$$

$$\int_s \nabla \times \mathbf{B} \cdot d\mathbf{S} = \int_s \mu_0 \mathbf{J} \cdot d\mathbf{S} = \mu_0 I$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$\longrightarrow 2\pi r B(r) = \mu_0 I \longrightarrow B(r) = \frac{\mu_0 I}{2\pi r}$$



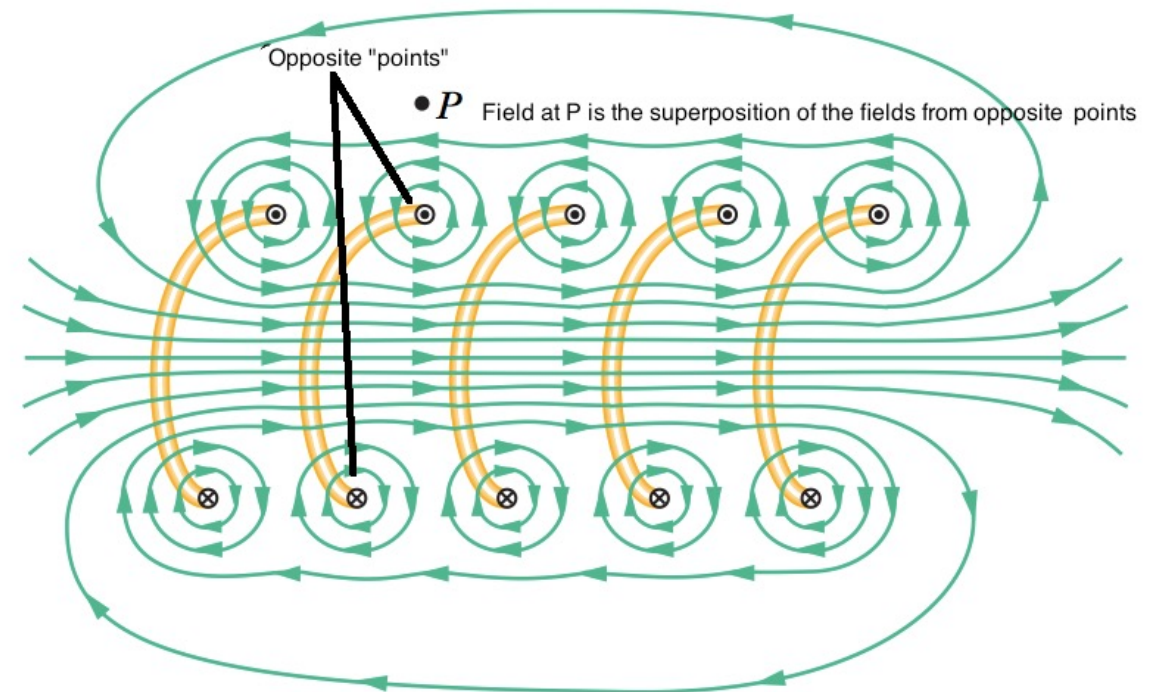
Solenoid

$$\int_S \nabla \times \mathbf{H} \cdot d\mathbf{S} = \oint \mathbf{H} \cdot d\mathbf{l} = Hl$$

$$\int_S \nabla \times \mathbf{H} \cdot d\mathbf{S} = \int_S \mathbf{J} \cdot d\mathbf{S} = NI$$

$$\mathbf{B} = \mu\mathbf{H}$$

$$H = \frac{NI}{l}$$



If the core is iron instead of air, the flux density **B** is much stronger.

Example: geomagnetic field

EARTH'S MAGNETIC FIELD

Our home is unique among our solar system's inner planets (Mercury, Venus, Earth and Mars) as it has a strong magnetic field that reaches millions of kilometres into space and shields us from solar winds and space radiation

COMPILED BY BURGERT BEHR
INFOGRAPHIC: MICHAEL DE LUCCHI

YOUNG YOU

THE WORDS 'NORTH' AND 'SOUTH' ORIGINATED BEFORE WE KNEW ABOUT EARTH'S MAGNETIC FIELD. THE SIDE THAT'S TRADITIONALLY AT THE TOP OF MAPS, WE CALL (GEOGRAPHIC) NORTH, AND THE OTHER SIDE WAS CALLED SOUTH.

BECAUSE OF EARTH'S INTERNAL 'MAGNET', THE MAGNETIC POLES ARE THE OPPOSITE OF THE GEOGRAPHIC ONES.

The magnetic field lines surrounding a permanent magnet always run from north to south. Earth's magnetic field lines run from the north magnetic pole to the south magnetic pole, or from "bottom" to top in our illustration. It's easy to see that the north and south magnetic poles are the opposites of the geographic poles.

CRUST
The outer 20km on land and 5km under the sea.

MANTLE
Mostly solid rock; just a small part of the mantle is molten.

INNER CORE
A solid ball of superhot iron and nickel which cause convection currents in the outer core.

OUTER CORE
Molten iron and nickel. These metals are good electricity conductors and are therefore crucial elements in generating the magnetic field.

THE AXES

EARTH'S AXIS OF ROTATION

- The geographical north and south poles are where Earth's rotational axis intersects the planet's surface. Earth rotates around an imaginary axis which runs through the north and south poles.
- The geographical poles are fixed. In other words are always in the same place and never shift.

MAGNETIC AXIS

- The magnetic north and south poles are situated at the top and bottom tip of the planet's magnetic axis.
- In reality, the geographic north pole is the magnetic field's "south pole", and vice versa. Two opposing poles (such as north and south) attract each other, so the northern tip of the magnet in a compass will point to the magnetic south pole, which is in the Northern Hemisphere.

DECLINATION

- The north and south magnetic poles aren't exactly where the geographic poles are. There is a difference in angle of about 11° - this is called the declination.
- This is because the magnetic poles are constantly shifting as the molten metals in the outer core move and the magnetic field strengthens and weakens.
- The south magnetic pole is currently in northern Canada and shifting in the direction of Russia, while the north magnetic pole is at the edge of Antarctica.

HOW IT WORKS

- The system of magnetic spheres around Earth, also known as the magnetosphere, is similar to the magnetic field around a magnet.
- Earth's inner core is made of iron and nickel, and pressure has shaped it into a dense ball.
- The outer core surrounds this ball. There's less pressure in the outer core, and Earth's fast rotation plus convection forces due to the extremely hot inner core make molten iron and nickel move in streams here. Convection forces are forces that distribute heat, such as those that evenly heat water in a boiling kettle.
- The movement in the outer core causes electric currents, because the outer core works like a giant electrical generator that creates direct current.
- Electrically charged particles, called ions, create a magnetic field. This field reaches deep into space. It's as if Earth has a huge magnet at its centre.

DID YOU KNOW?

Several animal species, from birds to tortoises, use Earth's magnetic field to determine their location and to set a course. For example during the annual bird migrations when some species fly from some part of the world to another with the changing seasons.

SOLAR WIND

- Solar winds are electrically charged ions forcefully radiating from the sun's intense heat. Solar winds are always heading towards Earth at speeds of 450km per second or faster.
- Earth's magnetosphere deflects most of the ions away from our planet.
- Solar wind also causes the magnetosphere to flatten on the sun's side and elongate into a long "tail" on the other side of Earth. This is because solar radiation has its own magnetism which repels Earth's magnetic field.

SO ARTH is like a gigantic magnet which generates a magnetic field, putting the planet in a cocoon of sorts where it's protected. The magnetic field shields the planet from deadly solar winds and radiation from space, providing us with a safe environment in which to live.

NOT TO SCALE. THE DISTANCE BETWEEN THE NORTH AND SOUTH MAGNETIC POLES IS NOT THE SAME AS THE DISTANCE BETWEEN THE GEOGRAPHIC NORTH AND SOUTH POLES. THE DISTANCE BETWEEN THE NORTH AND SOUTH MAGNETIC POLES IS ABOUT 11 DEGREES. THE DISTANCE BETWEEN THE GEOGRAPHIC NORTH AND SOUTH POLES IS ABOUT 180 DEGREES.

Example: Magnetic field

A coaxial line carrying current I on the inner conductor and $-I$ on the outer. Calculate the magnetic field \mathbf{H} at r distance, current evenly distributed in the two conductors.

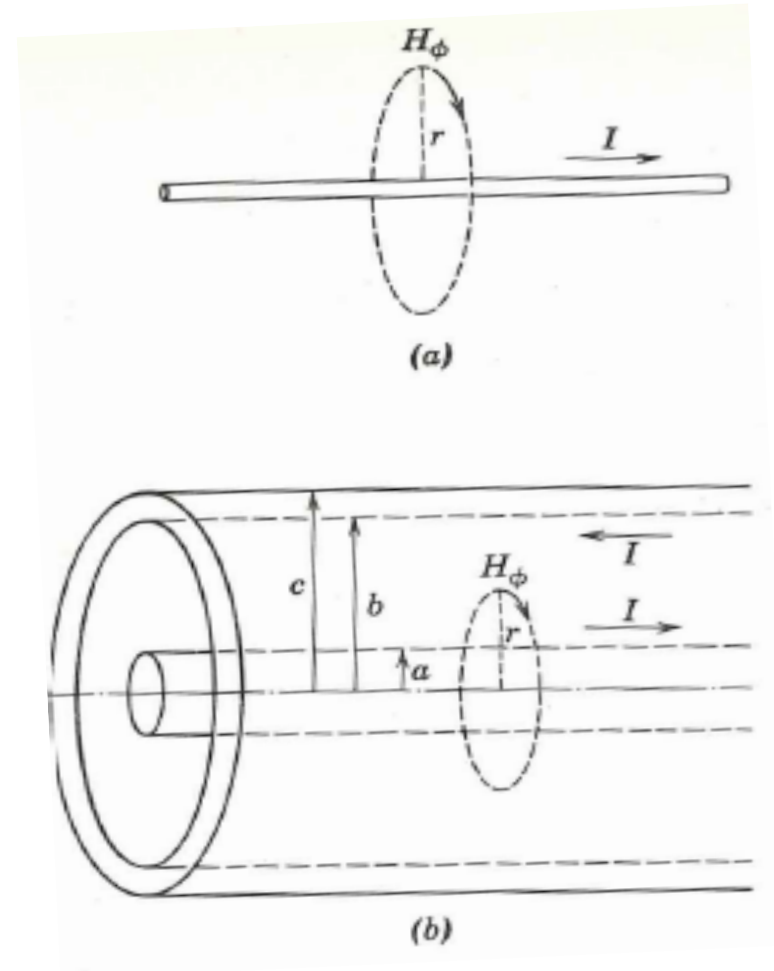
- 1) $0 < r < a$,
- 2) $a < r < b$

$$\int_S \mathbf{J} \cdot d\mathbf{S} = I(r) = J\pi r^2 \qquad \int_S \mathbf{J} \cdot d\mathbf{S} = I = J\pi a^2$$

$$I(r) = I \frac{r^2}{a^2}$$

$$H_\phi(r) = \frac{I(r)}{2\pi r} = \frac{Ir}{2\pi a^2}$$

$$H_\phi = \frac{I}{2\pi r} \quad (a < r < b)$$



Magnetic field in material

$$\mathbf{B} = \mu_0 \mathbf{H} \text{ in vacuum}$$

Once there is magnetic field applied to medium, the electronic spin motions in the atoms can be thought of as circulating current that produces a field \mathbf{M} , magnetization, which adds to magnetic field \mathbf{H} .

$$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M})$$

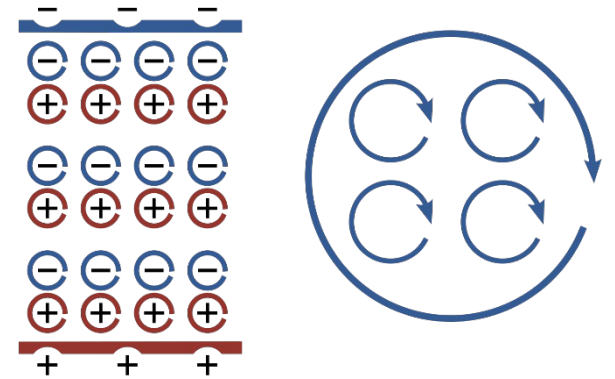
Magnetic susceptibility χ_m is used to quantify the additional field \mathbf{M} .

$$\mathbf{M} = \mu_0 \chi_m \mathbf{H}$$

$$\mathbf{B} = \mu_0 (1 + \chi_m) \mathbf{H} = \mu_0 \mu_r \mathbf{H} = \mu \mathbf{H}$$

μ_r relative permeability.

Magnetization increases in the magnetic flux density \mathbf{B} in ferro-magnetic materials compared to vacuum.

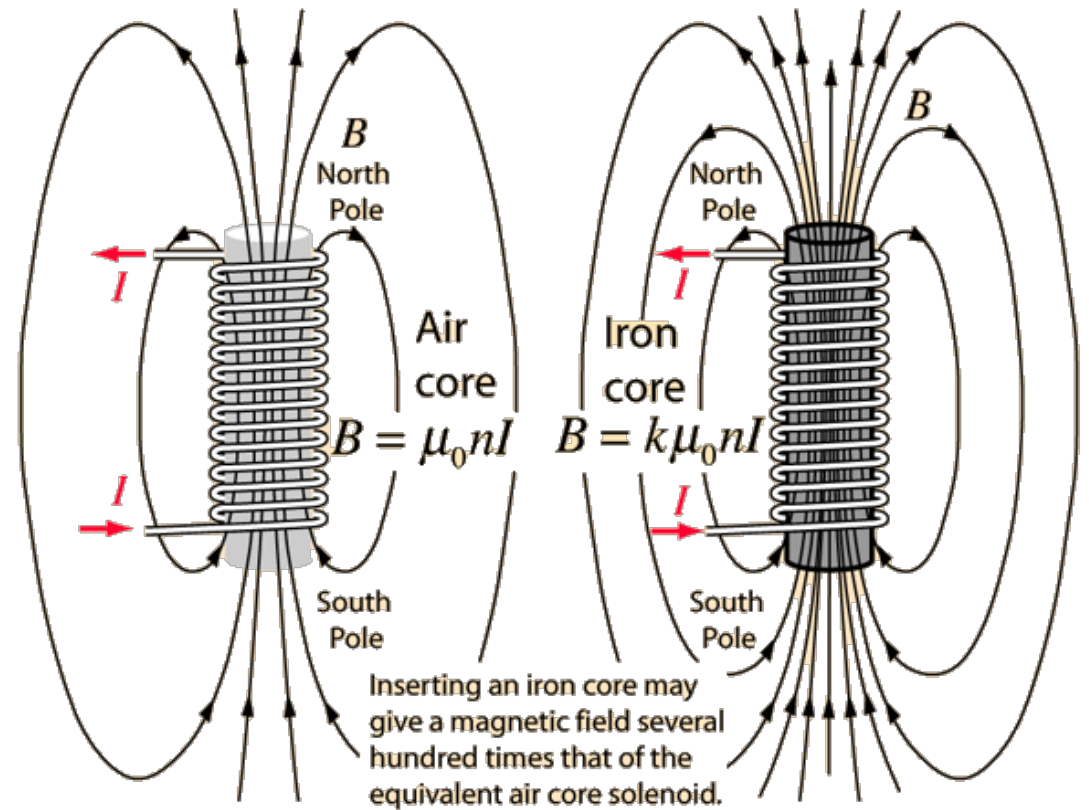
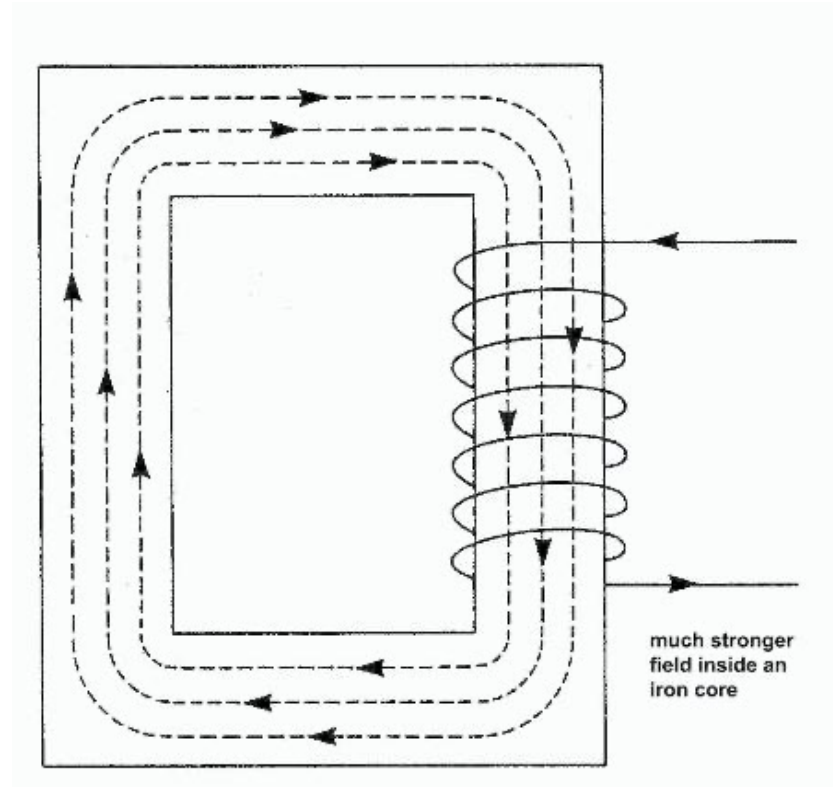


Good magnetic material
relative permeability
Iron: ~ 5000

Bad magnetic material
relative permeability
Silver: 1
Copper : 1
Gold: 1
Aluminium: 1

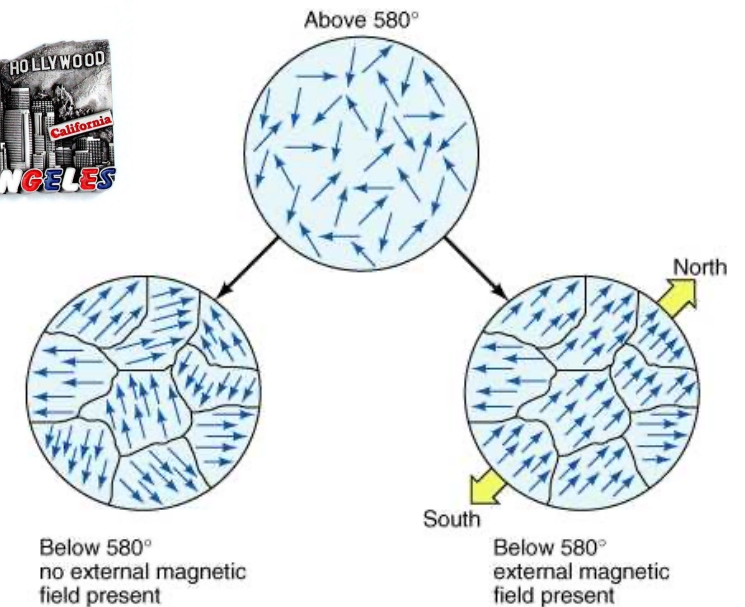
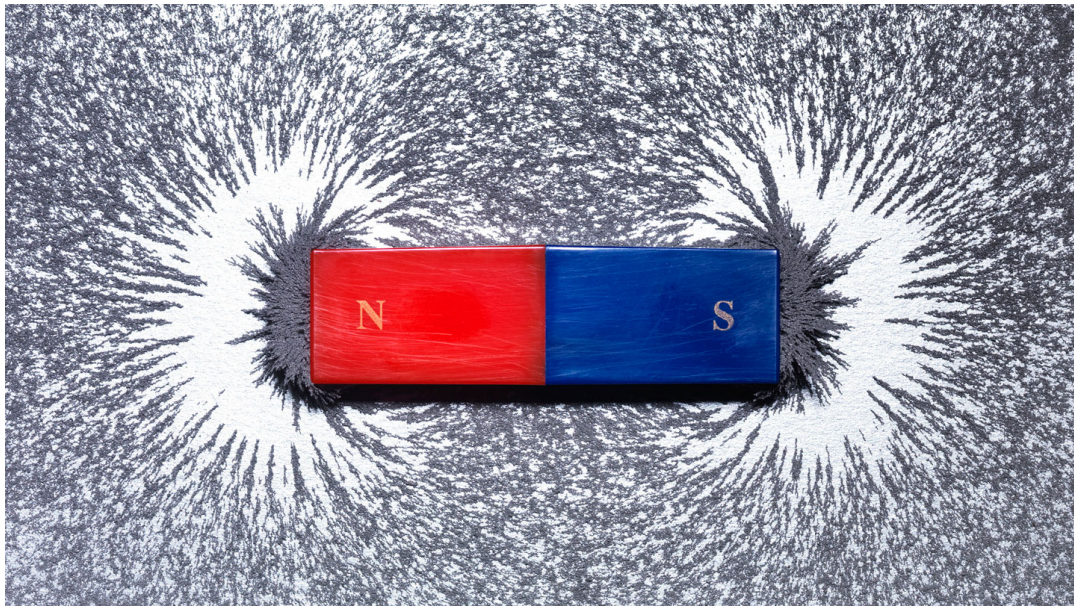
Field in magnetic material

Magnetic material can be used to guide magnetic field path.

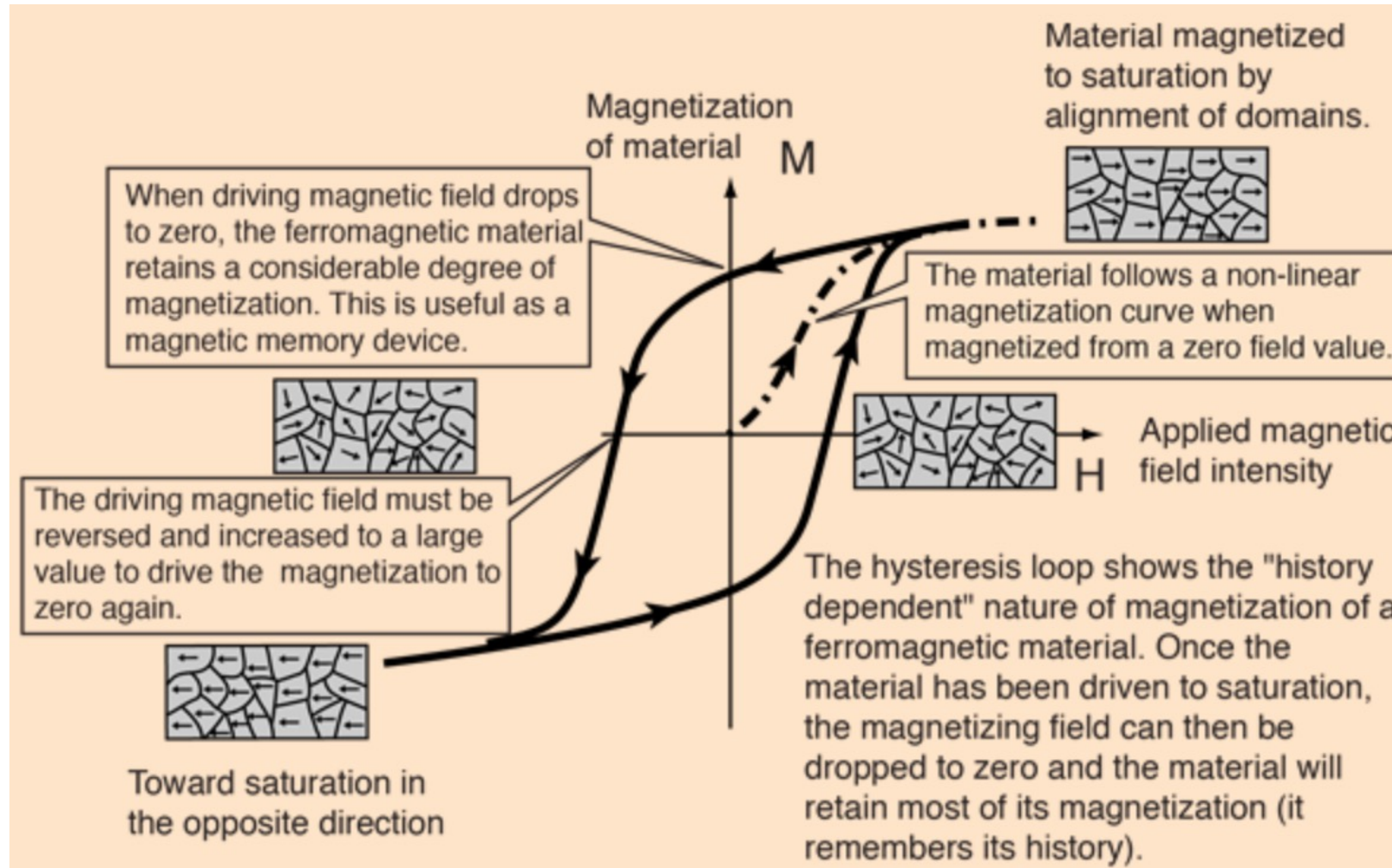


Permanent magnet

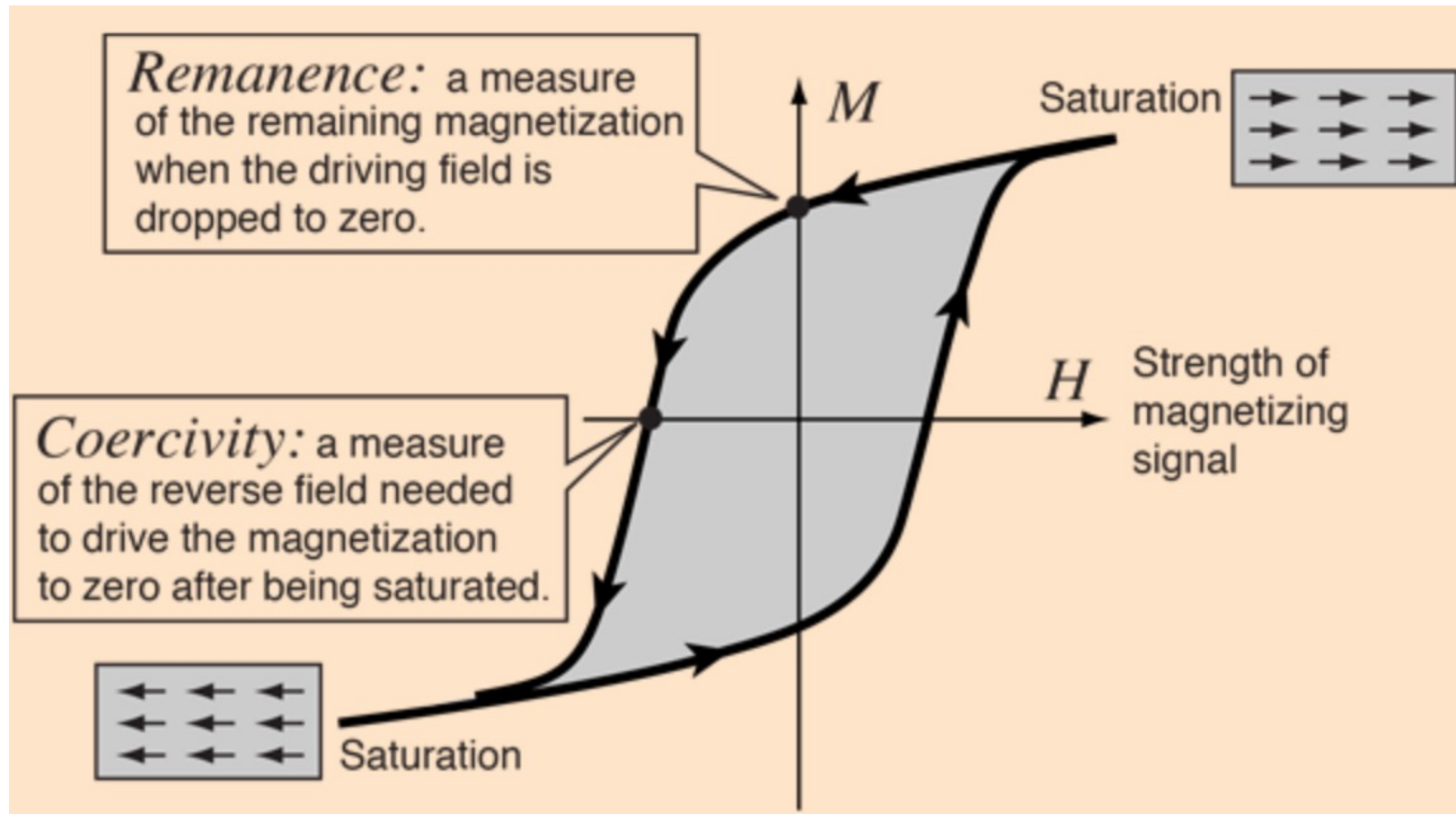
For some materials, after magnetization the electron movement can remain when external magnetic field disappears. The electron movement can be distorted at high temperatures, strong opposite external field or strong shock.



Hysteresis loop

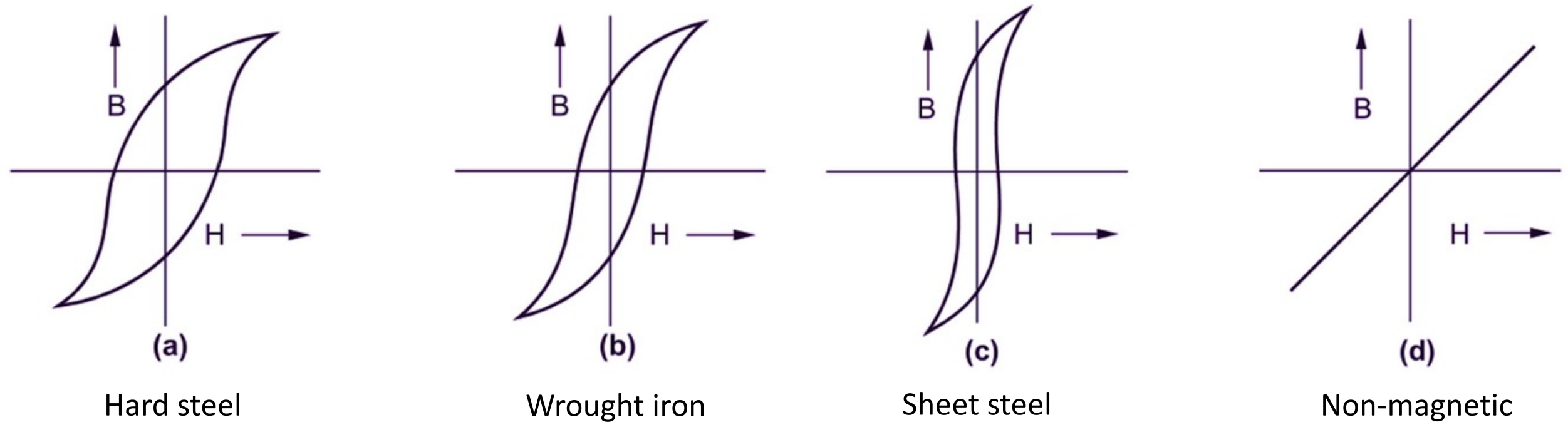


Hysteresis loop



Magnetic material: B-H curve

Suitability of magnetic material for use depends on the areas and shape of hysteresis loop



Hard steel

Wrought iron

Sheet steel

Non-magnetic

- For permanent magnet, the material should have a large HL to gain high remanence and coercive force.
- For electro-magnet, high permeability and low coercivity are required.
- For electrical equipment subjected to rapid reversals of magnetization, high permeability and low hysteresis loss are required.

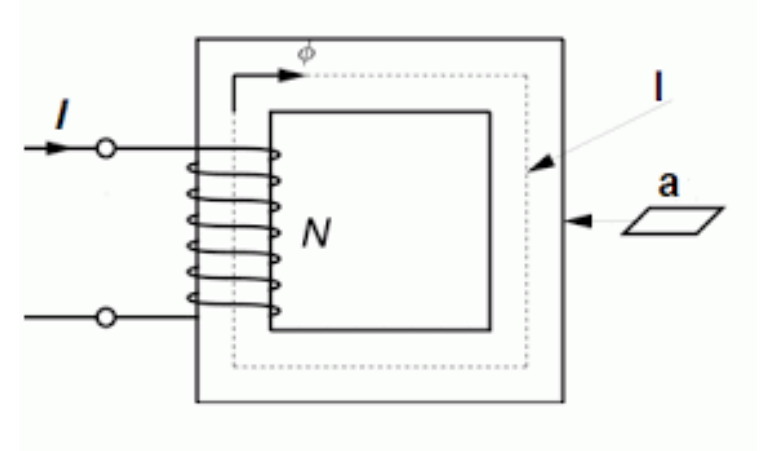
Magnetic circuit

Magnetomotive force (MMF): $F = NI = HL = \Phi R$

Φ , magnetic flux

$R = \frac{l}{\mu S}$, magnetic reluctance

- The magnet (HL) or current (NI) possesses a magneto-motive force (MMF).
- The MMF generates a magnetic flux.
- The flux exists within the magnet and the air gap between the poles. The enclosed flux path is called a magnetic circuit.
- A stronger MMF will produce more flux.
- The lower the reluctance of the magnetic circuit, the more flux will be produced.



Magnetic circuit and electric circuit

Magnetic Circuit

Electrical Circuit

1. The closed path for magnetic flux is called a magnetic circuit.	1. The closed path for electric circuit is called an electric circuit.
2. Flux, $\phi = \frac{mmf}{Reluctance}$	2. Current, $I = \frac{emf}{resistance}$
3. mmf (Ampere – turns)	3. emf (Volts)
4. Reluctance, $S = \frac{l}{a\mu_0\mu_r}$	4. Resistance, $R = \rho \frac{l}{a}$
5. Flux density, $B = \frac{\phi}{a}$ Wb/m ²	5. Current density $\delta = \frac{I}{a}$ A/m ²
6. mmf drop = ϕS	6. Voltage drop = $I R$
7. Magnetic Intensity, $H = \frac{NI}{l}$	7. Electric intensity, $E = \frac{V}{d}$

Boundary condition for static magnetic field

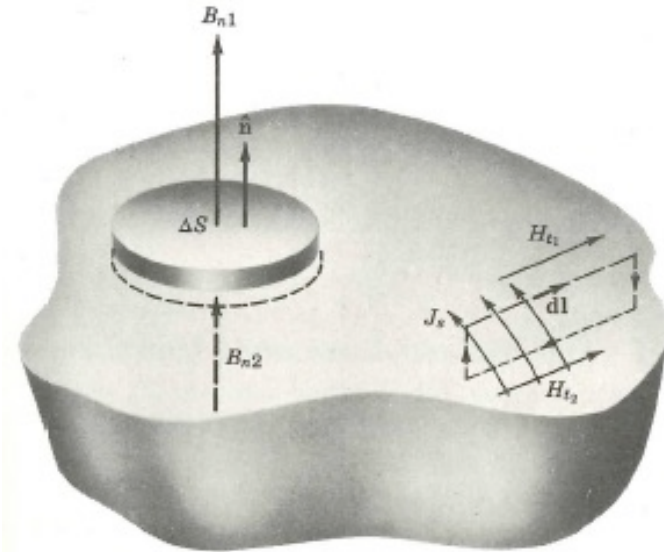
$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0.$$

$$B_{n1} \Delta S = B_{n2} \Delta S$$

$$B_{n1} = B_{n2}$$

$$\oint \mathbf{H} \cdot d\mathbf{l} = H_{t1} \Delta l - H_{t2} \Delta l = J_s \Delta l$$

$$H_{t1} - H_{t2} = J_s$$



$$\frac{1}{\mu_0} \nabla \times \mathbf{B} = \nabla \times \mathbf{H} = \mathbf{J} \quad \text{Line current density}$$

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{B} \cdot d\mathbf{S} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{S}.$$

Example: Magnetic flux density

$$\begin{aligned}
 \int \frac{1}{(a^2+x^2)^{3/2}} dx &= \int \frac{1}{(a^2 + a^2 \cdot \tan^2(u))^{3/2}} \cdot \frac{a}{\cos^2(u)} du = \int \frac{1}{(a^2(1 + \tan^2(u))^{3/2}} \cdot \frac{a}{\cos^2(u)} du = \\
 x &= a \cdot \tan(u) \rightarrow \frac{x}{a} = \tan(u) \rightarrow \arctan\left(\frac{x}{a}\right) = u \\
 dx &= \frac{a}{\cos^2(u)} du = \int \frac{1}{(a^2)^{3/2} \cdot (1 + \tan^2(u))^{3/2}} \cdot \frac{a}{\cos^2(u)} du = \frac{1}{a^3} \int \frac{1}{\left(\frac{1}{\cos^2(u)}\right)^{3/2}} \frac{a}{\cos^2(u)} du = \\
 &= \frac{a}{a^3} \int \frac{1}{\frac{1}{\cos^3(u)}} \cdot \frac{1}{\cos^2(u)} du = \frac{1}{a^2} \int \cos(u) du = \frac{1}{a^2} \sin(u) = \frac{1}{a^2} \sin\left(\arctan\left(\frac{x}{a}\right)\right) = \\
 &= \frac{1}{a^2} \cdot \frac{\frac{x}{a}}{\sqrt{1 + \left(\frac{x}{a}\right)^2}} = \frac{x}{a^3} \cdot \frac{1}{\sqrt{1 + \frac{1}{a^2}(a^2+x^2)}} = \frac{x}{a^3} \cdot \frac{1}{\left(\frac{1}{a}\right)\sqrt{a^2+x^2}} = \boxed{\frac{x}{a^2} \cdot \frac{1}{\sqrt{a^2+x^2}} + C'}
 \end{aligned}$$