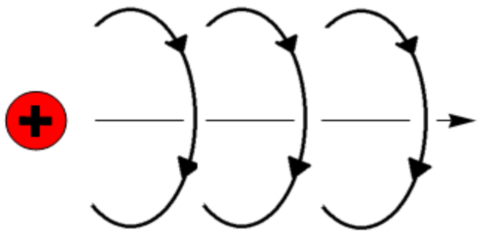
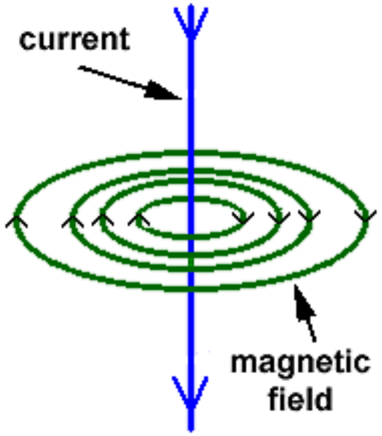
Lecture 4: Stationary magnetic field

- Charge in motion
- Magnetic field
- Gauss’s law for magnetic field
- Ampere’s Law

Charge in motion

Charges in *motion* (electrical current) produce a magnetic field



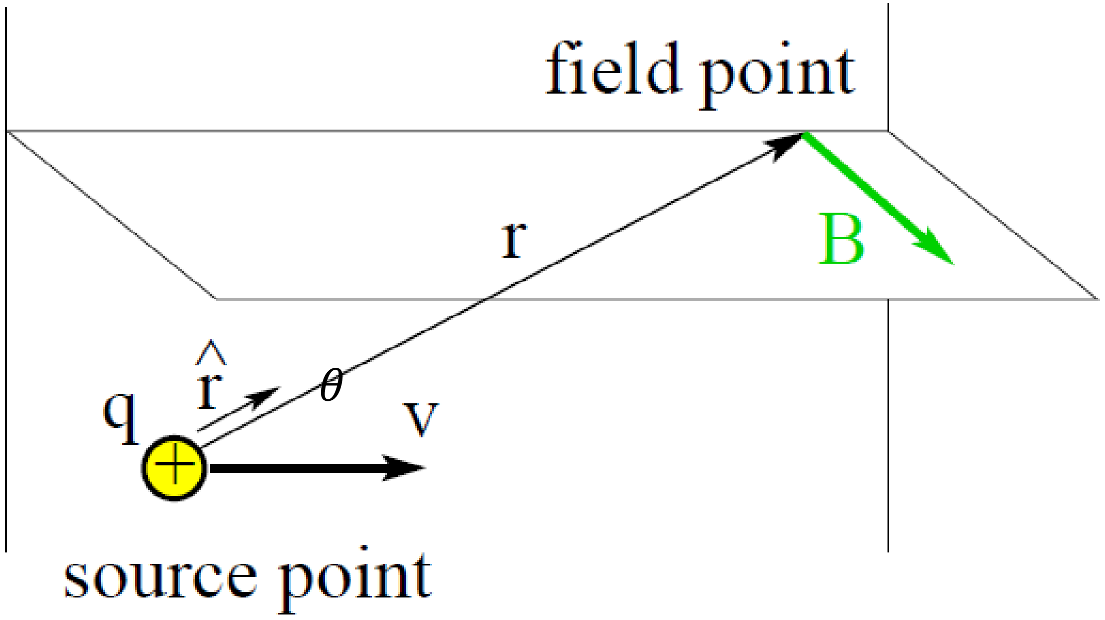


Magnetic flux density **B** in vacuum generated by moving charge q: $\mathbf{B} = \frac{\mu_0}{4\pi}\frac{q\boldsymbol{v}\times\hat{\boldsymbol{r}}}{r^2}$

$$\boldsymbol{v}\times\hat{\boldsymbol{r}} = |v|\sin(\theta)\,\boldsymbol{n}$$

n is a unit vector perpendicular to the plane containing ***v*** and ***r*** in the direction given by the right-hand rule

$\mu_0 = 4\pi\times10^{-7}H/m$, permeability of free space

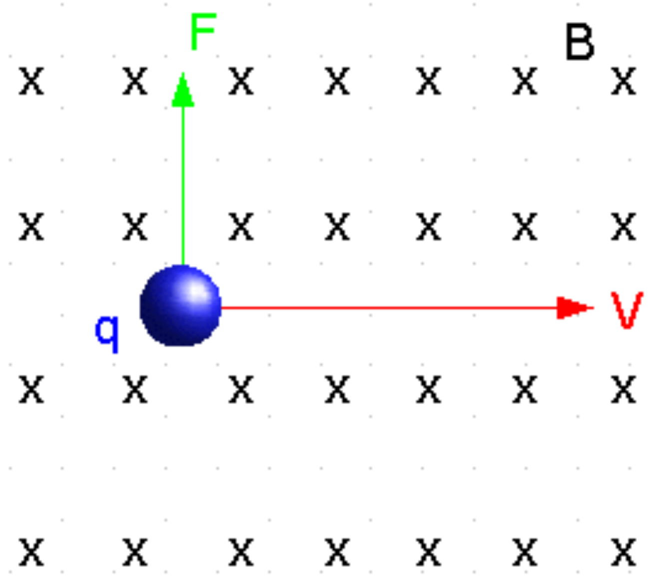


Electromagnetic (Lorentz) force

Force exerted by magnetic field **B** on a moving point charge Q is:

$$\boldsymbol{F} = Q\, \boldsymbol{v} \times \boldsymbol{B}$$

Compare to $\boldsymbol{F}_c$



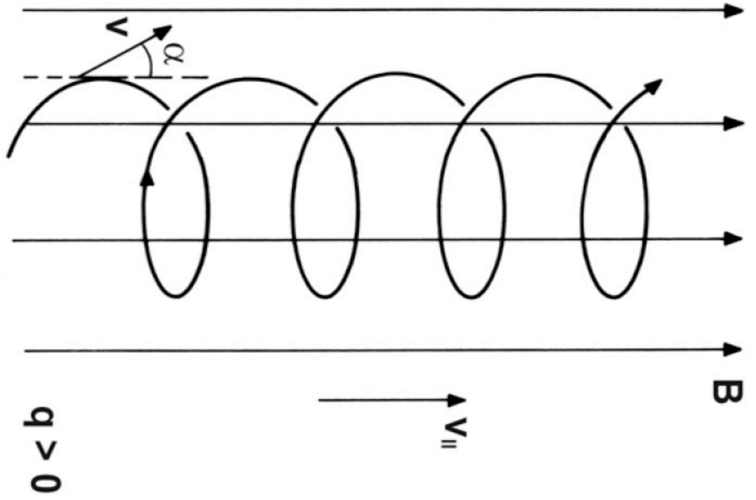
Lorentz law $\mathbf{F} = q\mathbf{E} + q(\mathbf{v} \times \mathbf{B})$

Magnetic force acting on a moving charge is always perpendicular to it's moving direction, so magnetic force does not work on the charges, but changing the charges' moving direction

$$W = \boldsymbol{F} \cdot \boldsymbol{l} = \int \boldsymbol{F} \cdot d\boldsymbol{l} = \int \boldsymbol{F} \cdot \boldsymbol{v} dt$$

*Work is application of force, **F**, to move an object over a distance, **l**,* in the direction that the force is applied.

Example: Point charge's movement in constant uniform magnetic filed



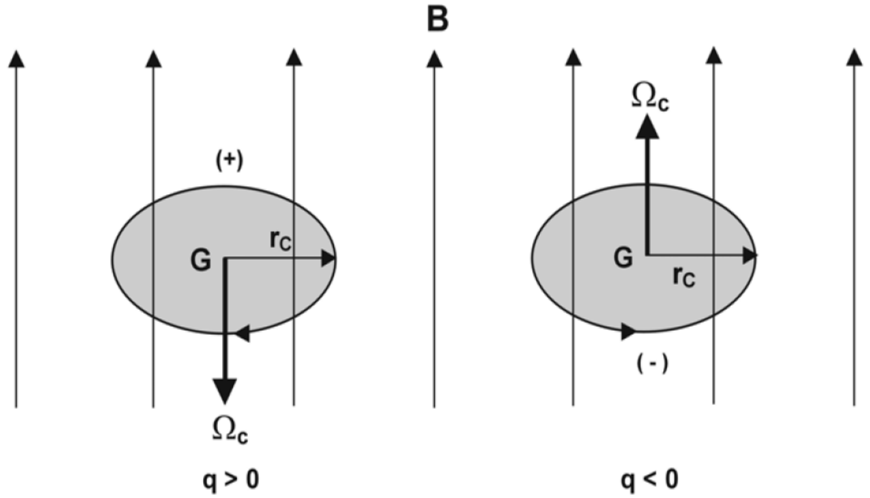
A constant uniform magnetic field **B**, a charge q with mass m is shot perpendicularly to the magnetic field with speed **V**(0), what is the radius of charge?

Centrifugal force

$$F = \frac{mv^2}{r} = qvBsin\theta$$ (θ is the angle between **v** and **B**, 90^o)

$$r = \frac{mv^2}{qvB} = \frac{mv}{qB}$$

https://www.youtube.com/watch?v=orsMYomjwIw

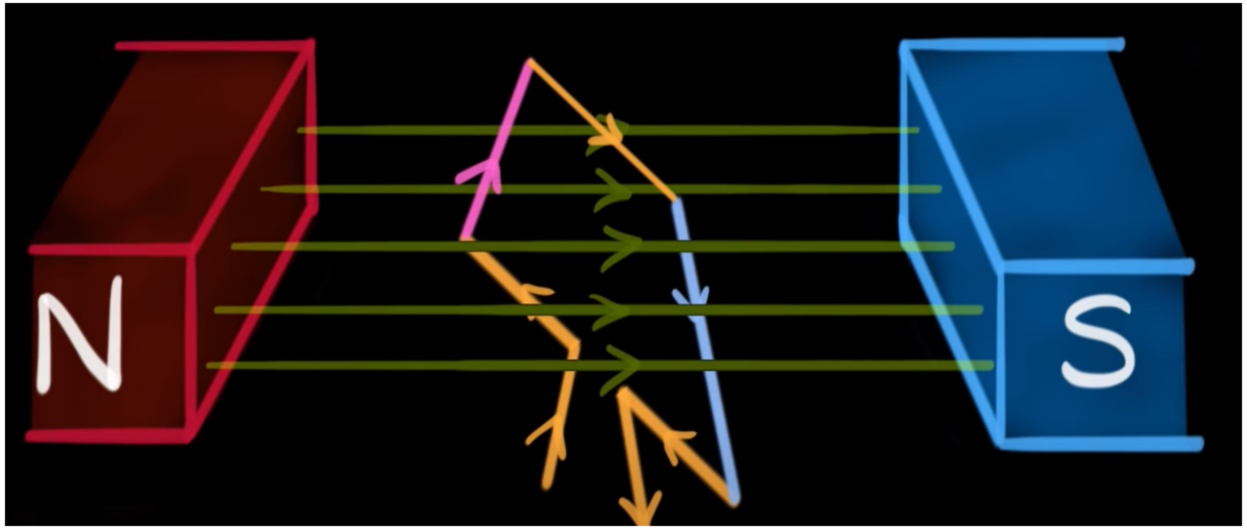


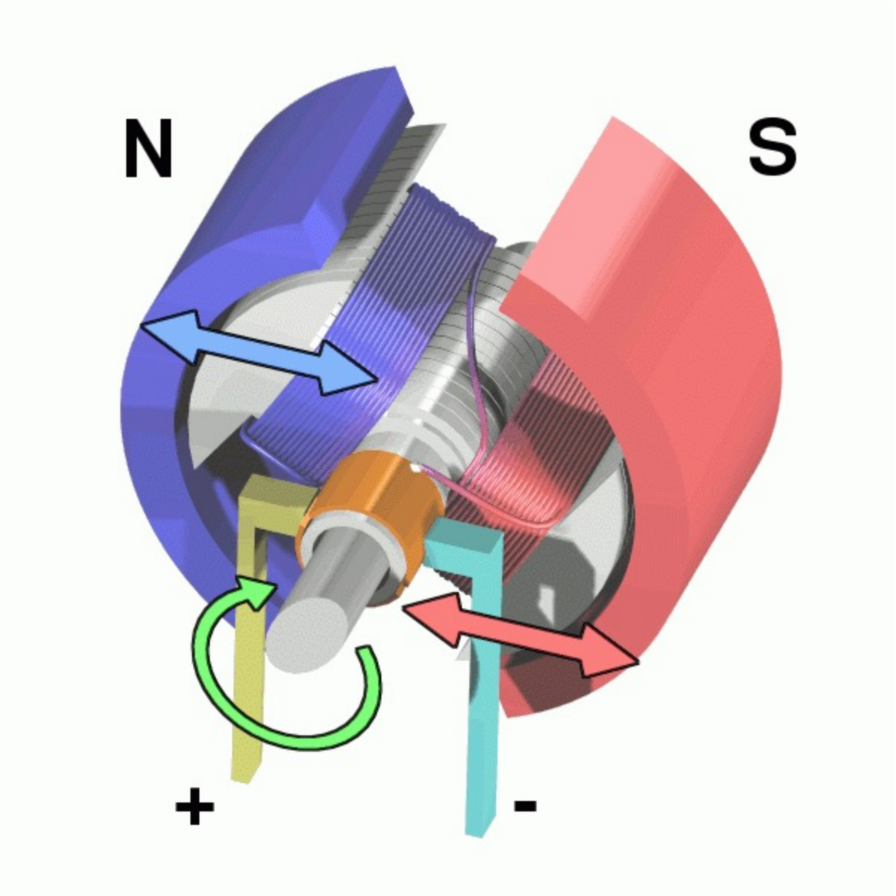
Electric motor

An electrical motor consist of: stator and rotor

One produces mechanical energy by **current and magnets**

Rotating windings: electric current (moving charge) flows





Magnetic field

Magnetic field **H**$= \frac{\boldsymbol{B}}{\mu}$, μ is called permeability and material dependent, the value of μ for free space is $\mu_0 = 4\pi\times10^{-7} H/m$.

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\frac{q\boldsymbol{v}\times\hat{\boldsymbol{r}}}{r^2}$$

$$\boldsymbol{H} = \frac{1}{4\pi}\frac{q\boldsymbol{v}\times\hat{\boldsymbol{r}}}{r^2}$$

The magnetic field intensity is independent of material property.

Biot-Savart law : Magnetic field generated by current

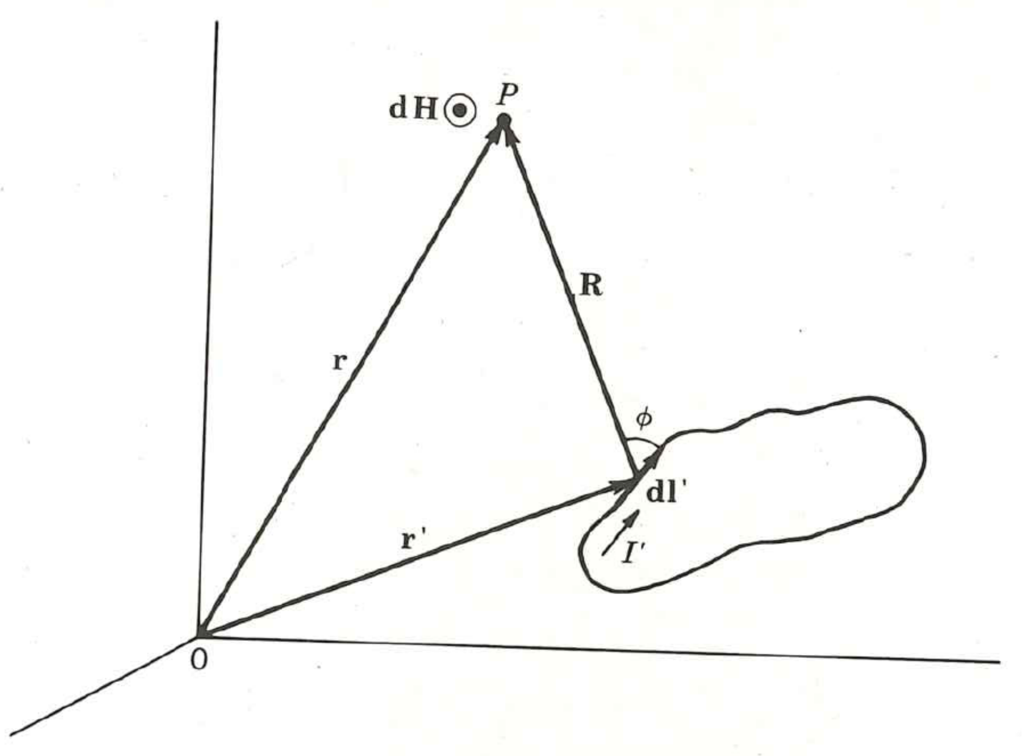
The Biot–Savart law is used for computing the resultant magnetic field **B** at position **r** in 3D-space generated by a *steady current I.*

Vector expression: $\boldsymbol{H}(\boldsymbol{r}) = \int_{c} \frac{I'(r)d\boldsymbol{l}'\times\hat{\boldsymbol{R}}}{4\pi R^2}$

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int_{c} \frac{I'(r)d\boldsymbol{l}'\times\hat{\boldsymbol{R}}}{R^2}$$

$$I'd\boldsymbol{l}'\times\hat{\boldsymbol{R}} = |I'|dl'\sin(\phi)\,\boldsymbol{n}$$

n is a unit **vector** **perpendicular** **to the plane containing I and R** in the direction given by the right-hand rule



Scalar calculation

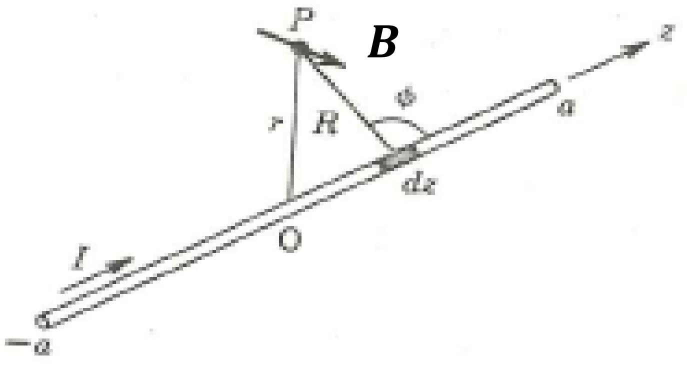
$$H(r) = \int_{c} \frac{I'(r)dl'sin\phi}{4\pi R^2}$$

$$B(r) = \frac{\mu_0}{4\pi}\int_{c} \frac{I'(r)dl'sin\phi}{R^2}$$

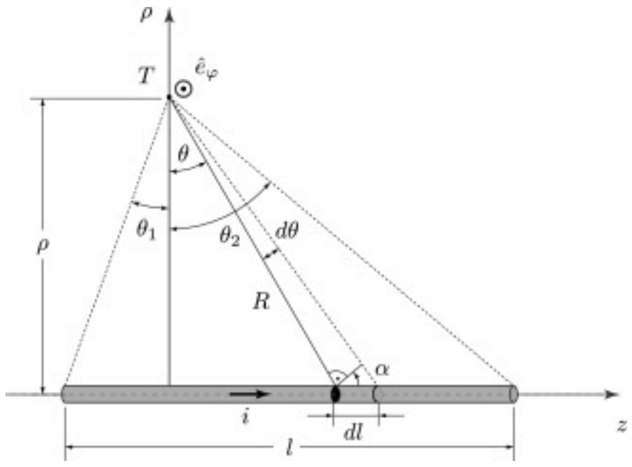
Example: Magnetic flux density

Find the magnetic field B at a point **P** at perpendicular distance r from the center of a finite length of current I, the total current length is 2a.

$$d\boldsymbol{B} = \frac{\mu_0}{4\pi}\frac{Id\boldsymbol{l}\times\widehat{\boldsymbol{R}}}{R^2}$$



$$\boldsymbol{B} = \int_{-a}^{a}\frac{\mu_0}{4\pi}\frac{Idz}{R^2}\hat{R} = \int_{-a}^{a}\frac{\mu_0 \boldsymbol{I}dz}{4\pi R^2}\hat{R} = \int_{-a}^{a}\frac{\mu_0 I\sin\phi dz}{4\pi R^2}$$



$$\sin(\phi) = \frac{r}{\sqrt{r^2+z^2}}$$

$$R^2 = r^2 + z^2$$

$$B = \int_{-a}^{a}\frac{\mu_0 I\sin\phi dz}{4\pi R^2} = \frac{\mu_0 Ir}{4\pi}\int_{-a}^{a}\frac{dz}{(r^2+z^2)^{3/2}} = \frac{\mu_0 I}{2\pi r}\frac{a}{(r^2+a^2)^{1/2}}$$

Assuming a>>r, what is B?

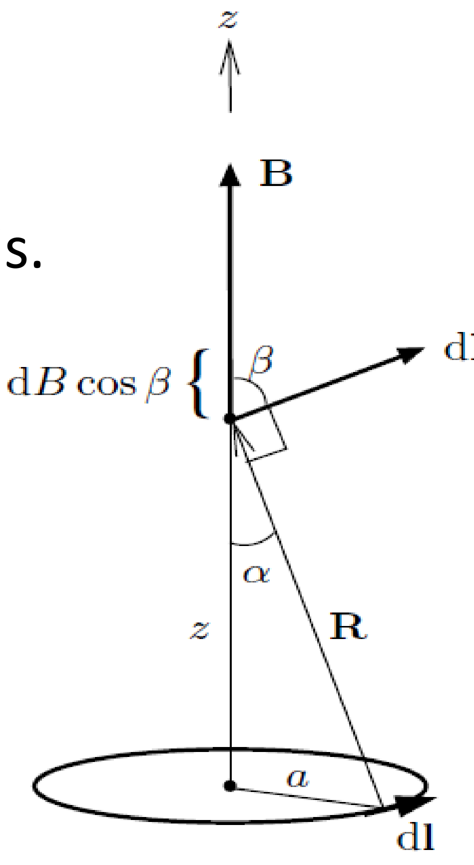
Field on axis of circular loop

A ring with radius *a* and current I, calculate B at point on the z axis.

$$\mathrm{d}\mathbf{B}\cdot\hat{\mathbf{z}} = \mathrm{d}B\cos\beta = \mathrm{d}B\sin\alpha,$$

B and d**B** directions are different

$$\mathrm{d}\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I\,\mathrm{d}\mathbf{l}\times\hat{\mathbf{R}}}{R^2} \qquad \mathrm{d}B = \frac{\mu_0}{4\pi}\frac{I\,\mathrm{d}l}{R^2}.$$



$$\mathbf{dl}\times\hat{\boldsymbol{R}} = \mathrm{dl}$$

$$B = \oint_{\text{ring}} \mathrm{d}B\sin\alpha = \frac{\mu_0 I\sin\alpha}{4\pi R^2}\oint \mathrm{d}l = \frac{\mu_0 I\sin\alpha(2\pi a)}{4\pi R^2} = \frac{\mu_0 I a^2}{2R^3}$$

$$R = \sqrt{z^2 + a^2}$$

$$\sin\alpha = \frac{a}{R}$$

$$\mathbf{B} = \frac{\mu_0 I a^2}{2R^3}\hat{\mathbf{z}} = \frac{\mu_0 I a^2}{2(z^2 + a^2)^{3/2}}\hat{\mathbf{z}}$$

Magnetic flux and flux continuity

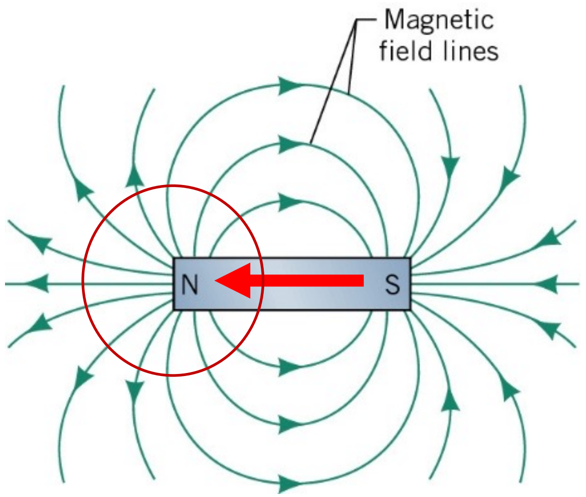
Magnetic flux ϕ is the integral of the flux density accross surface

$$\phi = \int_s \boldsymbol{B} \cdot d\boldsymbol{S},$$

For an enclosed surface, the flux is zero

$$\oint_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0.$$

$$\nabla \cdot B = 0$$



There is no magnetic monopole, continuous magnetic field.

$$\oint_S \mathbf{A} \cdot \mathrm{d}\mathbf{S} = \int_v \nabla \cdot \mathbf{A} \mathrm{d}v = 0$$

Ampere's law

Ampere's Law states that for any closed loop path, the line integral of the magnetic field around closed curve C is equal to the electric current enclosed in the loop.

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int_s \mathbf{J} \cdot d\mathbf{s}$$

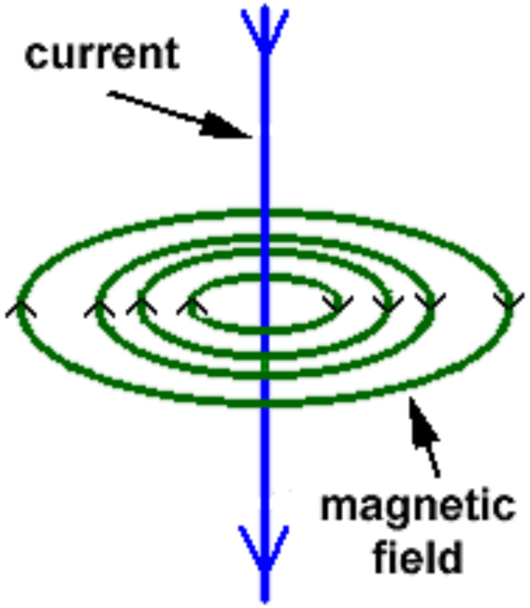
$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{B} \cdot d\mathbf{S} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{S}.$$

Stokes' theorem

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J},$$

$$\frac{1}{\mu_0} \nabla \times \mathbf{B} = \nabla \times \mathbf{H} = \boldsymbol{J}$$

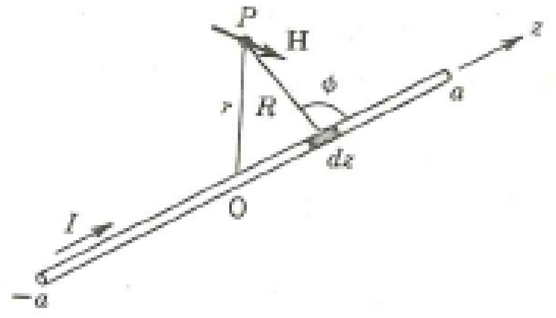
$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$$



In magnetostatic, such as constant DC current, $\frac{\partial \boldsymbol{D}}{\partial \boldsymbol{t}} = \mathbf{0}$

Ampere's law: describing the magnetic field around a conductor

$$B = \int_{-a}^{a} \frac{\mu_0 I \sin\phi dz}{4\pi R^2} = \frac{\mu_0 Ir}{4\pi} \int_{-a}^{a} \frac{dz}{(a^2+z^2)^{3/2}} = \frac{\mu_0 I}{2\pi r} \frac{a}{(r^2+a^2)^{1/2}}$$



Stokes' theorem: $\oint_{\partial\Sigma} \mathbf{B} \cdot \mathrm{d}\boldsymbol{l} = \iint_{\Sigma} \nabla \times \mathbf{B} \cdot \mathrm{d}\mathbf{S}$

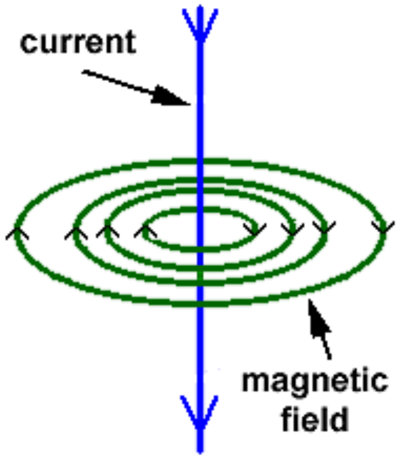
$$\int_s \nabla\times\mathbf{B} \cdot d\mathbf{S} = \oint \mathbf{B} \cdot d\mathbf{l} = 2\pi r B(r)$$

$$\int_s \nabla\times\mathbf{B} \cdot d\mathbf{S} = \int_s \mu_0 \mathbf{J} \cdot d\mathbf{S} = \mu_0 I$$

$\Rightarrow$ $2\pi r B(r) = \mu_0 I$ $\Rightarrow$ $B(r) = \frac{\mu_0 I}{2\pi r}$

$\uparrow$

$$\nabla\times\mathbf{B} = \mu_0 \mathbf{J}$$



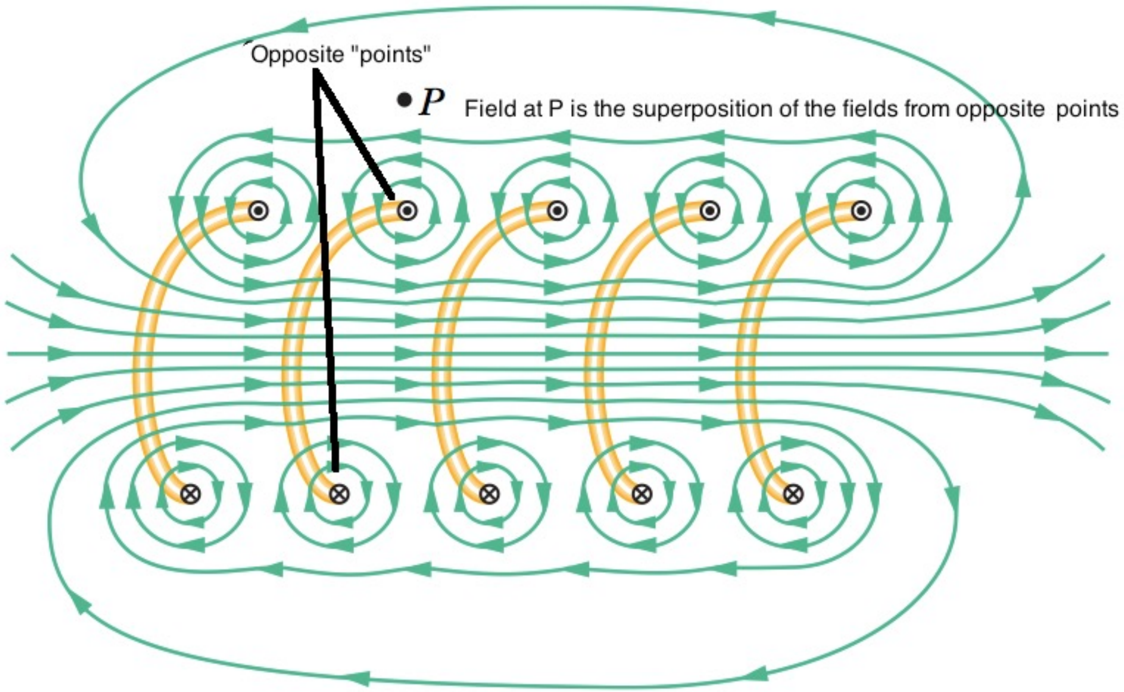
Solenoid

$$\int_S \nabla\times\mathbf{H}\cdot d\mathbf{S} = \oint \mathbf{H}\cdot d\mathbf{l} = Hl$$

$$\int_S \nabla\times\mathbf{H}\cdot d\mathbf{S} = \int_S \mathbf{J}\cdot d\mathbf{S} = NI$$

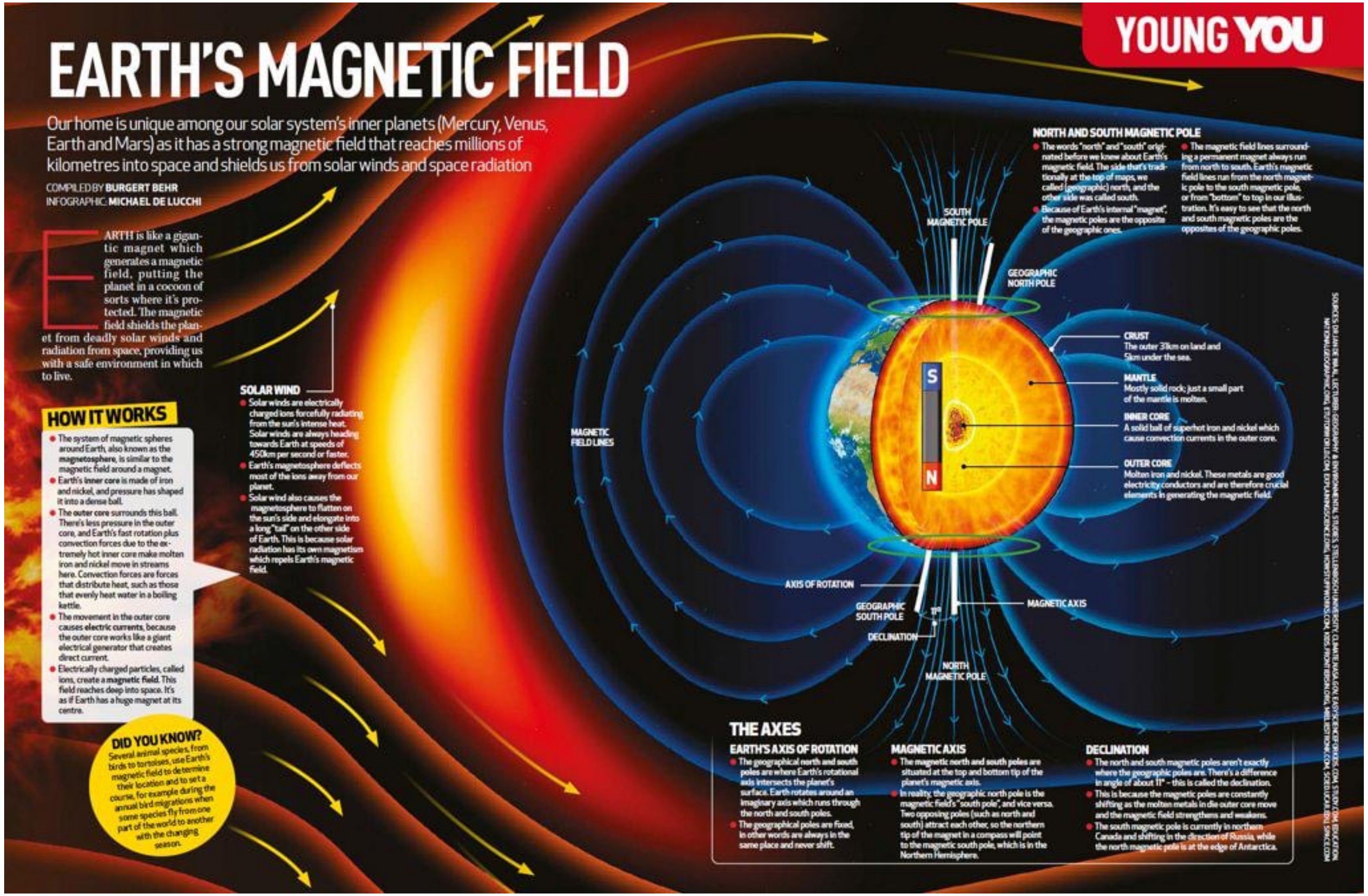
$$\boldsymbol{B} = \mu\boldsymbol{H}$$

$$H = \frac{NI}{l}$$



If the core is iron instead of air, the flux density **B** is much stronger.

Example: geomagnetic field



Example: Magnetic field

A coaxial line carrying current I on the inner conductor and –I on the outer. Calculate the magnetic field **H** at r distance, current evenly distributed in the two conductors.

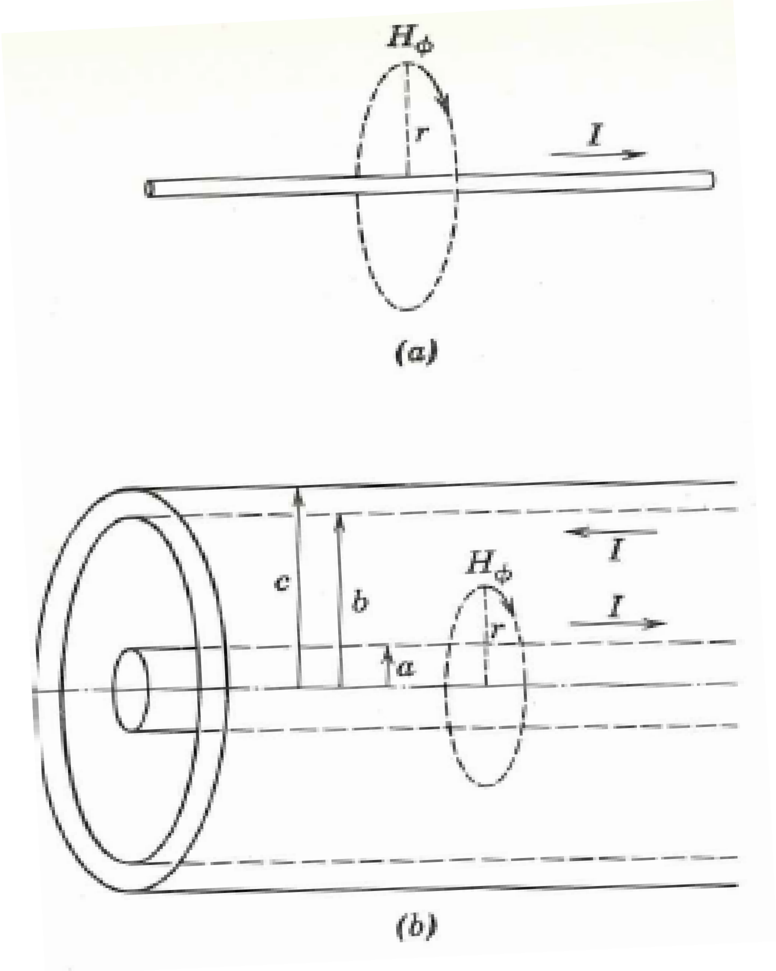
1) 0<r<a,
2) a<r<b

$$\int_s \mathbf{J} \cdot d\mathbf{S} = I(r) = J\pi r^2 \qquad \int_s \mathbf{J} \cdot d\mathbf{S} = I = J\pi a^2$$

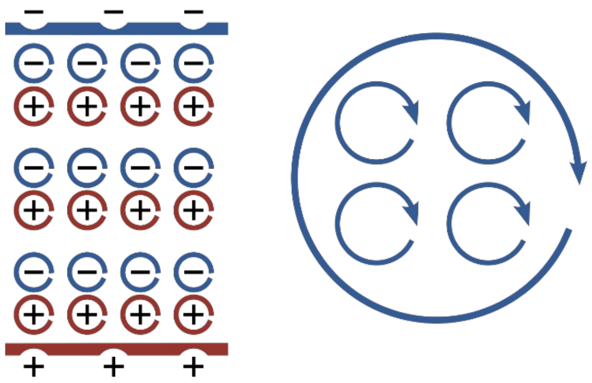
$$I(r) = I\frac{r^2}{a^2}$$

$$H_\phi(r) = \frac{I(r)}{2\pi r} = \frac{Ir}{2\pi a^2}$$

$$H_\phi = \frac{I}{2\pi r} \qquad (a< r < b)$$



Magnetic field in material



$\mathbf{B} = \mu_0 \mathbf{H}$ in vacuum

Once there is magnetic field applied to medium, the electronic spin motions in the atoms can be thought of as circulating current that produces a field **M**, magnetization, which adds to magnetic field **H**.

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$$

Magnetic susceptibility χ_m is used to quantify the additional field **M**.

$$\mathbf{M} = \mu_0 \chi_m \boldsymbol{H}$$

$$\mathbf{B} = \mu_0(1 + \chi_m)\boldsymbol{H} = \mu_0 \mu_r \mathbf{H} = \mu \mathbf{H}$$

μ_r *relative permeability.*

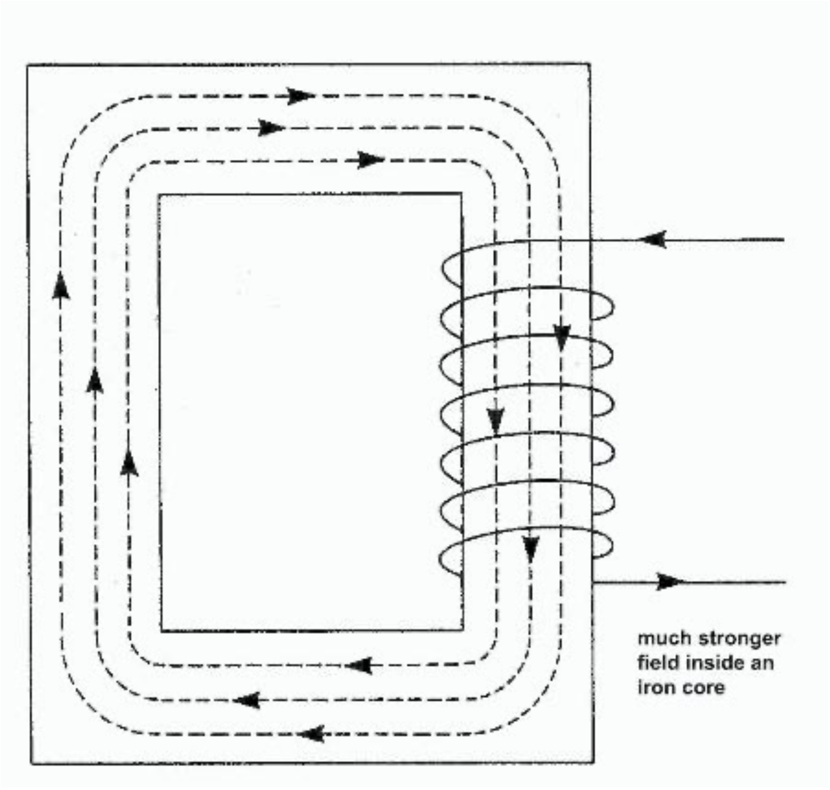
Good magnetic material relative permeability
Iron: ~ 5000

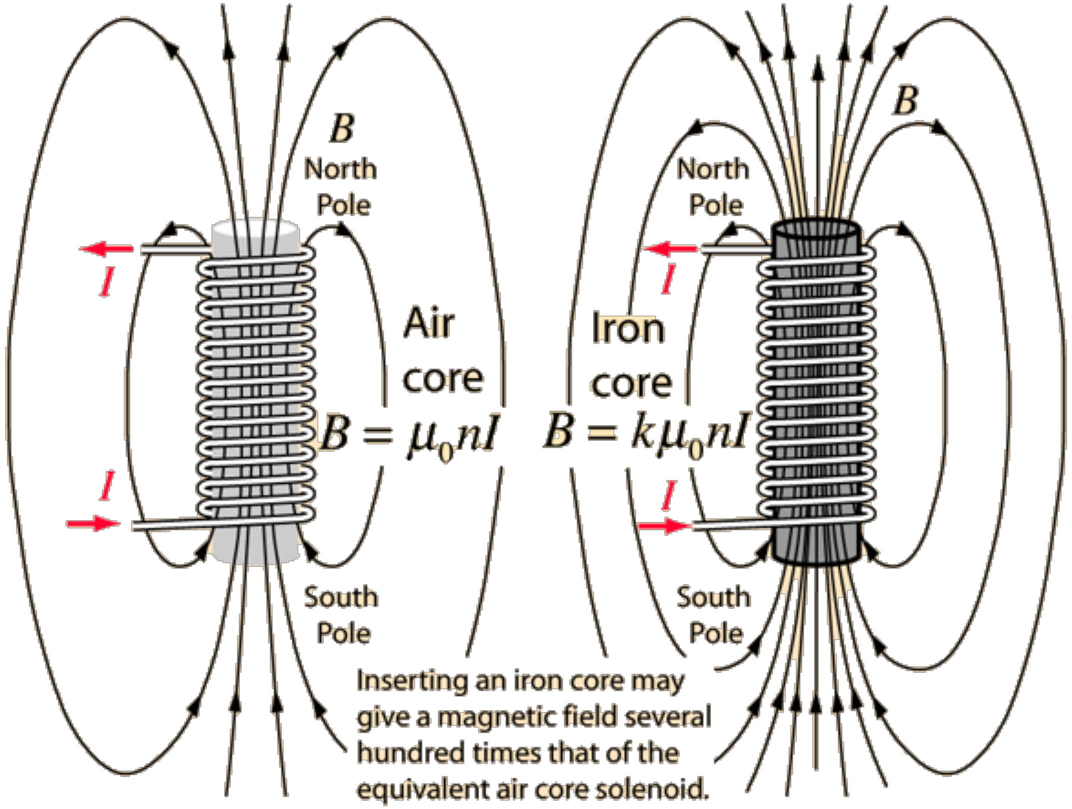
Bad magnetic material relative permeability
Silver:1
Copper : 1
Gold: 1
Aluminium: 1

Magnetization increases in the magnetic flux density **B** in ferro-magnetic materials compared to vacuum.

Field in magnetic material

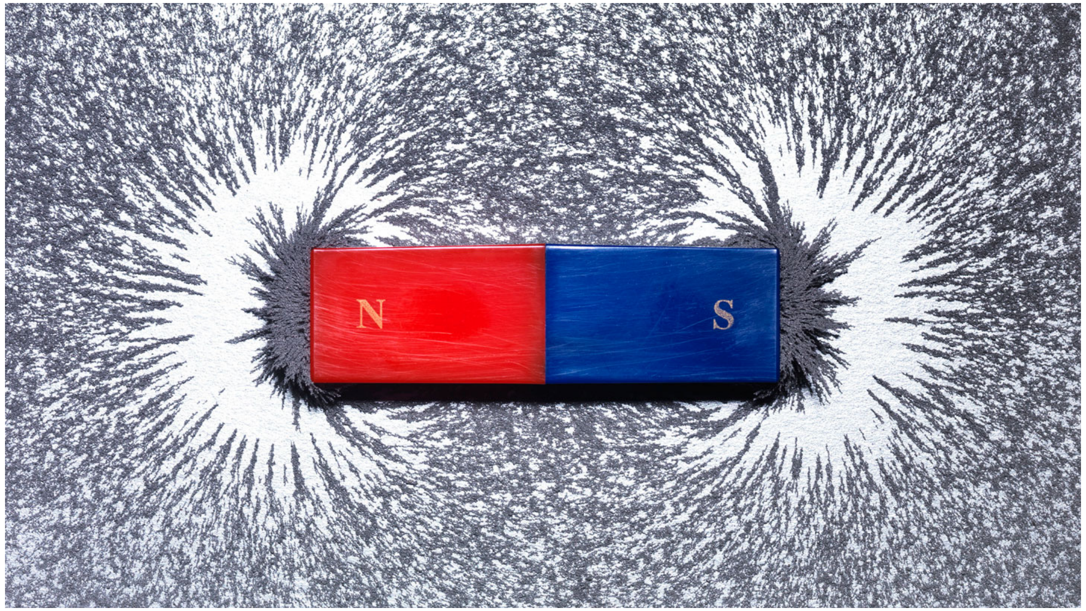
Magnetic material can be used to guide magnetic field path.

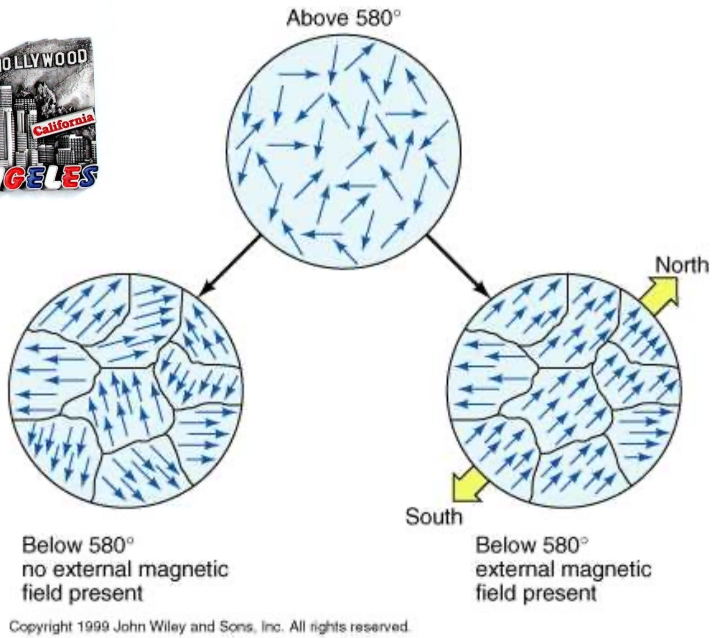




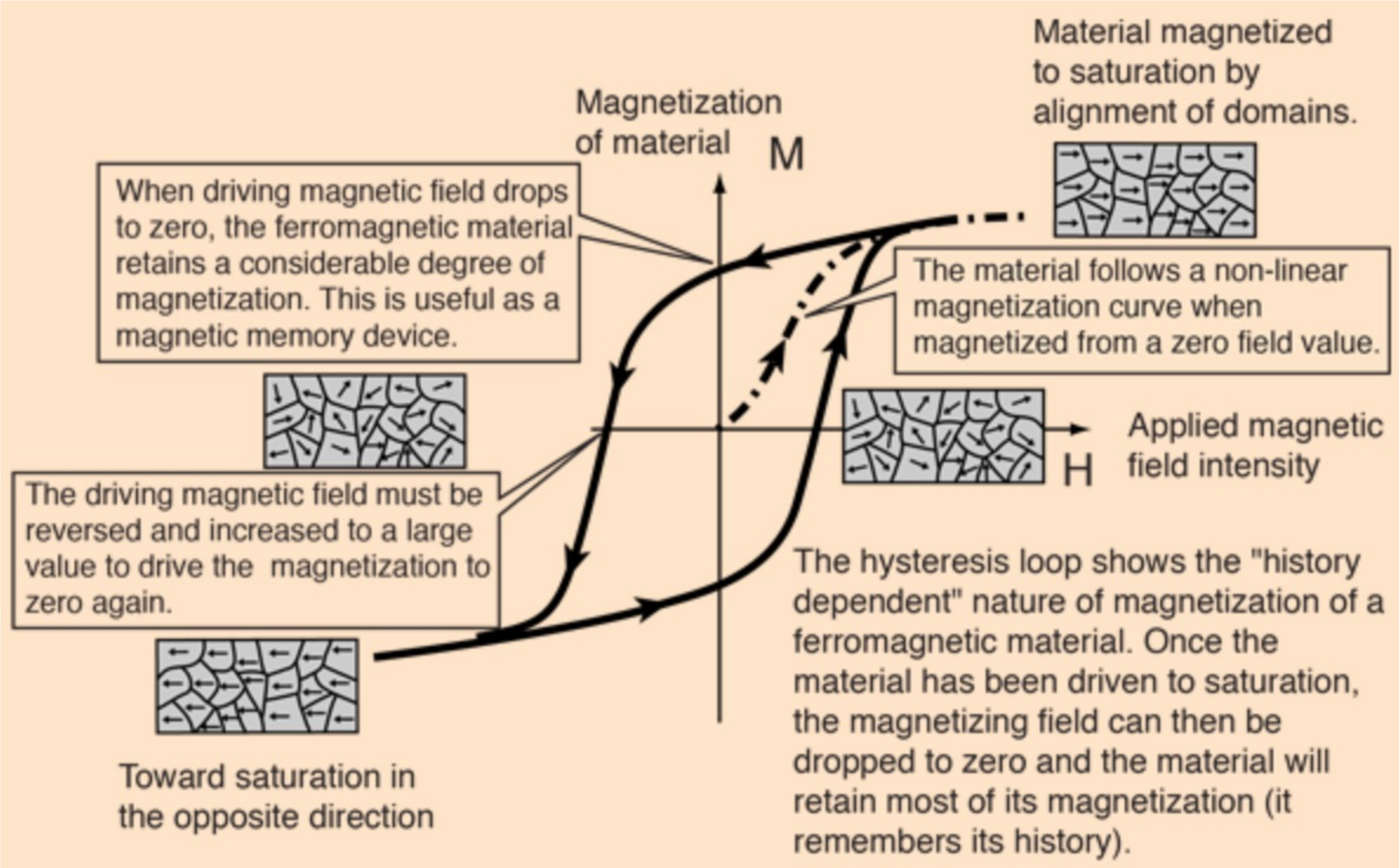
Permanent magnet

For some materials, after magnetization the electron movement can remain when external magnetic field disappears. The electron movement can be distorted at high temperatures, strong opposite external field or strong shock.

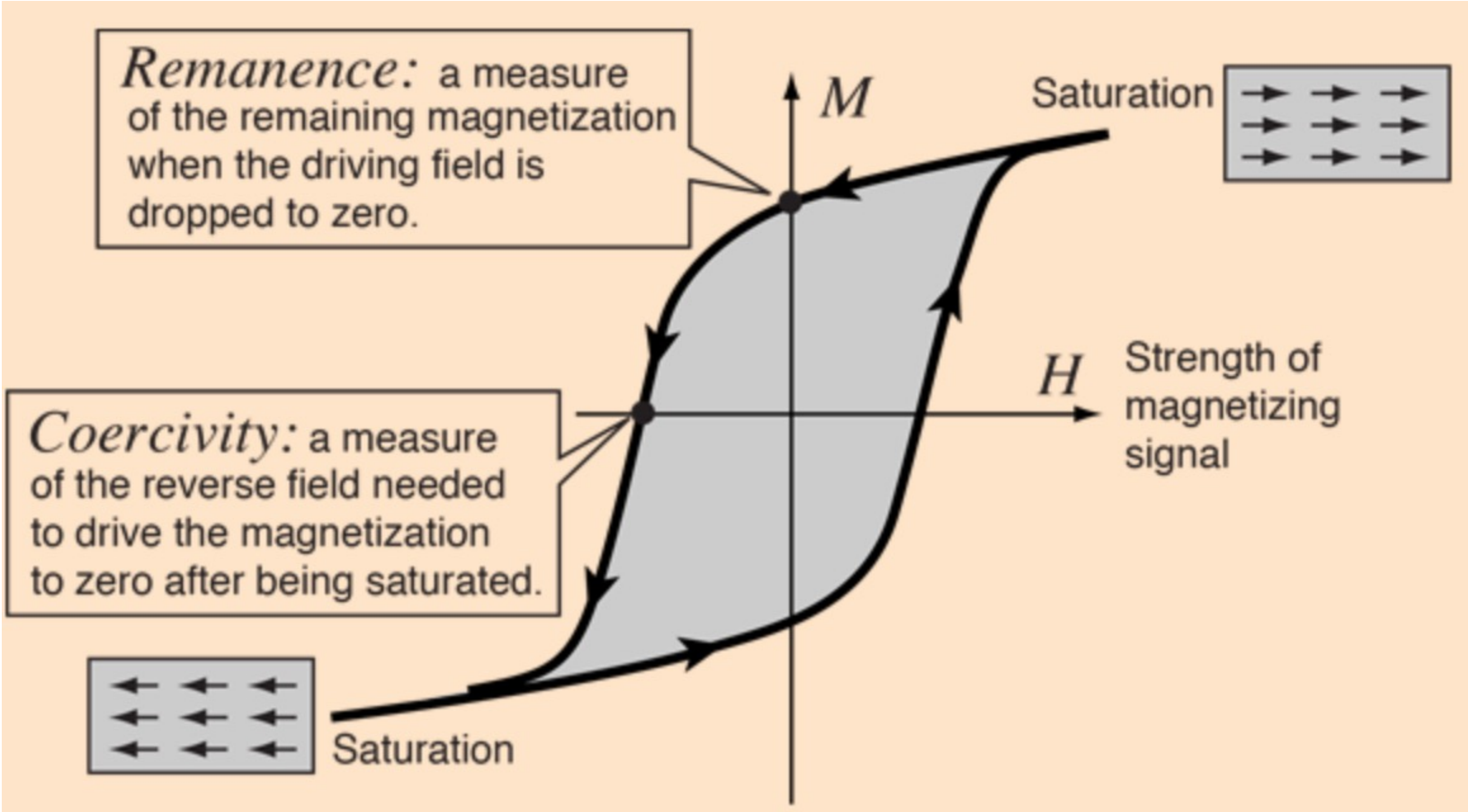




Hysteresis loop

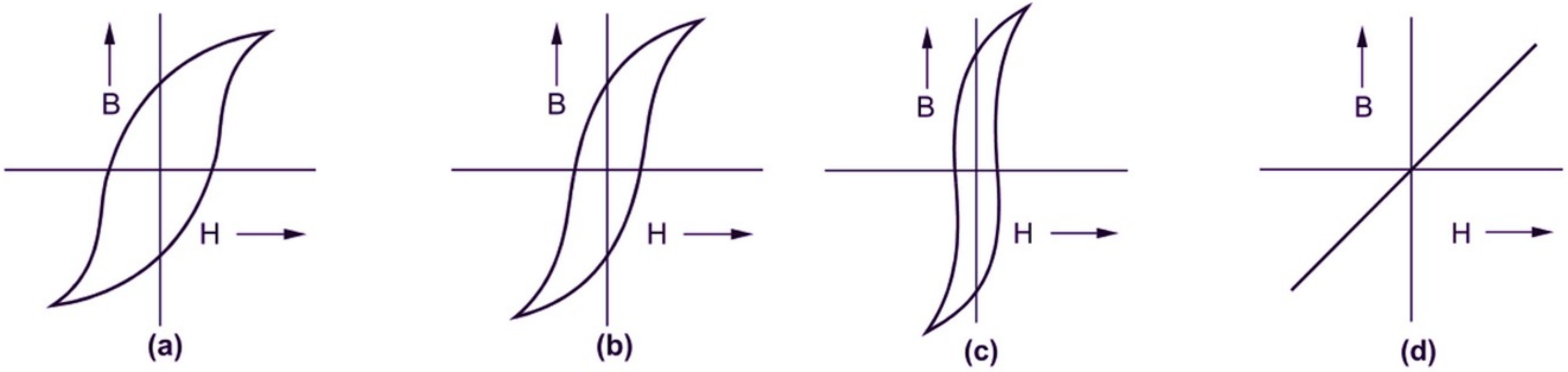


Hysteresis loop



Magnetic material: B-H curve

Suitability of magnetic material for use depends on the areas and shape of hysteresis loop



Hard steel | Wrought iron | Sheet steel | Non-magnetic

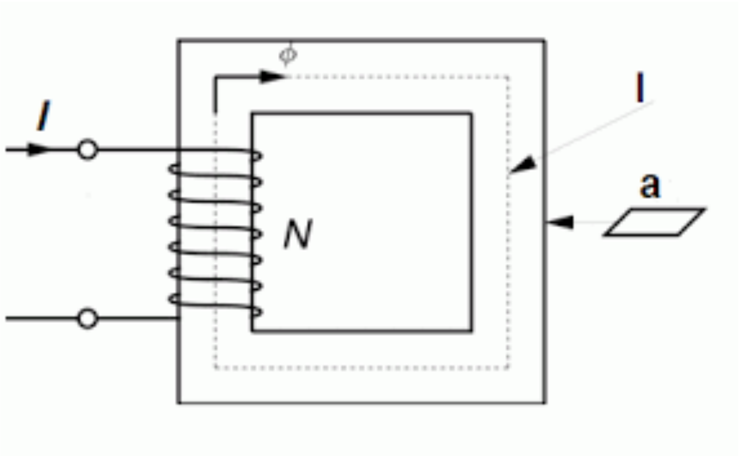
a. For permanent magnet, the material should have a large HL to gain high remanence and coercive force.

b. For electro-magnet, high permeability and low coercivity are required.

c. For electrical equipment subjected to rapid reversals of magnetization, high permeability and low hysteresis loss are required.

Magnetic circuit

Magnetomotive force (MMF): $F = NI = HL = \Phi R$



Φ, magnetic flux

$R = \frac{l}{\mu S}$, magnetic reluctance

- The magnet (*HL*) or current (*NI*) possesses a magneto-motive force (MMF).
- The MMF generates a magnetic flux.
- The flux exists within the magnet and the air gap between the poles. The enclosed flux path is called a magnetic circuit.
- A stronger MMF will produce more flux.
- The lower the reluctance of the magnetic circuit, the more flux will be produced.

Magnetic circuit and electric circuit

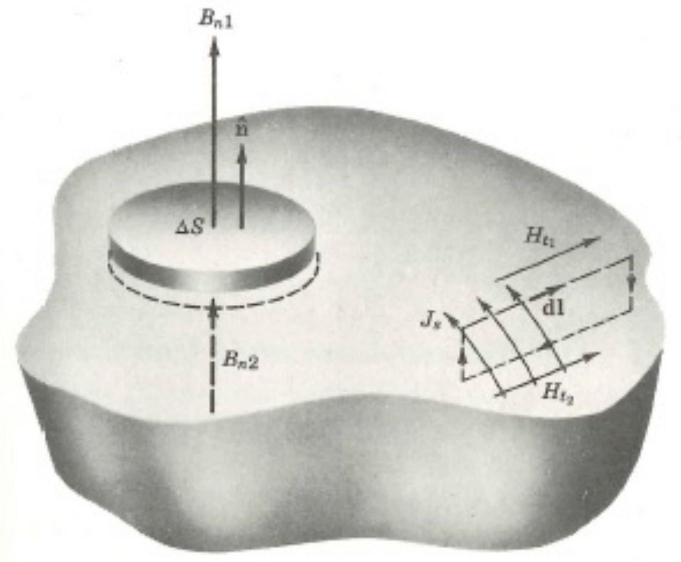
Magnetic Circuit	***Electrical Circuit***
1. The closed path for magnetic flux is called a magnetic circuit.	1. The closed path for electric circuit is called an electric circuit.
2. Flux , ø = $\frac{mmf}{Reluctance}$	2. Current, $I = \frac{emf}{resistance}$
3. mmf (Ampere – turns)	3. emf (Volts)
4. Reluctance, $S = \frac{l}{a\mu_0\mu_r}$	4. Resistance, $R = \rho\,\frac{l}{a}$
5. Flux density, $B = \frac{\varnothing}{a}$ Wb/m^2	5. Current density $\delta = \frac{I}{a}$ A/m^2
6. mmf drop = $\varnothing\ S$	6. Voltage drop = $I\ \ R$
7. Magnetic Intensity, $H = \frac{N\,I}{l}$	7. Electric intensity, $E = \frac{V}{d}$

Boundary condition for static magnetic field

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0.$$

$$B_{n1}\,\Delta S = B_{n2}\,\Delta S$$

$$B_{n1} = B_{n2}$$



$$\oint \mathbf{H} \cdot d\mathbf{l} = H_{t1}\,\Delta l - H_{t2}\,\Delta l = J_s\,\Delta l$$

$$H_{t1} - H_{t2} = J_s$$

$$\frac{1}{\mu_0}\nabla\times\mathbf{B} = \nabla\times\mathbf{H} = \mathbf{J}$$

Line current density

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{B} \cdot d\mathbf{S} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{S}.$$

Example: Magnetic flux density

$$\int \frac{1}{(a^2+x^2)^{3/2}}\,dx = \int \frac{1}{\left(a^2 + a^2\cdot \tan^2(u)\right)^{3/2}} \cdot \frac{a}{\cos^2(u)}\,du = \int \frac{1}{\left(a^2(1+\tan^2(u))\right)^{3/2}} \cdot \frac{a}{\cos^2(u)}\,du =$$

$$x = a\cdot\tan(u) \rightarrow \frac{x}{a} = \tan(u) \rightarrow \arctan\left(\frac{x}{a}\right) = u$$

$$dx = \frac{a}{\cos^2(u)}\,du$$

$$= \int \frac{1}{\underbrace{(a^2)^{3/2}}_{=\,a^3} \cdot \underbrace{(1+\tan^2(u))^{3/2}}_{=\,\frac{1}{\cos^2(u)}}} \cdot \frac{a}{\cos^2(u)}\,du = \frac{1}{a^3}\int \frac{1}{\left(\frac{1}{\cos^2(u)}\right)^{3/2}}\,\frac{a}{\cos^2(u)}\,du =$$

$$= \frac{a}{a^3}\int \frac{1}{\frac{1}{\cos^3(u)}} \cdot \frac{1}{\cos^2(u)}\,du = \frac{1}{a^2}\int \cos(u)\,du = \frac{1}{a^2}\sin(u) = \frac{1}{a^2}\sin\left(\arctan\left(\frac{x}{a}\right)\right) =$$

$$= \frac{1}{a^2} \cdot \frac{\frac{x}{a}}{\sqrt{1+\left(\frac{x}{a}\right)^2}} = \frac{x}{a^3} \cdot \frac{1}{\sqrt{\frac{1}{a^2}(a^2+x^2)}} = \frac{x}{a^3} \cdot \frac{1}{\left(\frac{1}{a}\right)\sqrt{a^2+x^2}} = \boxed{\frac{x}{a^2} \cdot \frac{1}{\sqrt{a^2+x^2}} + C'}$$