Lecture 3: Stationary electric field

1. Energy in electric field
2. Boundary condition in electro-statics
3. Perfect conductor
4. Ohm’s law

Electric field and electric displacement field

Electric field: $\vec{\boldsymbol{E}} = \frac{\boldsymbol{Q}}{\boldsymbol{4\pi\varepsilon_0 r^2}}\hat{\boldsymbol{r}}$ $\quad\quad$ $\vec{\boldsymbol{E}} = \vec{\boldsymbol{F}}/\boldsymbol{q}$

Electric displacement field: $\mathbf{D} = \varepsilon_0 \mathbf{E}$ $\quad\quad$ $\vec{\boldsymbol{D}} = \frac{Q}{4\pi r^2}\hat{\boldsymbol{r}}$

Electric field and potential

$$\boxed{\mathbf{E} = -\nabla V, \ \mathrm{V/m}}$$

The potential difference between two points A and B:

$$V_{AB} = -\int_{\boldsymbol{r}_A}^{\boldsymbol{r}_B} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\varepsilon_0 {r_i}^2} dr_i = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$

$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}}$$

Poisson's equation

Inserting $\mathbf{E} = -\nabla V$ into Maxwell's equation $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ gives

$$-\nabla \cdot (\nabla V) = \frac{\rho}{\epsilon_0},$$

which gives

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}.$$

Gauss's law

Electric flux flowing out of a closed surface = Enclosed total charges

divided by the permittivity ⇨ $\oint \mathbf{E} ds = Q/\varepsilon$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon}; \qquad \nabla \cdot \mathbf{D} = \rho$$

Electric polarization and material permittivity

The influence of electric polarization: $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \varepsilon_0 \chi_e \mathbf{E} = (1 + \chi_e)\varepsilon_0 \mathbf{E} = \varepsilon_r \varepsilon_0 \mathbf{E} = \varepsilon \mathbf{E}$$

- χ_e is ***electric susceptibility***
- $1 + \chi_e = \varepsilon_r$, ***relative permittivity***
- $\varepsilon = \varepsilon_r \varepsilon_0$, electric permittivity

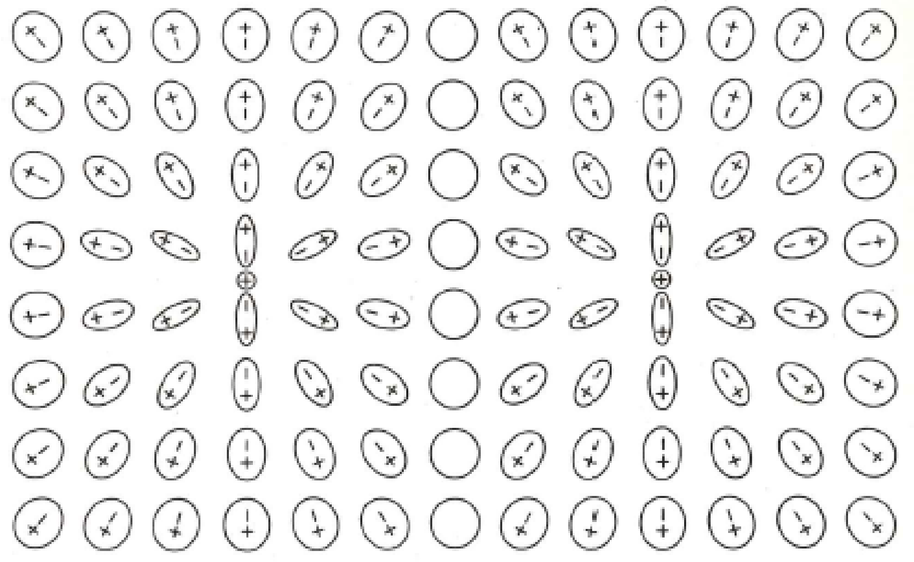


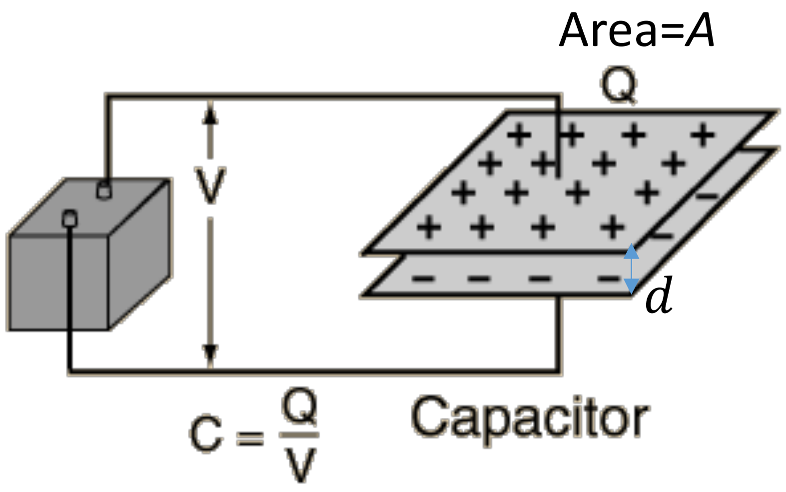
Fig. 1.3c Polarization of the atoms of a dielectric by a pair of equal positive charges.

Capacitor

A two-terminal electrical device that can store energy in the form of electric charges.

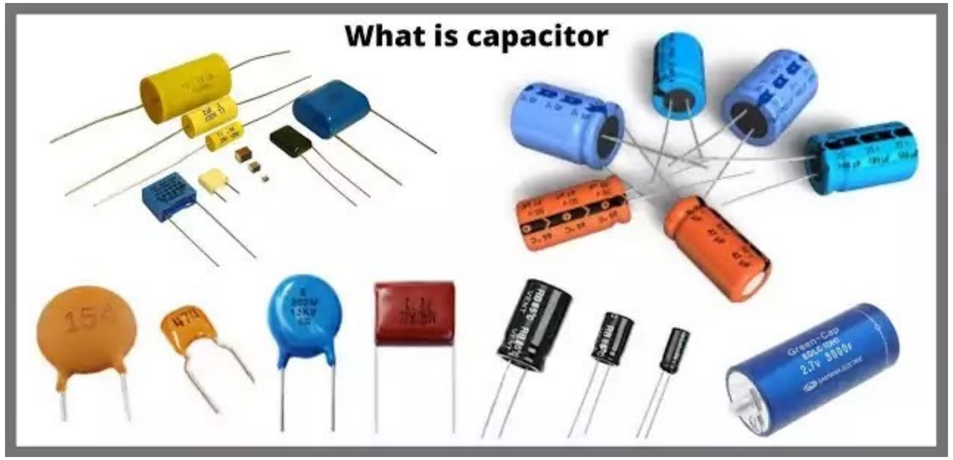
It consists of two electrical conductors that are separated by a distance.

The space between the conductors may be filled by vacuum or dielectric.



Capacitance is the ability of an object to store electrical charge.

$$C = \frac{Q}{V} = \frac{\varepsilon A}{d}$$



Electric energy in capacitor

How much energy capacitor can store?

Voltage represents energy per unit charge

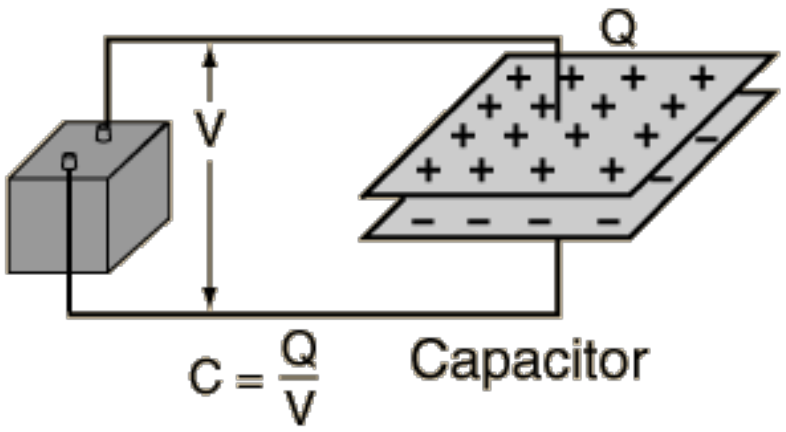
$$W_e = \int_{\boldsymbol{a}}^{\boldsymbol{b}} F d\boldsymbol{l} = \int_{\boldsymbol{a}}^{\boldsymbol{b}} qE d\boldsymbol{l} = qV$$

The work to move a charge element d*q* from the negative plate to the positive plate is equal to Vd*q*

$$\mathrm{d}W_e = V dq \qquad Q = CV$$

$$W_e = \int_{\boldsymbol{0}}^{\boldsymbol{Q}} V dq = \int_{\boldsymbol{0}}^{\boldsymbol{Q}} \frac{q}{C} dq = \frac{Q^2}{2\mathrm{C}} = \frac{1}{2} CV^2$$

$$\boxed{W_e = \frac{1}{2} CV^2}$$

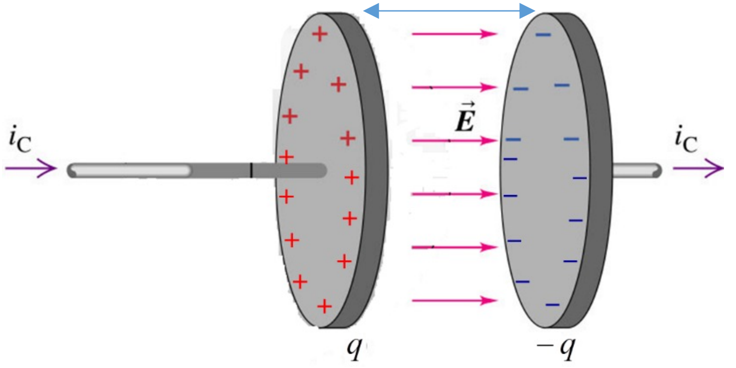


Energy density in electric field

$C = \frac{\varepsilon A}{\mathrm{d}}$ Volume=Ad

$$W_e = \frac{1}{2}CV^2 = \frac{1}{2}\frac{\varepsilon A}{\mathrm{d}}V^2 = \frac{1}{2}\varepsilon Ad\frac{V^2}{d^2} = \frac{1}{2}\varepsilon V_{vol}E^2 = \frac{1}{2}V_{vol}DE$$

$\frac{V^2}{d^2} = E^2$ **D**= ε**E**



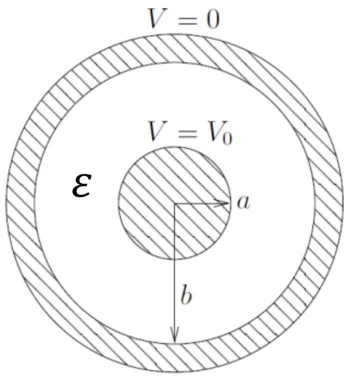
Electric energy density

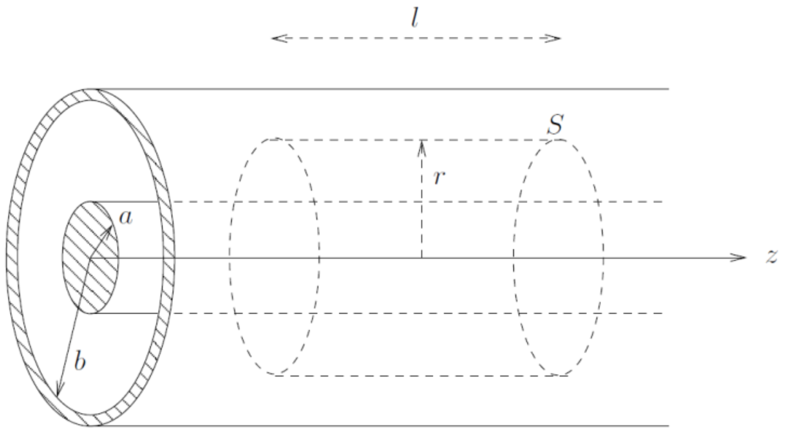
$$\boxed{\eta_e = \frac{W_e}{V_{vol}} = \frac{1}{2}ED = \frac{1}{2}\varepsilon E^2}$$

Example

Assuming: infinite long cable and the permittivity of the dielectric in between is ε

1) calculate the capacitance per unit length
2) calculate the electric energy stored in unit length





$$C' = \frac{Q'}{V_0} \qquad V_0 = \frac{Q'}{2\pi\epsilon} \ln \frac{b}{a}$$

$$V_0 = \int_a^b \mathbf{E} \cdot \mathrm{d}\mathbf{l} = \int_a^b E \mathrm{d}r = \frac{Q'}{2\pi\epsilon_0} \int_a^b \frac{\mathrm{d}r}{r} = \frac{Q'}{2\pi\epsilon_0} \ln \frac{b}{a}.$$

$$C' = \frac{Q'}{V_0} = \frac{2\pi\epsilon}{\ln \frac{b}{a}}. \qquad W_\mathrm{e}' = \frac{1}{2} C' V_0^2 = \frac{\pi\epsilon}{\ln \frac{b}{a}} V_0^2.$$

Boundary conditions in electrostatics

Electric field involves more than one materials

$Normal\ component$

$$D_{n1}\Delta S - D_{n2}\Delta S = \rho_s\ \Delta S \qquad \oint \mathbf{D}\cdot d\mathbf{S} = Q_S$$

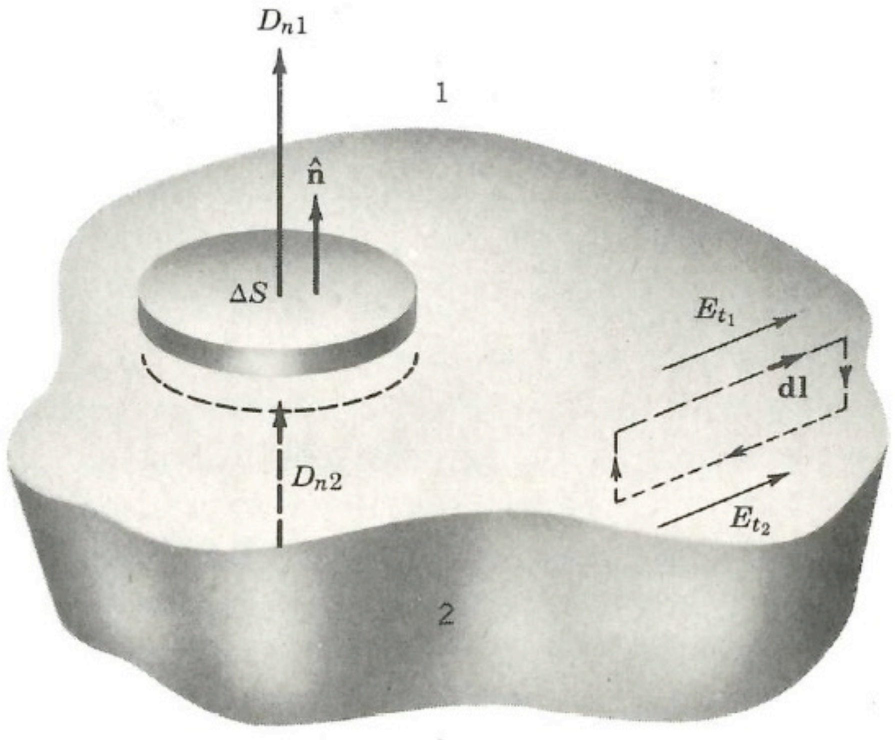
$$D_{n1} - D_{n2} = \rho_s$$

Conservative field: electrostatic electric field

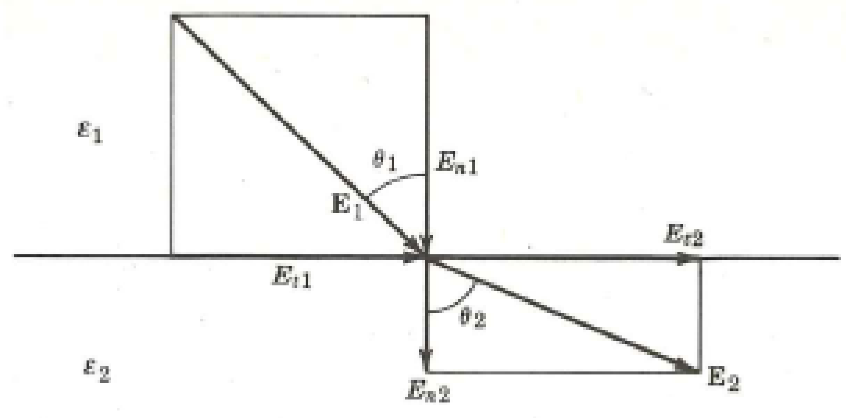
$Tangential\ component$

$$E_{t1}\Delta l - E_{t2}\Delta l = 0 \qquad \oint \mathbf{E}\cdot d\mathbf{l} = 0$$

$$E_{t1} = E_{t2}$$



Electric field change direction across two different dielectric materials



Considering, no charge on the surface between two dielectric

$$D_{n1} - D_{n2} = \rho_s = 0$$

$$\varepsilon_1 E_{n1} = \varepsilon_2 E_{n2}$$

$$E_{n2} = \frac{\varepsilon_1 E_{n1}}{\varepsilon_2}$$

$$E_{t1} = E_{t2}$$

$$\tan\theta_1 = \frac{E_{t1}}{E_{n1}}$$

$$E_{t2} = E_{t1} = E_{n1}\tan\theta_1$$

$$\Rightarrow$$

$$\theta_2 = \tan^{-1}\frac{E_{t2}}{E_{n2}}$$

$$\theta_2 = \tan^{-1}\left(\frac{\varepsilon_2}{\varepsilon_1}\tan\theta_1\right)$$

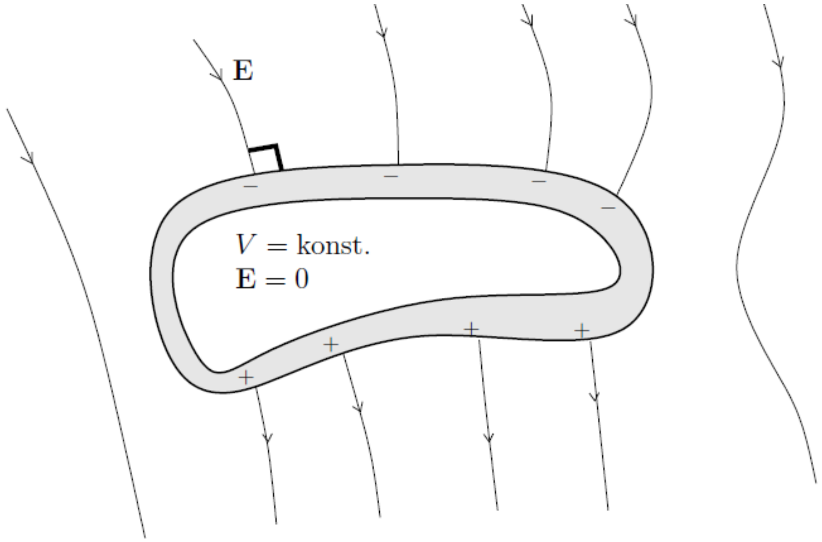
Perfect conductor in electric field

1) **E** = 0 inside the conductor.
2) $\rho_{in} = 0$, no charge inside the conductor.
3) $\rho_s \neq 0$, there is surface charge.
4) The electric field outside the boundary of the perfect conductor is

$$E_n = \frac{\rho_s}{\varepsilon},\ E_t = 0,$$

5) The conductor is an equipotential surface.

$$V_{AB} = -\int_A^B \boldsymbol{E} d\mathbf{l} = 0.$$



$$D_{n1} - D_{n2} = \rho_s$$

$$E_{t1} = E_{t2}$$

Faraday cage:

https://www.youtube.com/watch?v=QU0fLnucE6A&ab_channel=MITxVideos

Material conductivity

Good conductive material:

Silver: $6.2\times10^7 S/m$

Copper: $5.8\times10^7 S/m$

Gold: $4.1\times10^7 S/m$

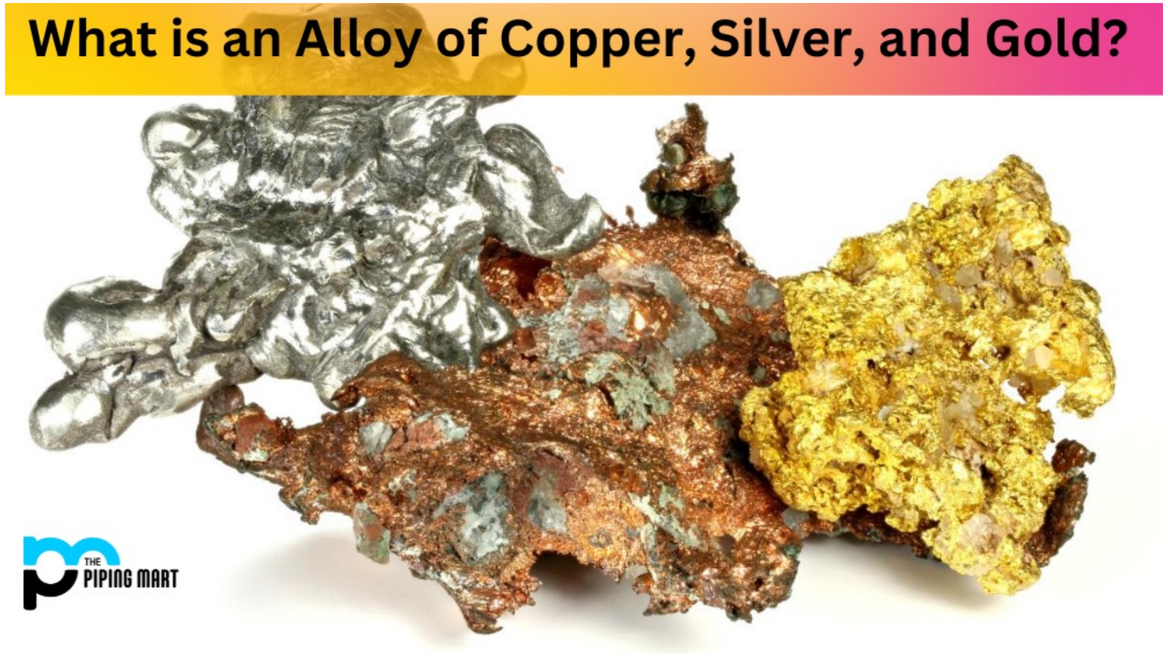
Aluminium: $3.5\times10^7 S/m$

Non-conductive material:

Glass $\times10^{-12} S/m$

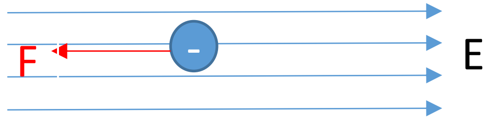
Rubber $\times10^{-13} S/m$

Air $\times10^{-14} S/m$



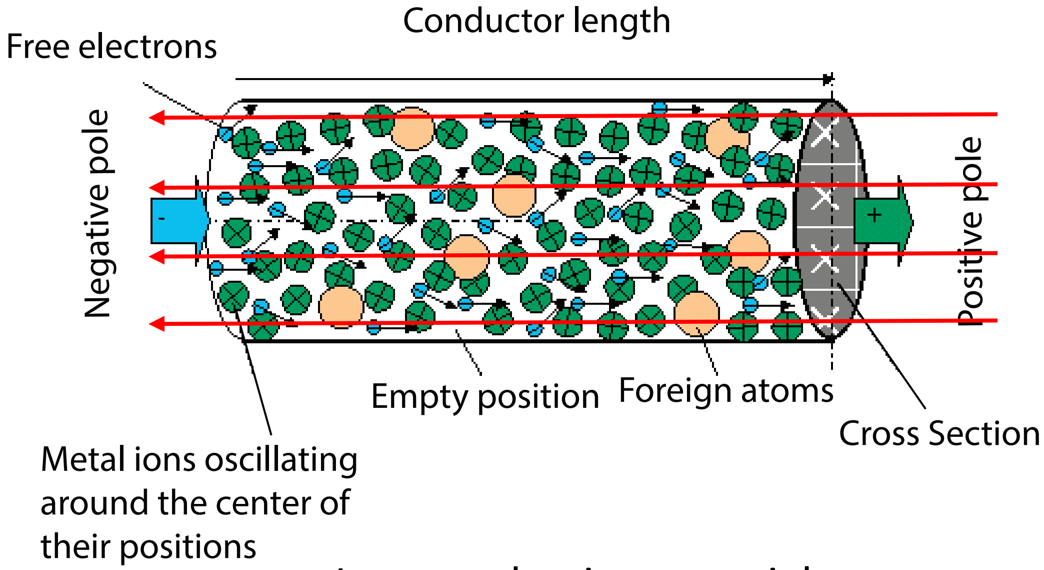
Material conductivity

Once there are free charges in an electric field, the charges can move along the electric field.

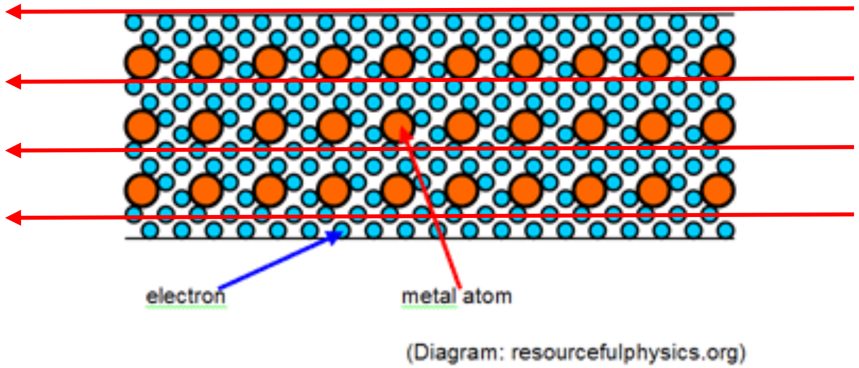


The capability of allowing current to flow is defined by conductivity: $\sigma = 1/\rho$.
That how many charges can move is dependent of material and electric field.

When electrons move, they collide atoms and lost energy. $\Rightarrow$ Resistance: $R = \rho\,\frac{l}{s}$



Low conductive material



High conductive material

Ohm's law

An electric current is a flow of electric charge

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t}$$

Current density **J** is the current per unit area

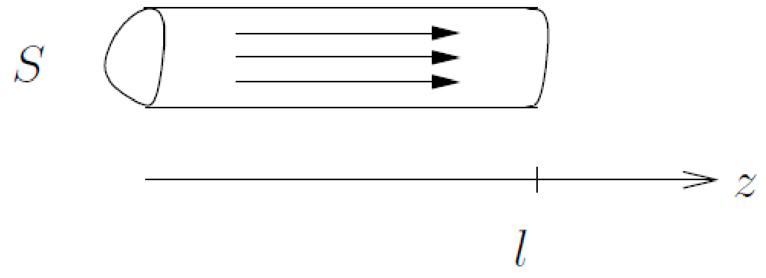
$$I = \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S}.$$

Ohm's law states that the current through a conductor between two points is proportional to the voltage across the two points, and is inversely proportional to the conductor's resistance

$$I = \frac{V}{R} \qquad \mathbf{J} = \frac{\mathbf{E}}{\rho} \qquad R = \frac{V}{I} = \rho \frac{l}{S}$$

Example

A conductor with constant conductivity σ, and cross − sectional area S , the length is l and the constant current is I



1) Calculating the resistance of the conductor and deriving $R = \frac{V}{I}$.
2) Calculating the power done by the current.

Area S, and I is also constant. $J = I/S$

Constant σ , the electric field is constant: $E = J/\sigma$

$$V = \int_0^l E\mathrm{d}z = \int_0^l (J/\sigma)\mathrm{d}z = Jl/\sigma$$

$$R = \frac{l}{S\sigma} = \boxed{\frac{Jl}{\sigma}}\,\frac{1}{\boxed{JS}} = \frac{V}{I}$$

Coulomb's law $\vec{\boldsymbol{F}} = \vec{\boldsymbol{E}} \cdot \boldsymbol{q}$

$$V = -\int \boldsymbol{E}.\boldsymbol{dl}$$

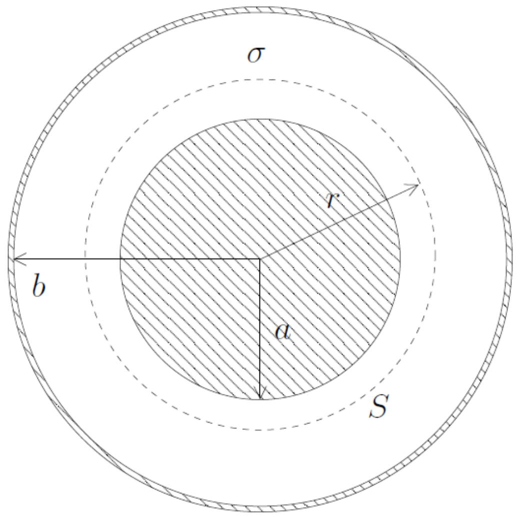
*Work is application of force **F** to move an object for a distance **L**.*

$$W_e = \int_0^l Fdl = \int_0^l q\boldsymbol{E}dl = qV \Rightarrow W_e = \int VIdt$$

q= $\int Idt$

$$P = VI = I^2R$$

Example



A solid conductive ball with a radius *a* is put into a hollow conductive ball with inner radius *b*, between the two is a material with conductivity *σ*.

a) What is the resistance between the two balls?
b) If the solid ball is buried deeply into earth, what is the earth resistance?

$$a = 0.5\ \mathrm{m} \qquad \sigma = 10^{-2}\ \mathrm{m}^{-1}\Omega^{-1}$$

$$\mathbf{J} = \frac{I}{4\pi r^2}\hat{\mathbf{r}}, \quad \text{for } a < r < b.$$

$$\mathbf{E} = \mathbf{J}/\sigma = I\hat{\mathbf{r}}/(4\pi\sigma r^2)$$

$$V = \int_a^b E\mathrm{d}r = \int_a^b \frac{I}{4\pi\sigma r^2}\mathrm{d}r = \frac{I}{4\pi\sigma}\left(-\frac{1}{b} + \frac{1}{a}\right)$$

$$R = \frac{V}{I} = \frac{1}{4\pi\sigma}\left(\frac{1}{a} - \frac{1}{b}\right)$$

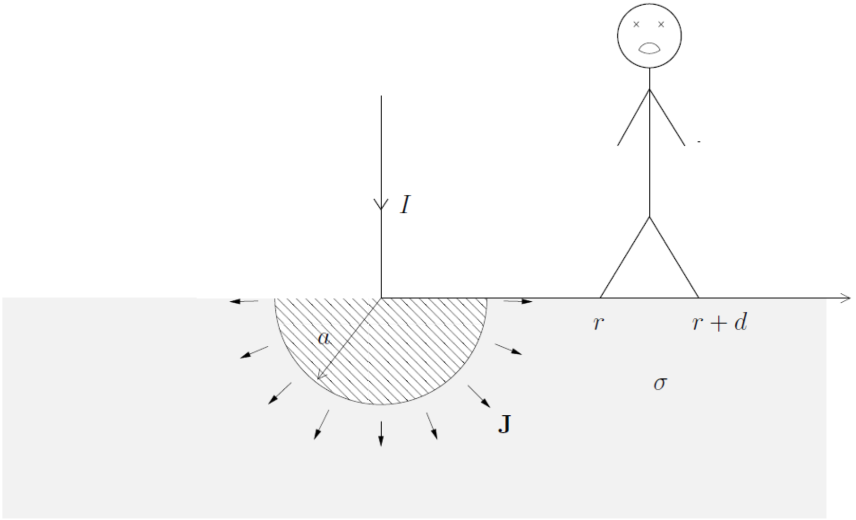
Example

Now if a half solid ball is buried in earth as shown in picture, recalculate the resistance?
If the current is 1000A $\sigma = 10^{-2}\,\mathrm{m}^{-1}\Omega^{-1}$, $r = 1\,\mathrm{m}$ og $d = 0.75\,\mathrm{m}$
What is the voltage between the two legs of the people?

$$\mathbf{J} = \frac{I}{2\pi r^2}\hat{\mathbf{r}}, \quad \mathbf{E} = \frac{I}{2\pi\sigma r^2}\hat{\mathbf{r}}, \quad \text{for } r > a.$$

$$R = \frac{1}{2\pi\sigma a}$$

$$V = \int_r^{r+d} E(r)\mathrm{d}r = \int_r^{r+d} \frac{I}{2\pi\sigma r^2}\mathrm{d}r = \frac{I}{2\pi\sigma}\left(\frac{1}{r} - \frac{1}{r+d}\right)$$



Kirchhoff’s law

It shows current conservation:

At any node of an electric circuit, **the sum of currents flowing into** that node is **equal to the sum of currents flowing out** of that node.

$$\oint_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} = 0.$$