Lecture 3: Stationary electric field

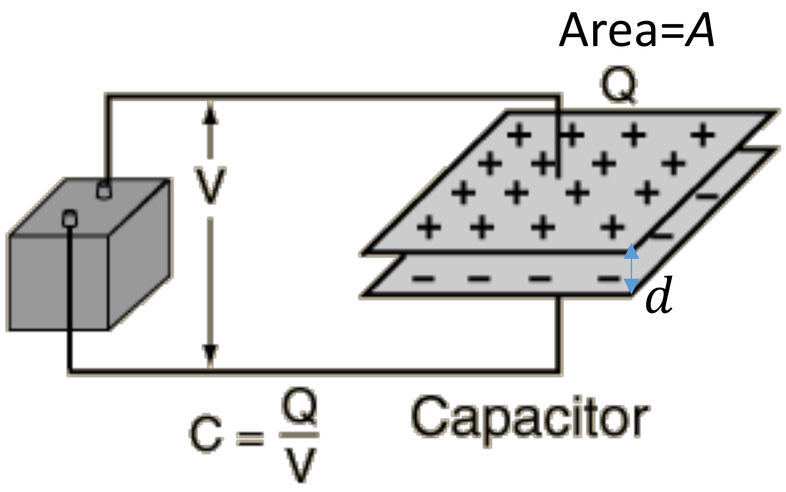
1. Energy in electric field
2. Boundary condition in electro-statics
3. Perfect conductor
4. Ohm’s law

Capacitor

A two-terminal electrical device that can store energy in the form of an electric charge

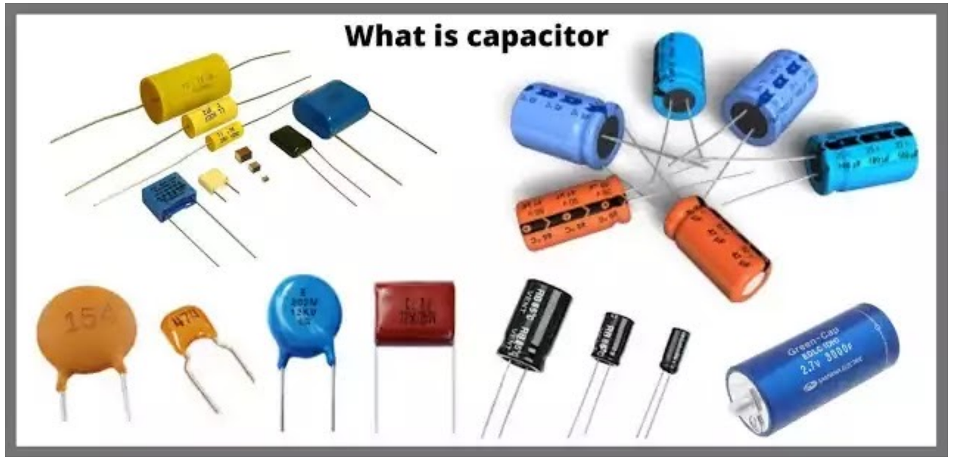
It consists of two electrical conductors that are separated by a distance.

The space between the conductors may be filled by vacuum or dielectric material.



Capacitance is the ability of an object to store an electrical charge.

$$C = \frac{Q}{V} = \frac{\varepsilon A}{d}$$



Electric energy in capacitor

The energy stored on a capacitor can be expressed in terms of the work done by the battery.

Voltage represents energy per unit charge

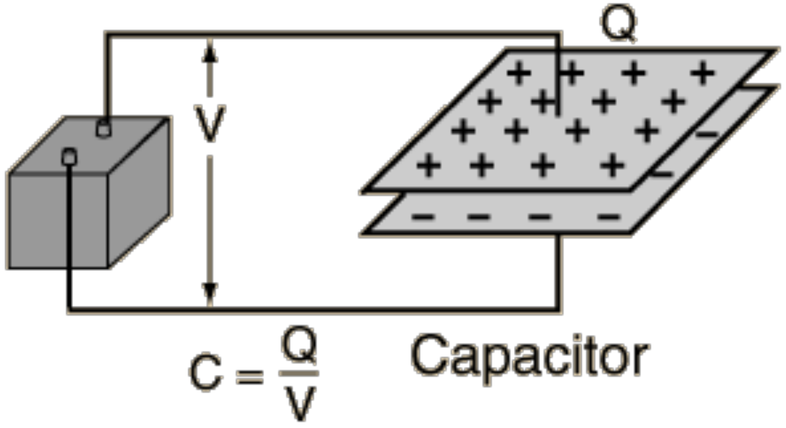
$$W_e = \int_{a}^{b} F d\boldsymbol{l} = \int_{a}^{b} qE d\boldsymbol{l} = qV$$

The work to move a charge element dq from the negative plate to the positive plate is equal to Vdq

$$\mathrm{d}W_e = V dq \qquad Q = CV$$

$$W_e = \int_{0}^{Q} V dq = \int_{0}^{Q} \frac{q}{C} dq = \frac{Q^2}{2C} = \frac{1}{2} C V^2$$

$$\boxed{W_e = \frac{1}{2} C V^2}$$



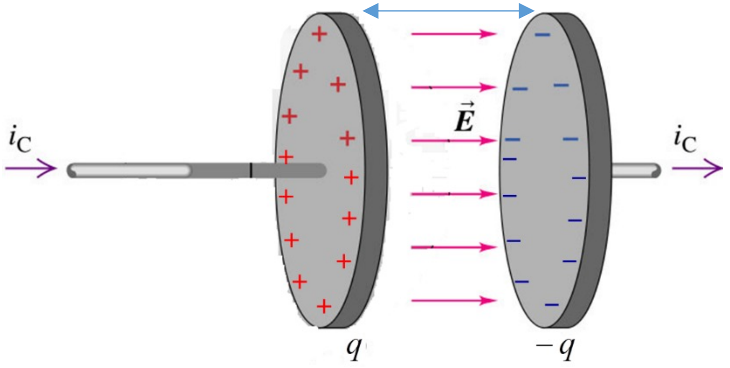
Energy density in electric field

$C = \frac{\varepsilon A}{\mathrm{d}}$ Volume=Ad

$$W_e = \frac{1}{2}CV^2 = \frac{1}{2}\frac{\varepsilon A}{\mathrm{d}}V^2 = \frac{1}{2}\varepsilon Ad\frac{V^2}{d^2} = \frac{1}{2}\varepsilon V_{vol}E^2 = \frac{1}{2}V_{vol}DE$$

$\frac{V^2}{d^2} = E^2$ $\boldsymbol{D} = \varepsilon\boldsymbol{E}$

d Area=A



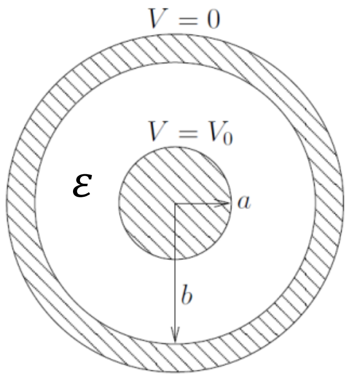
Electric energy density

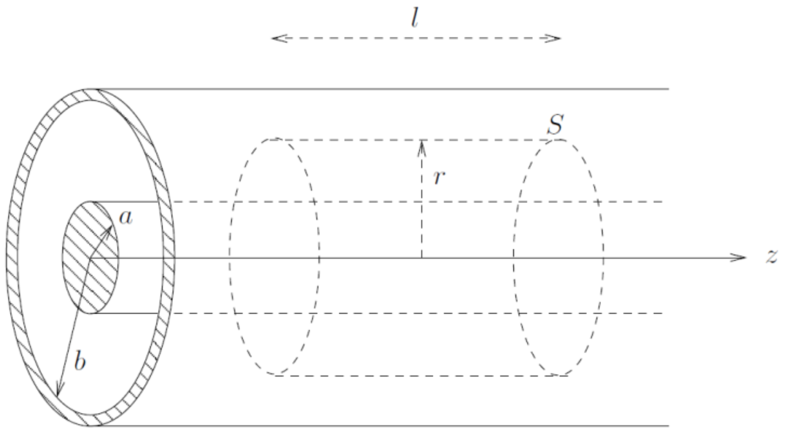
$$\eta_e = \frac{W_e}{V_{vol}} = \frac{1}{2}ED = \frac{1}{2}\varepsilon E^2$$

Example:

Assuming: infinite long cable and the permittivity of the dielectric in between is ε

1) calculate the capacitance per unit length.
2) calculate the electric energy stored in unit length





$$C' = \frac{Q'}{V_0} \qquad V_0 = \frac{Q'}{2\pi\epsilon}\ln\frac{b}{a}$$

$$V_0 = \int_a^b \mathbf{E}\cdot \mathrm{d}\mathbf{l} = \int_a^b E\mathrm{d}r = \frac{Q'}{2\pi\epsilon_0}\int_a^b \frac{\mathrm{d}r}{r} = \frac{Q'}{2\pi\epsilon_0}\ln\frac{b}{a}.$$

$$C' = \frac{Q'}{V_0} = \frac{2\pi\epsilon}{\ln\frac{b}{a}} \qquad W_e' = \frac{1}{2}C'V_0^2 = \frac{\pi\epsilon}{\ln\frac{b}{a}}V_0^2$$

Boundary conditions in electrostatics

Electric field involves more than one materials

$$D_{n1}\,\Delta S - D_{n2}\,\Delta S = \rho_s\,\Delta S$$

Gauss' law: $\epsilon_0 \oint_S \mathbf{E}\cdot \mathrm{d}\mathbf{S} = Q_{\mathrm{total\ i}\ S}$

$$D_{n1} - D_{n2} = \rho_s$$

Conservative field : electrostatic electric field

$$\oint \mathbf{E}\cdot \mathbf{dl} = E_{t1}\,\Delta l - E_{t2}\,\Delta l = 0$$

$$E_{t1} = E_{t2}$$

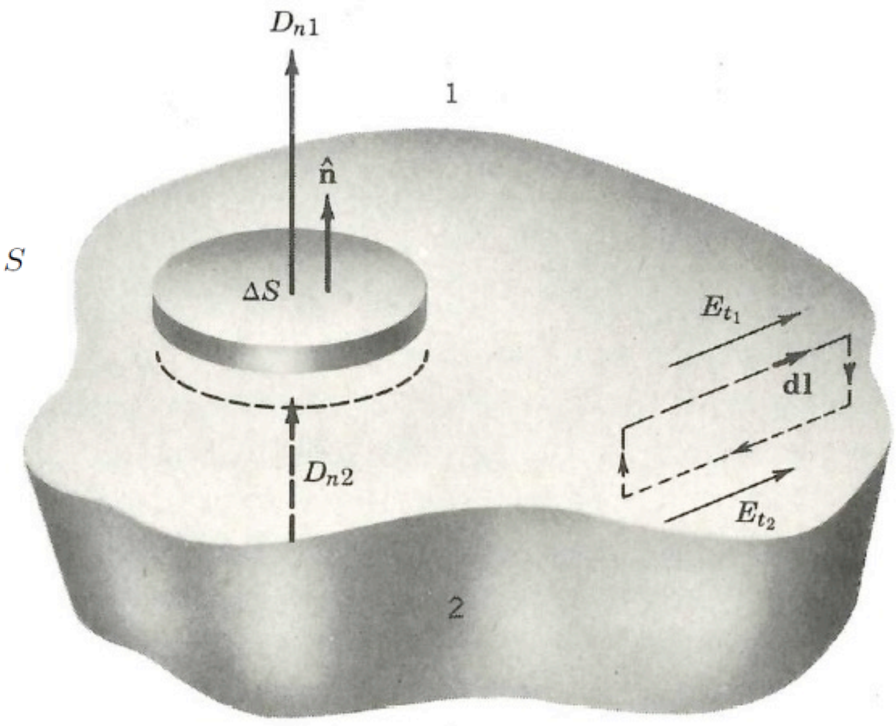


Fig. 1.14*a* Boundary between two different media.

Electric field change direction across two different dielectric materials

Considering, no charge on the surface between two dielectric

$$D_{n1} - D_{n2} = \rho_s = 0$$

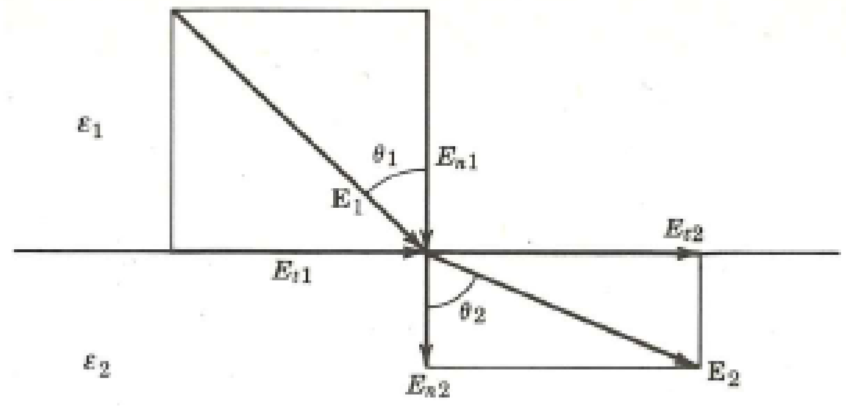
$$\varepsilon_1 E_{n1} = \varepsilon_2 E_{n2}$$

$$E_{n2} = \frac{\varepsilon_1 E_{n1}}{\varepsilon_2}$$

$$E_{t1} = E_{t2}$$

$$\tan\theta_1 = \frac{E_{t1}}{E_{n1}}$$

$$E_{t2} = E_{t1} = E_{n1}\tan\theta_1$$



$$\theta_2 = \tan^{-1}\frac{E_{t2}}{E_{n2}}$$

$$\theta_2 = \tan^{-1}\left(\frac{\varepsilon_2}{\varepsilon_1}\tan\theta_1\right)$$

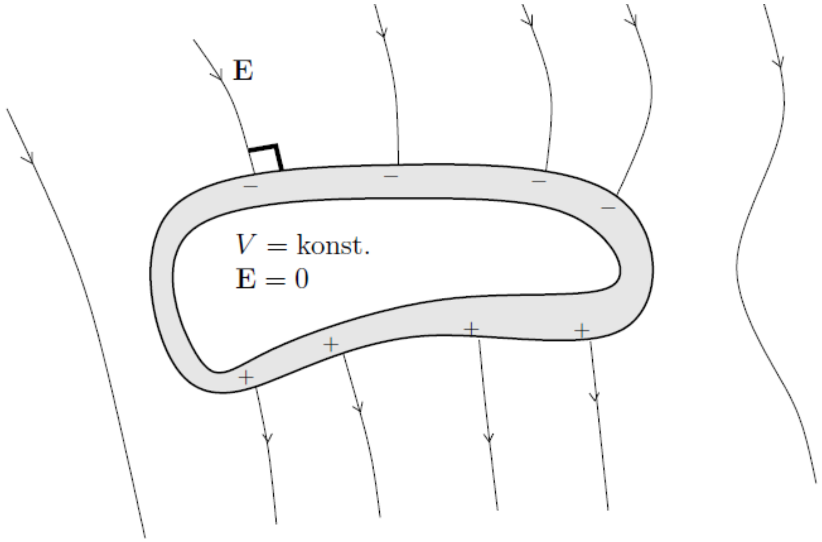
Perfect conductor in electric field

1) **E** = 0 inside the conductor. Both $E_t = E_n$ = 0.
2) $\rho_{\text{in}} = 0$, no charge inside the conductor.
3) $\rho_s \neq 0$, there is surface charge.
4) The electric field outside the boundary of the perfect conductor is

$$E_n = \frac{\rho_s}{\varepsilon}, \; E_t = 0,$$

5) The conductor is an equipotential surface.

$$V_{AB} = -\int_A^B \boldsymbol{E} d\mathbf{l} = 0.$$



$$D_{n1} - D_{n2} = \rho_s$$

$$E_{t1} = E_{t2}$$

Faraday cage:

https://www.youtube.com/watch?v=YQDqVUNdWiY&ab_channel=TritoxHD

Material conductivity

Good conductive material:

Silver: $6.2\times10^{7}S/m$

Copper: $5.8\times10^{7}S/m$

Gold: $4.1\times10^{7}S/m$

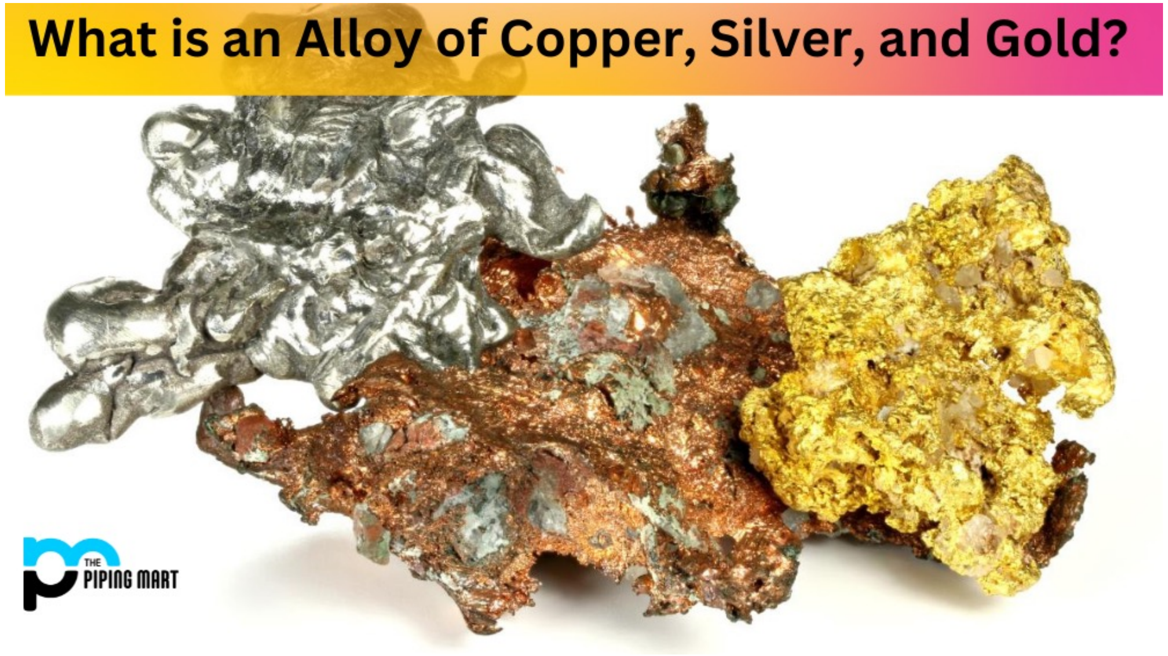
Aluminium: $3.5\times10^{7}S/m$

Non-conductive material:

Glass $\times10^{-12}S/m$

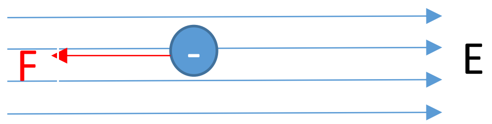
Rubber $\times10^{-13}S/m$

Air $\times10^{-14}S/m$

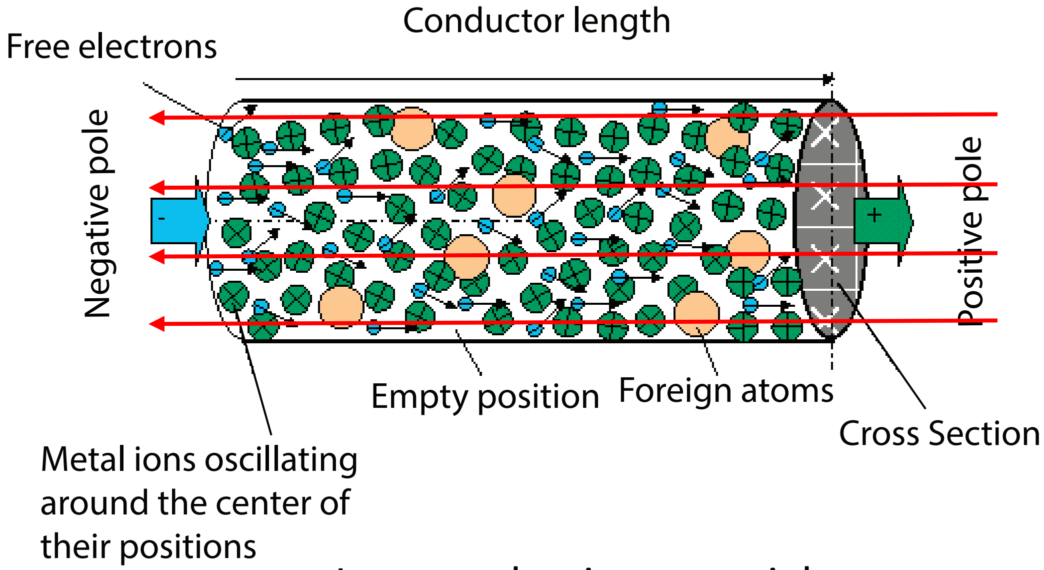


Material conductivity

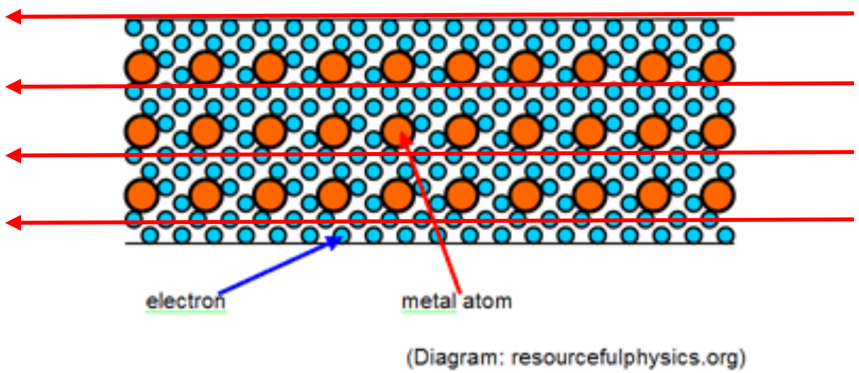
Once there are free charges in an electric field, the charges can move along the electric field direction.



That how many charges can move is dependent of material and electric field
When electrons move, they collide atoms and lost energy.
The capability is represented by σ, conductivity.
J is called current density. The charges go through the material per unit area per second.



Low conductive material



High conductive material

Ohm’s law

An electric current is a flow of electric charge

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t}$$

Current density **J** is the current per unit area

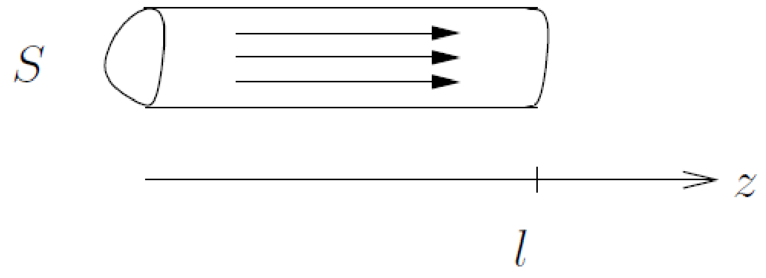
$$I = \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S}.$$

Ohm’s law states that the current through a conductor between two points is proportional to the voltage across the two points

$$R = \frac{V}{I} \qquad \mathbf{J} = \sigma \mathbf{E}$$

Example

A conductor with constant conductivity σ, and cross − sectional area S , the length is l and the constant current is I



1) Calculating the resistance of the conductor and deriving $R = \frac{V}{I}$.
2) Calculating the power done by the current.

Area *S*, and I is also constant. $J = I/S$

Constant σ , the electric field is constant: $E = J/\sigma$

$$V = \int_0^l E\mathrm{d}z = \int_0^l (J/\sigma)\mathrm{d}z = Jl/\sigma$$

$$R = \frac{l}{S\sigma} = \frac{Jl}{\sigma}\frac{1}{JS} = \frac{V}{I}$$

Coulomb's law $\boldsymbol{\vec{F} = \vec{E} \cdot q}$

$$V = -\int \boldsymbol{E.dl}$$

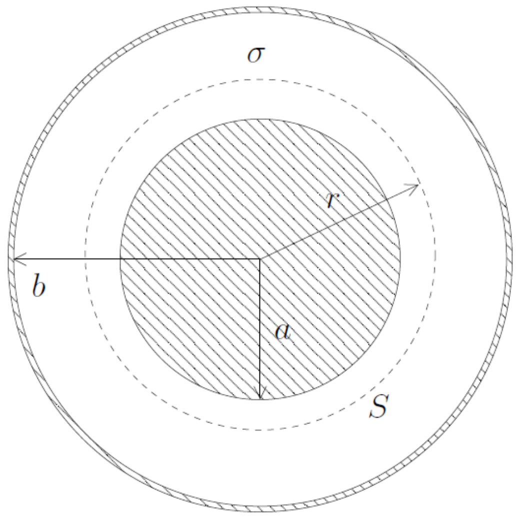
*Work is application of force **F** to move an object for a distance **L**.*

$$W_e = \int_0^l Fdl = \int_0^l q\boldsymbol{E}d\boldsymbol{l} = qV \Rightarrow W_e = \int VIdt$$

q= $\int Idt$

$$P = VI = I^2R$$

Example



A solid conductive ball with a radius *a* is put into a hollow conductive ball with inner radius *b*, between the two is a material with conductivity *σ*.

a) What is the resistance between the two balls?
b) If the solid ball is buried deeply into earth, what is the earth resistance?

$$a = 0.5\ \mathrm{m} \qquad \sigma = 10^{-2}\ \mathrm{m}^{-1}\Omega^{-1}$$

$$\mathbf{J} = \frac{I}{4\pi r^2}\hat{\mathbf{r}}, \quad \text{for } a < r < b.$$

$$\mathbf{E} = \mathbf{J}/\sigma = I\hat{\mathbf{r}}/(4\pi\sigma r^2)$$

$$V = -\int_a^b E\mathrm{d}r = -\int_a^b \frac{I}{4\pi\sigma r^2}\mathrm{d}r = \frac{I}{4\pi\sigma}\left(-\frac{1}{b} + \frac{1}{a}\right)$$

$$R = \frac{V}{I} = \frac{1}{4\pi\sigma}\left(\frac{1}{a} - \frac{1}{b}\right)$$

Example

Now if a half solid ball is buried in earth as shown in picture, recalculate the resistance?

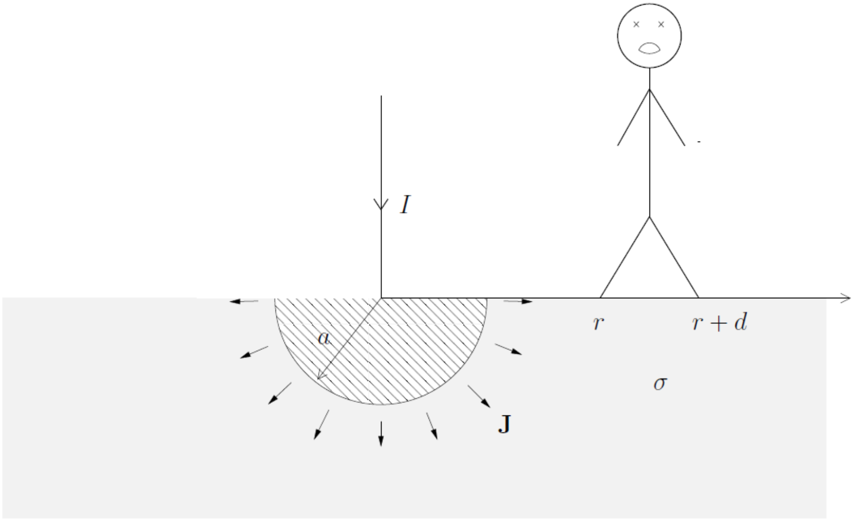
If the current is 1000A $\sigma = 10^{-2}\,\mathrm{m}^{-1}\Omega^{-1}$, $r = 1\,\mathrm{m}$ og $d = 0.75\,\mathrm{m}$

What is the voltage between the two legs of the people

$$\mathbf{J} = \frac{I}{2\pi r^2}\hat{\mathbf{r}}, \quad \mathbf{E} = \frac{I}{2\pi\sigma r^2}\hat{\mathbf{r}}, \quad \text{for } r > a.$$

$$R = \frac{1}{2\pi\sigma a}$$

$$V = -\int_r^{r+d} E(r)\mathrm{d}r = -\int_r^{r+d} \frac{I}{2\pi\sigma r^2}\mathrm{d}r = \frac{I}{2\pi\sigma}\left(\frac{1}{r} - \frac{1}{r+d}\right)$$



Kirchhoff’s law

Current conservation:

Kirchhoff’s law: at any node of an electric circuit, the sum of currents flowing into that node is equal to the sum of currents flowing out of that node.

$$\oint_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} = 0.$$