Lecture 3: Stationary electric field

1. Capacitor and energy in electric field

2. Boundary conditions for electric field

3. Perfect conductor

4. Conductivity, current, and Ohm's law

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Electric field and electric displacement field

Electric field: $\vec{E} = \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$ $\vec{E} = \vec{F}/q$

Electric displacement field: $\mathbf{D} = \varepsilon_0\mathbf{E}$ $\vec{D} = \dfrac{Q}{4\pi r^2}\hat{r}$

Electric field and potential

$$\mathbf{E} = -\nabla V, \ \mathrm{V/m}$$

The potential difference between two points A and B:

$$V_{AB} = -\int_{r_A}^{r_B} \mathbf{E}\cdot \mathrm{d}\mathbf{l} = -\int_{r_A}^{r_B}\dfrac{Q}{4\pi\varepsilon_0 r_i^2}\,dr_i = \dfrac{Q}{4\pi\varepsilon_0}\left(\dfrac{1}{r_B} - \dfrac{1}{r_A}\right)$$

$$V = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r}$$

Poisson's equation

Inserting $\mathbf{E} = -\nabla V$ into Maxwell's equation $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ gives

$$-\nabla \cdot (\nabla V) = \frac{\rho}{\epsilon_0},$$

which gives

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}.$$

Gauss's law

Electric flux flowing out of a closed surface $=$ Enclosed total charges

divided by the permittivity $\Longrightarrow$ $\oint \mathbf{E} d\mathbf{s} = Q/\varepsilon$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon}; \qquad \nabla \cdot \mathbf{D} = \rho$$

Electric polarization and material permittivity

The influence of electric polarization: $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \varepsilon_0 \chi_e \mathbf{E} = (1 + \chi_e)\varepsilon_0 \mathbf{E} = \varepsilon_r \varepsilon_0 \mathbf{E} = \varepsilon \mathbf{E}$$

- χ_e is *electric susceptibility*
- $1 + \chi_e = \varepsilon_r$, *relative permittivity*
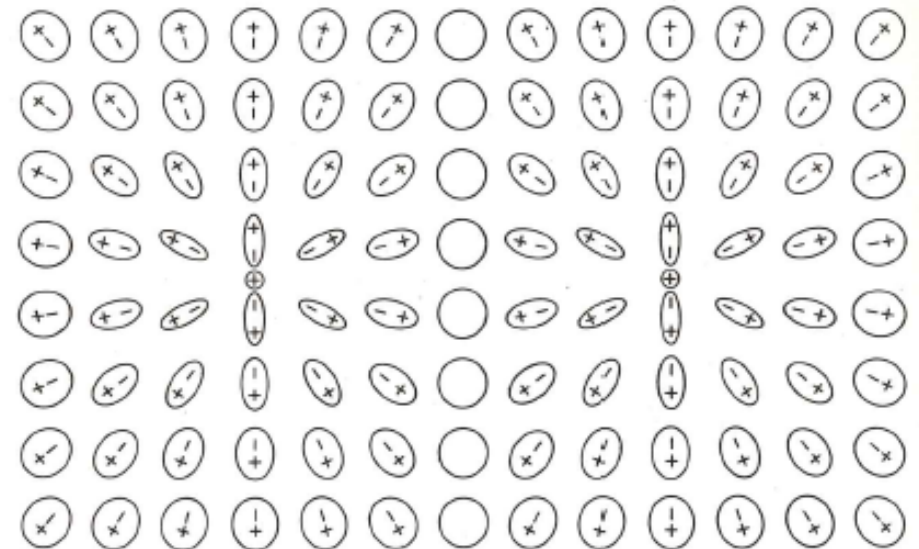- $\varepsilon = \varepsilon_r \varepsilon_0$, electric permittivity

Fig. 1.3c Polarization of the atoms of a dielectric by a pair of equal positive charges.
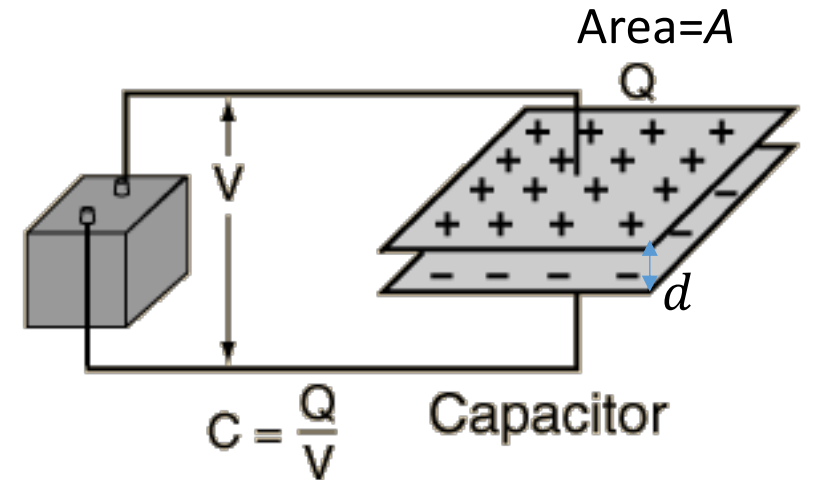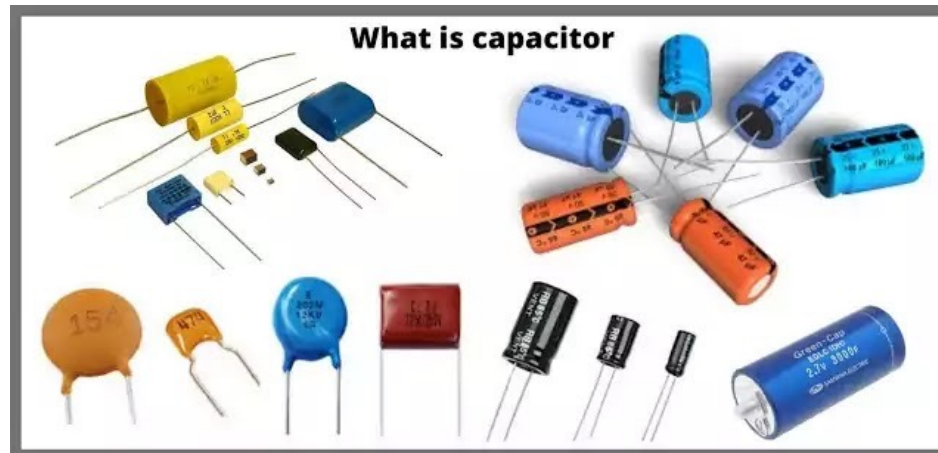
Capacitor

A two-terminal electrical device that can store energy in the form of electric charges.

It consists of two electrical conductors that are separated by a distance.

The space between the conductors may be filled by vacuum or dielectric.

Capacitance is the ability of an object to store electrical charge.

$$C = \frac{Q}{V} = \frac{\varepsilon A}{d}$$

Area=A

$$C = \frac{Q}{V}$$ Capacitor

Electric energy in capacitor

How much energy capacitor can store?

Voltage represents energy per unit charge

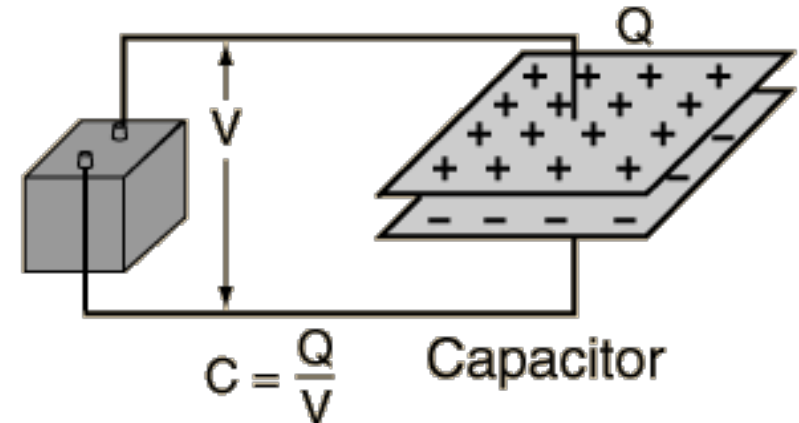$$W_e = \int_a^b F dl = \int_a^b qE dl = qV$$

The work to move a charge element dq from the negative plate to the positive plate is equal to V dq

$$dW_e = V dq \qquad\qquad Q = CV$$

$$W_e = \int_0^Q V dq = \int_0^Q \frac{q}{C} dq = \frac{Q^2}{2C} = \frac{1}{2} CV^2$$

$$\boxed{W_e = \frac{1}{2} CV^2}$$

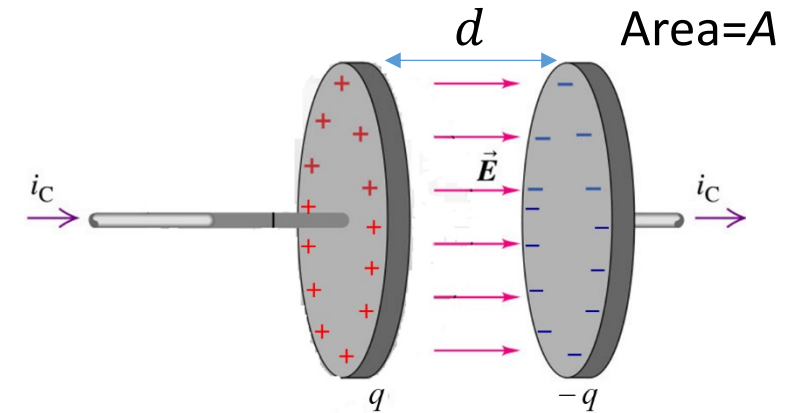$$C = \frac{Q}{V} \qquad \text{Capacitor}$$

Energy density in electric field

$$C = \frac{\varepsilon A}{d}$$

Volume=Ad

$$W_e = \frac{1}{2}CV^2 = \frac{1}{2}\frac{\varepsilon A}{d}V^2 = \frac{1}{2}\varepsilon Ad\frac{V^2}{d^2} = \frac{1}{2}\varepsilon V_{vol}E^2 = \frac{1}{2}V_{vol}DE$$

$$\frac{V^2}{d^2} = E^2$$

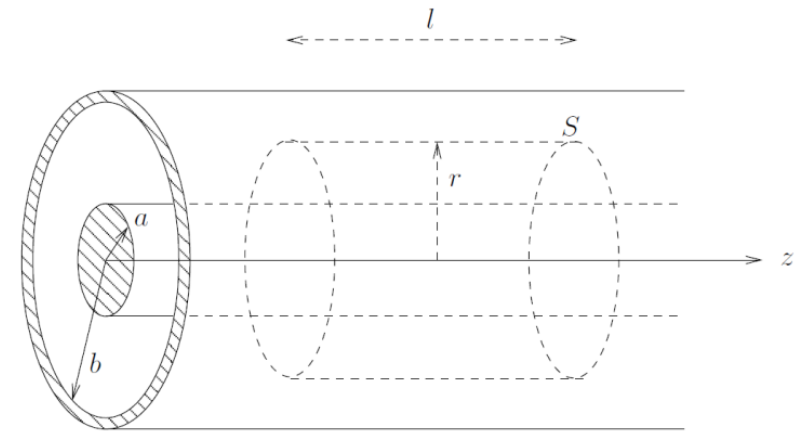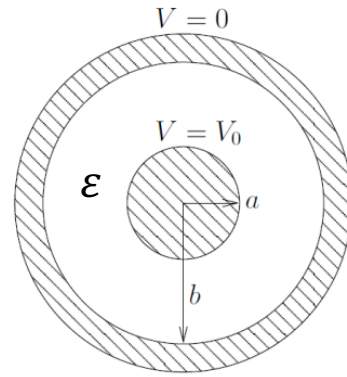$$\mathbf{D} = \varepsilon\mathbf{E}$$

Electric energy density

$$\eta_e = \frac{W_e}{V_{vol}} = \frac{1}{2}ED = \frac{1}{2}\varepsilon E^2$$

Example

Assuming: infinite long cable and the permittivity of
the dielectric in between is ε
1) calculate the capacitance per unit length
2) calculate the electric energy stored in unit length

$$C' = \frac{Q'}{V_0} \qquad V_0 = \frac{Q'}{2\pi\epsilon} \ln\frac{b}{a}$$

$$V_0 = \int_a^b \mathbf{E} \cdot d\mathbf{l} = \int_a^b E dr = \frac{Q'}{2\pi\epsilon_0} \int_a^b \frac{dr}{r} = \frac{Q'}{2\pi\epsilon_0} \ln\frac{b}{a}.$$

$$C' = \frac{Q'}{V_0} = \frac{2\pi\epsilon}{\ln\frac{b}{a}}. \qquad W_e' = \frac{1}{2} C' V_0^2 = \frac{\pi\epsilon}{\ln\frac{b}{a}} V_0^2.$$

Boundary conditions in electrostatics

Electric field involves more than one materials

Normal component

$$D_{n1}\Delta S - D_{n2}\Delta S = \rho_s\,\Delta S$$

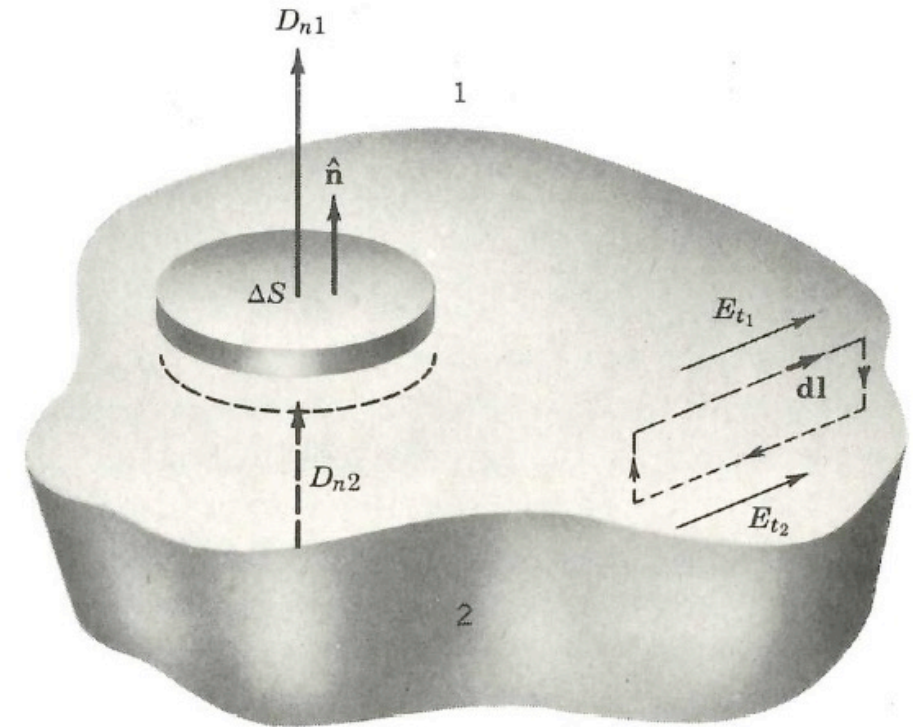$$\oint \mathbf{D} \cdot d\mathbf{S} = Q_S$$

$$D_{n1} - D_{n2} = \rho_s$$

Conservative field: electrostatic electric field

Tangential component

$$E_{t1}\Delta l - E_{t2}\Delta l = 0$$

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0$$

$$E_{t1} = E_{t2}$$

Electric field change direction across two different dielectric materials

Considering, no charge on the surface between two dielectric

$$D_{n1} - D_{n2} = \rho_s = 0$$

$$\varepsilon_1 E_{n1} = \varepsilon_2 E_{n2}$$

$$E_{n2} = \frac{\varepsilon_1 E_{n1}}{\varepsilon_2}$$
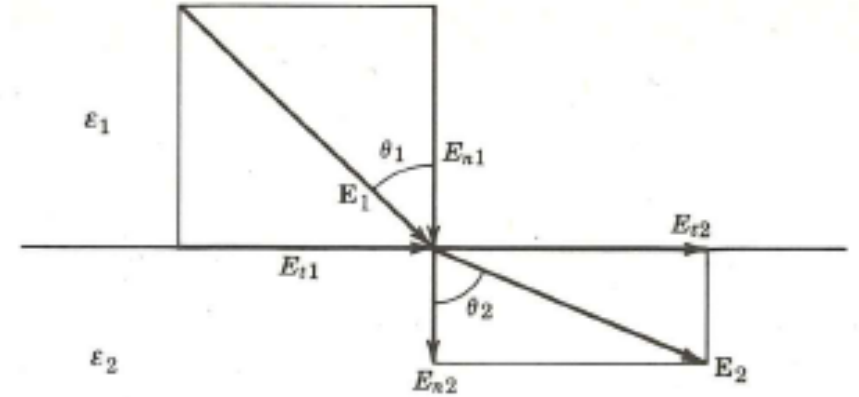

$$E_{t1} = E_{t2}$$

$$\tan\theta_1 = \frac{E_{t1}}{E_{n1}}$$

$$E_{t2} = E_{t1} = E_{n1}\tan\theta_1$$

$$\theta_2 = \tan^{-1}\frac{E_{t2}}{E_{n2}}$$

$$\theta_2 = \tan^{-1}\left(\frac{\varepsilon_2}{\varepsilon_1}\tan\theta_1\right)$$

Perfect conductor in electric field

1) $\mathbf{E} = 0$ inside the conductor.
2) $\rho_{in} = 0$, no charge inside the conductor.
3) $\rho_s \neq 0$, there is surface charge.
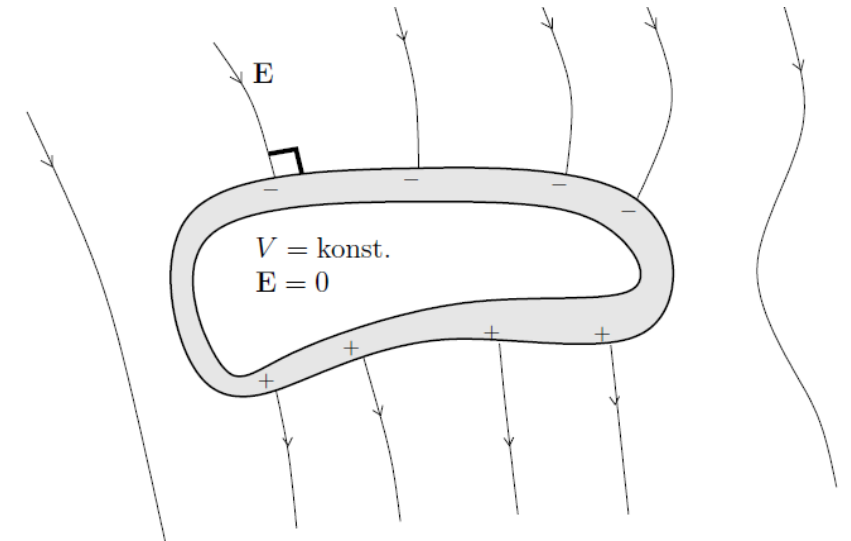4) The electric field outside the boundary of the perfect conductor is

$$E_n = \frac{\rho_s}{\varepsilon}, \quad E_t = 0,$$

5) The conductor is an equipotential surface.

$$V_{AB} = -\int_A^B \mathbf{E}d\mathbf{l} = 0.$$

$$D_{n1} - D_{n2} = \rho_s$$

$$E_{t1} = E_{t2}$$

$\mathbf{E}$

$V = \text{konst.}$
$\mathbf{E} = 0$

Faraday cage:

https://www.youtube.com/watch?v=QU0fLnucE6A&ab_channel=MITxVideos

Material conductivity

Good conductive material:

$Silver$: $6.2 \times 10^7 S/m$

$Copper$: $5.8 \times 10^7 S/m$

$Gold$: $4.1 \times 10^7 S/m$
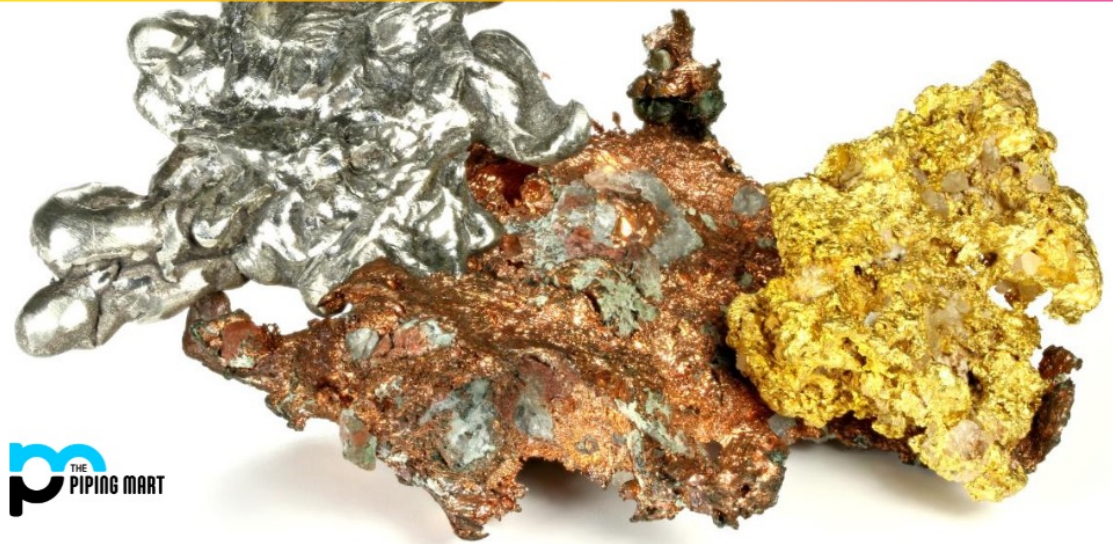
$Aluminium$: $3.5 \times 10^7 S/m$

Non-conductive material:

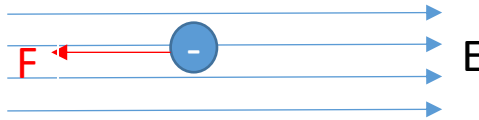$Glass$ $\times 10^{-12} S/m$

$Rubber$ $\times 10^{-13} S/m$

Air $\times 10^{-14} S/m$

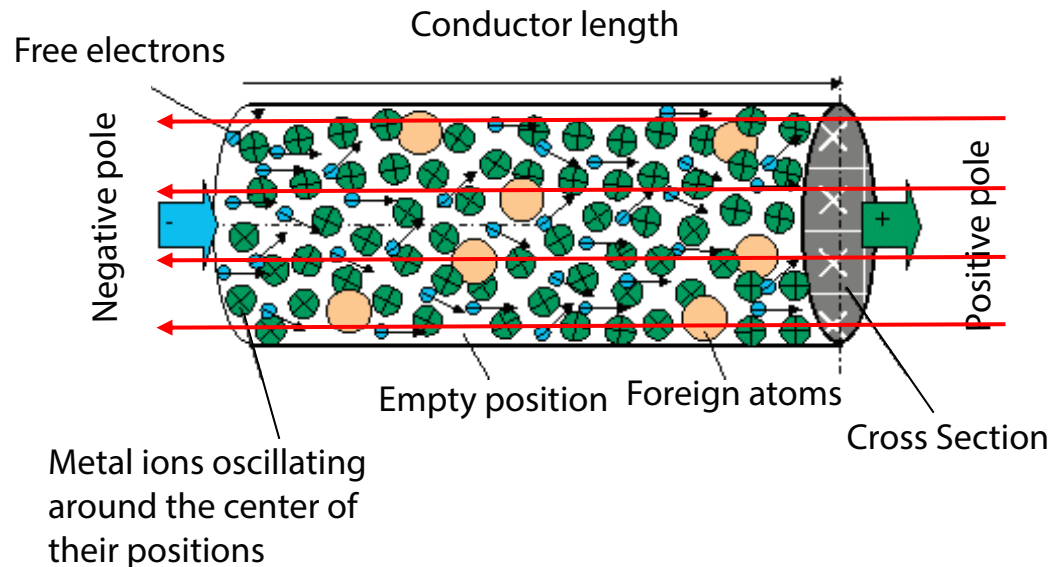What is an Alloy of Copper, Silver, and Gold?

Material conductivity

Once there are free charges in an electric field, the charges can move along the electric field.
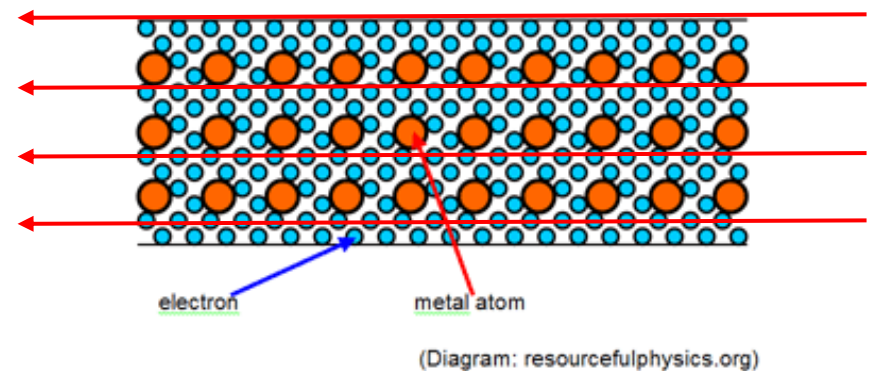

The capability of allowing current to flow is defined by conductivity: $\sigma = 1/\rho$.
That how many charges can move is dependent of material and electric field.

When electrons move, they collide atoms and lost energy. ➡️ Resistance: $R = \rho \dfrac{l}{S}$

Free electrons

Conductor length

Negative pole

Positive pole

Empty position   Foreign atoms

Cross Section

Metal ions oscillating around the center of their positions

Low conductive material

electron   metal atom

(Diagram: resourcefulphysics.org)

High conductive material

Ohm's law

An electric current is a flow of electric charge

$$I = \frac{dQ}{dt}$$

Current density **J** is the current per unit area

$$I = \int_S \mathbf{J} \cdot d\mathbf{S}.$$

Ohm's law states that the current through a conductor between two points is proportional to the voltage across the two points, and is inversely proportional to the conductor's resistance
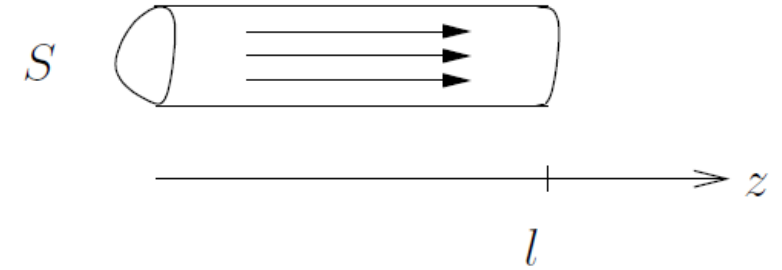
$$I = \frac{V}{R} \qquad \mathbf{J} = \frac{\mathbf{E}}{\rho} \qquad R = \frac{V}{I} = \rho\frac{l}{S}$$

Example

A conductor with constant conductivity σ, and cross $-$ sectional area S , the length is l and the constant current is I

S

z

l

1) Calculating the resistance of the conductor and deriving $R = \frac{V}{I}$.

2) Calculating the power done by the current.

Coulomb's law $\vec{F} = \vec{E} \cdot q$

Area S, and I is also constant. $J = I/S$

$$V = -\int E.\,dl$$

Constant σ , the electric field is constant: $E = J/\sigma$

Work is application of force **F** *to move an object for a distance* **L.**

$$V = \int_0^l E\mathrm{d}z = \int_0^l (J/\sigma)\mathrm{d}z = Jl/\sigma$$

$$W_e = \int_0^l Fdl = \int_0^l qEdl = qV \implies W_e = \int VIdt$$

$$R = \frac{l}{S\sigma} = \boxed{\frac{Jl}{\sigma}} \boxed{\frac{1}{JS}} = \frac{V}{I}$$

q$= \int Idt$

$P = VI = I^2 R$

Example

A solid conductive ball with a radius a is put into a hollow conductive ball with inner radius b, between the two is a material with conductivity σ.

a) What is the resistance between the two balls?
b) If the solid ball is buried deeply into earth, what is the earth resistance?
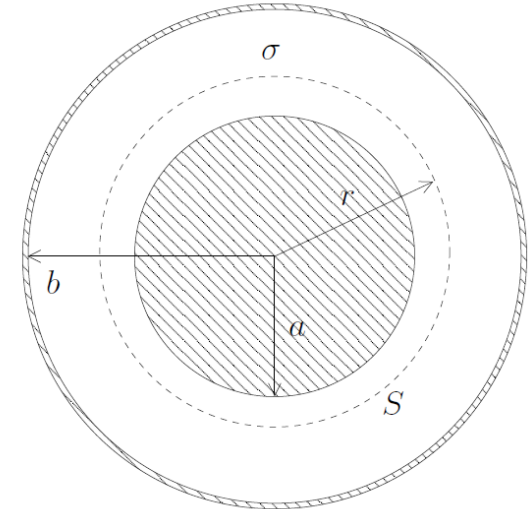
$$a = 0.5 \text{ m} \qquad \sigma = 10^{-2} \text{ m}^{-1}\Omega^{-1}$$

$$\mathbf{J} = \frac{I}{4\pi r^2}\hat{\mathbf{r}}, \quad \text{for } a < r < b.$$

$$\mathbf{E} = \mathbf{J}/\sigma = I\hat{\mathbf{r}}/(4\pi\sigma r^2)$$

$$V = \int_a^b E dr = \int_a^b \frac{I}{4\pi\sigma r^2} dr = \frac{I}{4\pi\sigma}\left(-\frac{1}{b} + \frac{1}{a}\right)$$

$$R = \frac{V}{I} = \frac{1}{4\pi\sigma}\left(\frac{1}{a} - \frac{1}{b}\right)$$

Example

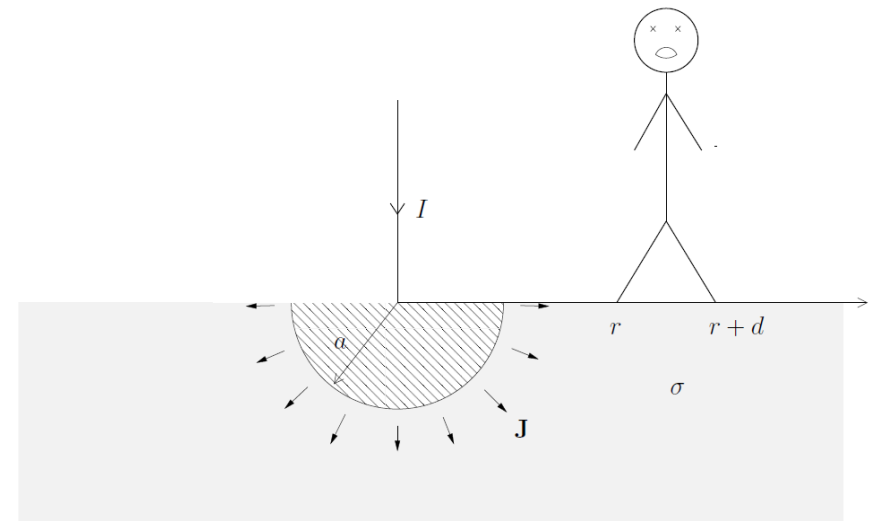Now if a half solid ball is buried in earth as shown in picture, recalculate the resistance?

If the current is 1000A $\quad \sigma = 10^{-2}\,\mathrm{m}^{-1}\Omega^{-1}$, $r = 1\,\mathrm{m}$ og $d = 0.75\,\mathrm{m}$

What is the voltage between the two legs of the people?

$$\mathbf{J} = \frac{I}{2\pi r^2}\hat{\mathbf{r}}, \quad \mathbf{E} = \frac{I}{2\pi\sigma r^2}\hat{\mathbf{r}}, \quad \text{for } r > a.$$

$$R = \frac{1}{2\pi\sigma a}$$

$$V = \int_r^{r+d} E(r)\mathrm{d}r = \int_r^{r+d} \frac{I}{2\pi\sigma r^2}\mathrm{d}r = \frac{I}{2\pi\sigma}\left(\frac{1}{r} - \frac{1}{r+d}\right)$$

Kirchhoff's law

It shows current conservation:

At any node of an electric circuit, **the sum of currents flowing into** that node is **equal to the sum of currents flowing out** of that node.

$$\oint_{S} \mathbf{J} \cdot \mathrm{d}\mathbf{S} = 0.$$