Lecture 2: Stationary electric field (Electrostatics)

- Electric field, electric displacement field
- Electric potential
- Conservative vector field
- Gauss’s law

Divergence and Stokes' theorem

Gradient: fastest rate of increase in spatial

$$\nabla f = \frac{\partial f}{\partial x}\hat{\boldsymbol{x}}+\frac{\partial f}{\partial y}\hat{\boldsymbol{y}}+\frac{\partial f}{\partial z}\hat{\boldsymbol{z}}$$

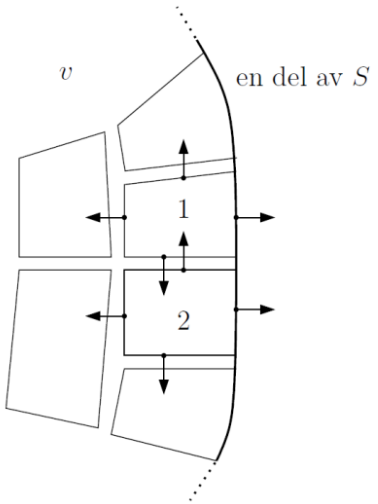
Divergence: Flux out of a point

$$\nabla \cdot \boldsymbol{E} = \frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}$$

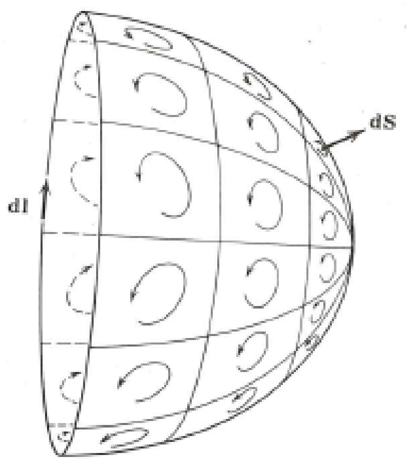
How much does a field circulate around a point

$$\nabla\times\boldsymbol{A} = \left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\hat{\boldsymbol{z}}$$

$$\boxed{\oint_S \mathbf{A}\cdot d\mathbf{S} = \int_v \nabla\cdot\mathbf{A}\,dv}$$

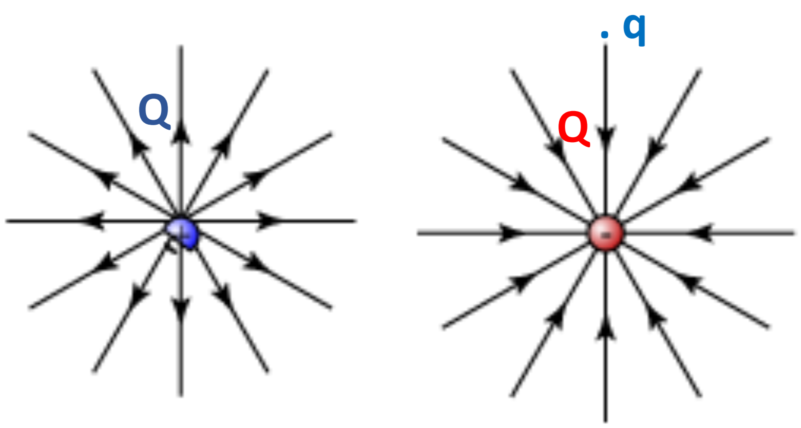


$$\boxed{\oint_C \mathbf{A}\cdot d\mathbf{l} = \int_S \nabla\times\mathbf{A}\cdot d\mathbf{S}.}$$



Electric field and electric displacement field

A stationary distribution of charges produces an electric field **E** in vacuum



Field pattern of
a pointed electrode

Vector: $\vec{F} = \frac{Qq}{4\pi\varepsilon_0 r^2}\hat{r}$ $\quad$ ***Coulomb's law***

Vector: $\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$

$\vec{E} = \vec{F}/q$

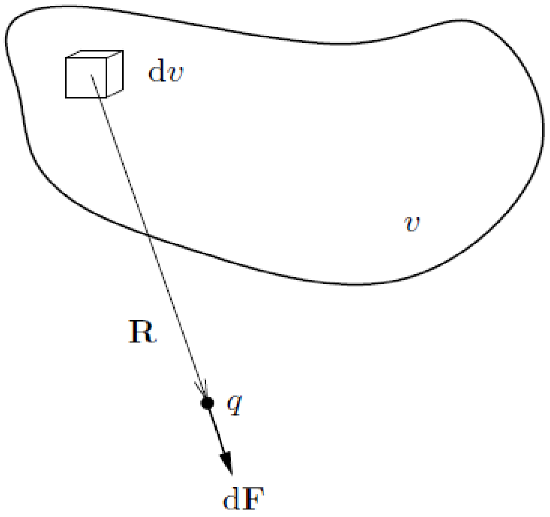
ε_0 is vacuum permittivity (physical property) $\qquad$ ε_0 = 8.85e^{-12} F/m

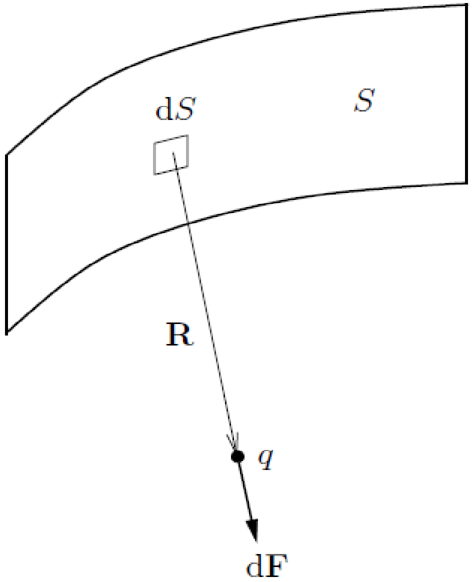
Electric displacement field: $\mathbf{D} = \varepsilon_0 \mathbf{E}$

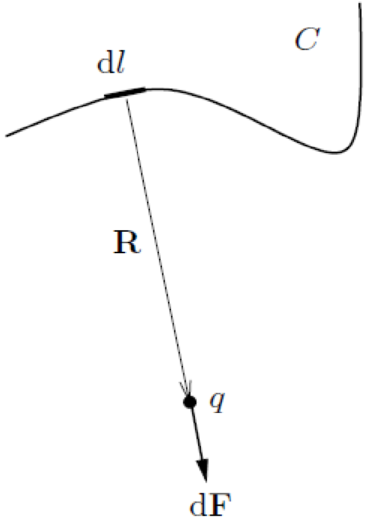
Vector: $\vec{D} = \frac{Q}{4\pi r^2}\hat{r}$

Electric displacement field: the equation is material independent.

Superposition: Vector **E**







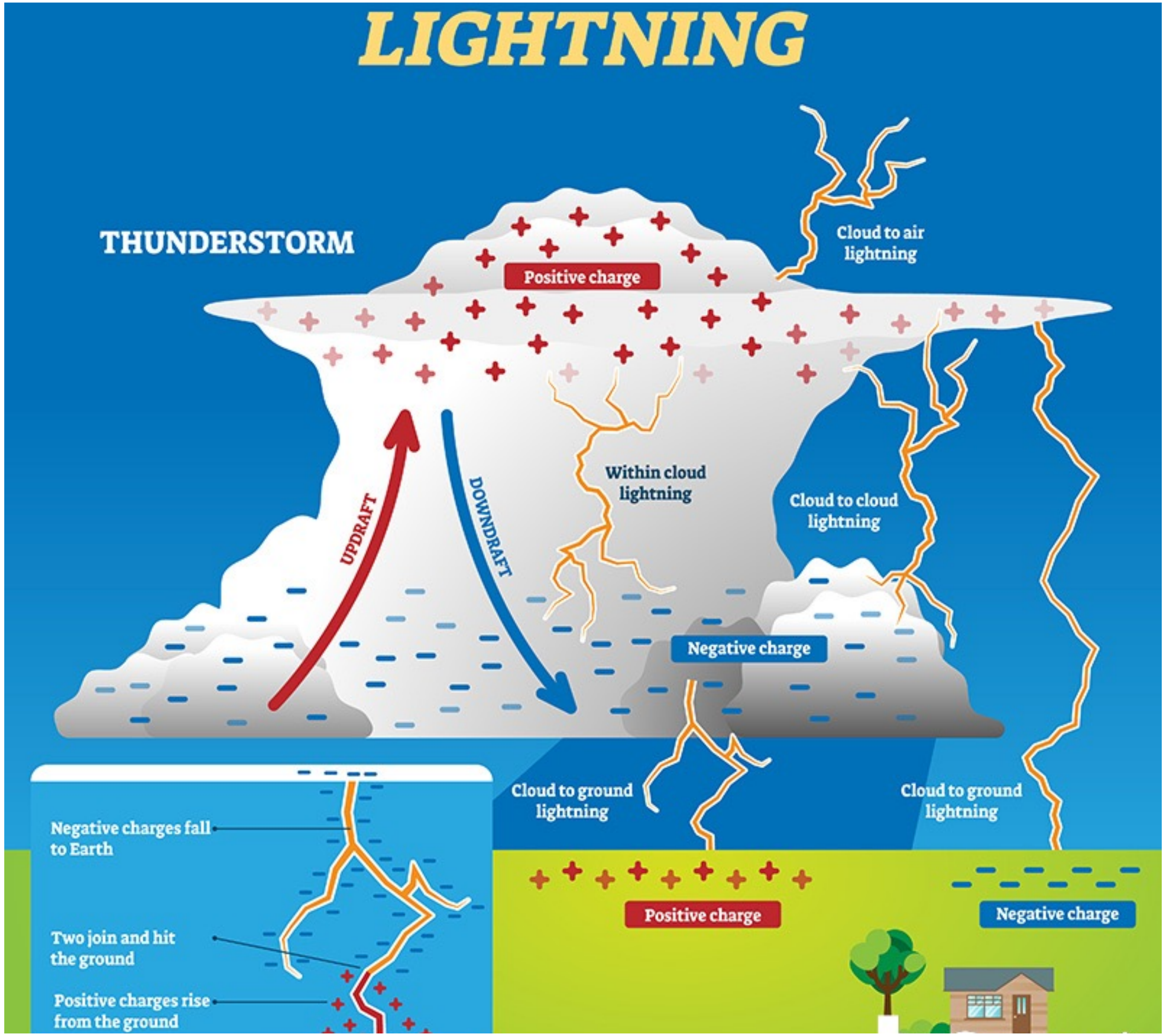
$$\boxed{\mathbf{E} = \int_v \frac{\hat{\mathbf{R}}\rho \, \mathrm{d}v}{4\pi\epsilon_0 R^2}.}$$

$$\boxed{\mathbf{E} = \int_S \frac{\hat{\mathbf{R}}\rho_{\mathrm{s}} \, \mathrm{d}S}{4\pi\epsilon_0 R^2}.}$$

$$\boxed{\mathbf{E} = \int_C \frac{\hat{\mathbf{R}}Q' \, \mathrm{d}l}{4\pi\epsilon_0 R^2}.}$$

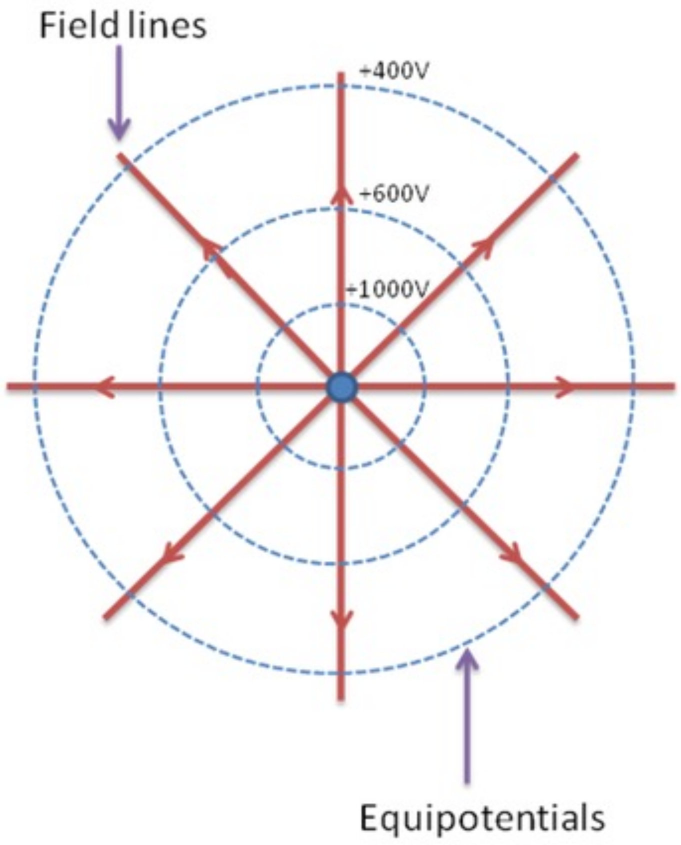
Electric potential

Work needed per unit of charge

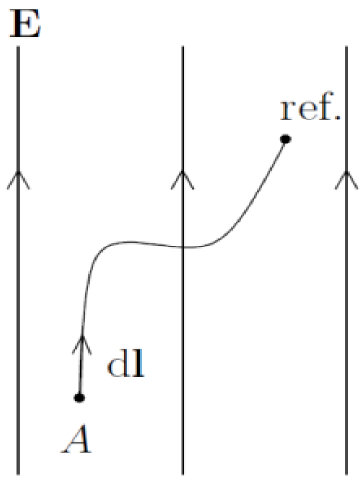


Electric potential

Electric potential definition $V = \frac{W}{q} = \frac{\int \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{l}}{q} = -\int \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{l}$



Vector: $\vec{\mathbf{E}} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$ Potential: $V = \frac{Q}{4\pi\varepsilon_0 r}$



$$V = \int_{\boldsymbol{r}}^{\infty} \boldsymbol{E} d\boldsymbol{l} = \int_{\boldsymbol{r}}^{\infty} \frac{Q}{4\pi\varepsilon_0 l^2} d\boldsymbol{l} = \frac{Q}{4\pi\varepsilon_0 r}$$

The potential difference between two points A and B:

$$V_{AB} = -\int_{\boldsymbol{rA}}^{\boldsymbol{rB}} \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{l} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\varepsilon_0 {r_i}^2} dr_i = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$

Electric field and potential

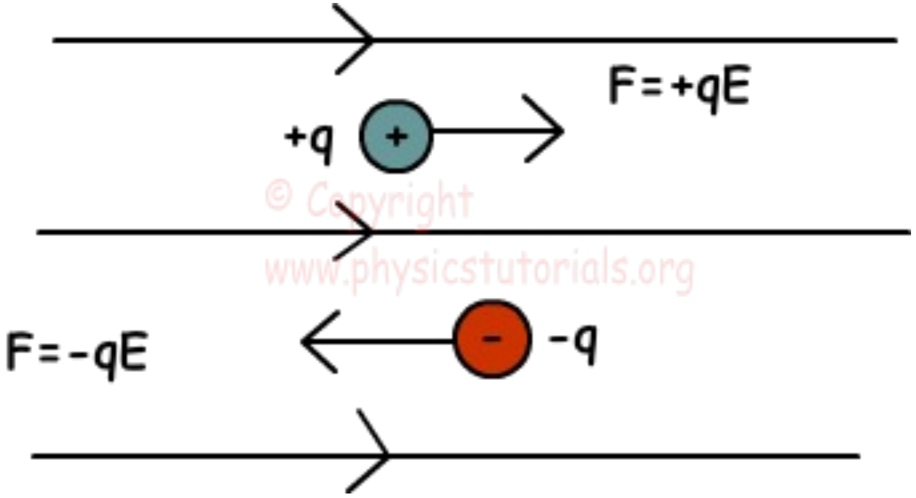
- Potential is a scalar
- $V = -\int \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l}$

$$\nabla f = \frac{\partial f}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}}$$

What happens to V when a positive and negative charge moves, respectively?

$$\boxed{\mathbf{E} = -\nabla V, \ \mathrm{V/m}}$$

$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}}$$

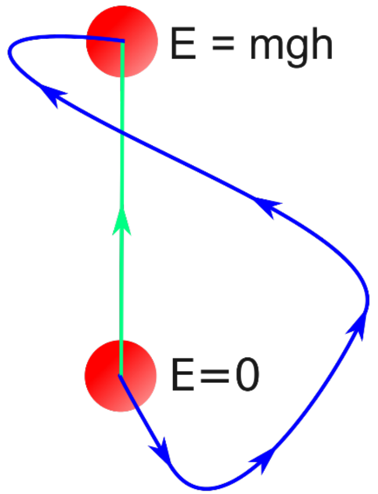


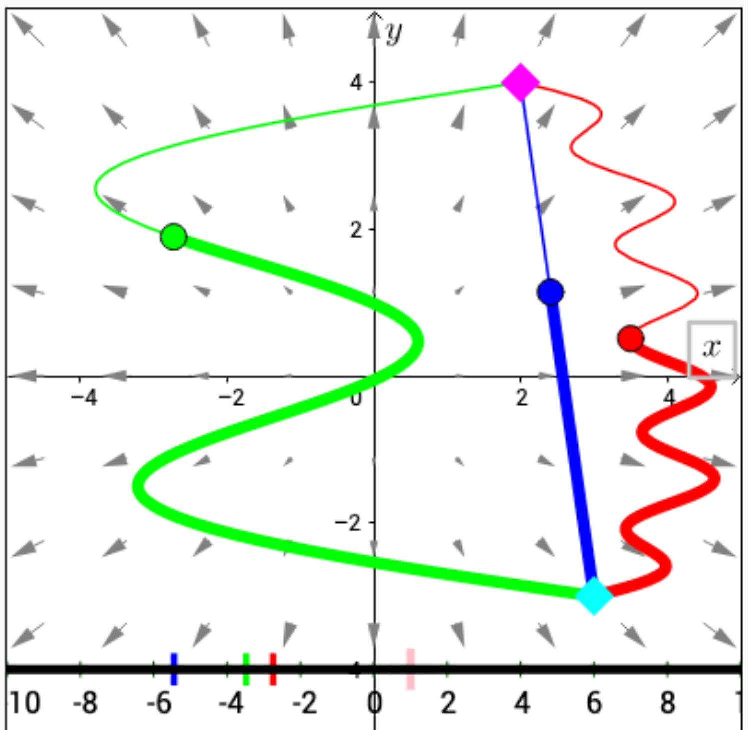
Conservative vector field

Stationary electric field is a conservative vector.

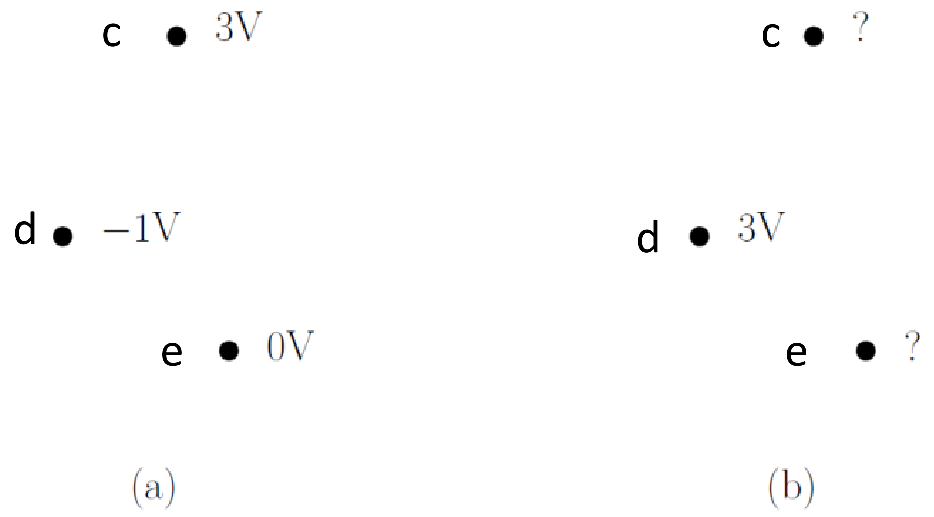
* Conservative vector fields have the property that the line integral is path independent.

* A conservative vector field is also irrotational. In three dimensions, it has vanishing curl. $\nabla \times \mathbf{E} = 0$





Example: find out the unknown potentials

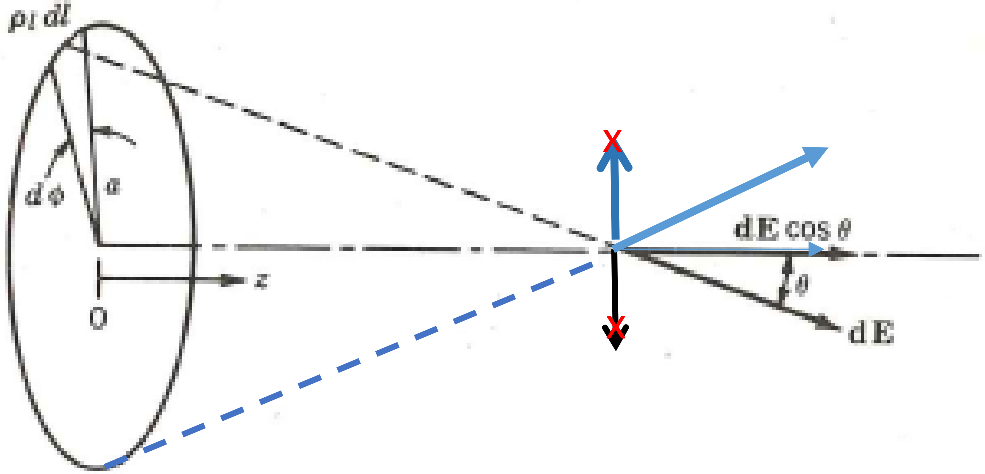


In a stationary electric field, the potential at each point is listed in (a). In the same stationary field and the potential at point d is 3V with another reference point.

What are the potential that are missed in b)?

Example: field of a ring of charge

Charge ring with radius *a* and line charge density: ρ_l
What is **E** at a point on the z axis?



$$\boxed{\vec{E} = \sum_{i=1}^{n} \frac{q_i}{4\pi\varepsilon_0 r_i^2}\hat{r}_i}$$

$$\boxed{\hat{r}_i = \hat{z}_i cos\theta}$$

$$\sum_{i=1}^{n} q_i = \oint \rho_l dl = \int_0^{2\pi a} \rho_l dl = \int_0^{2\pi} \rho_l a d\phi$$

$$\boxed{dl = a d\phi}$$

$$\mathbf{E} = \int_0^{2\pi} \rho_l a d\phi \frac{cos\theta}{4\pi\varepsilon_0 r^2}\hat{\mathbf{z}} = \int_0^{2\pi} \frac{\rho_l z a \;\; d\phi}{4\pi\varepsilon_0 (a^2+z^2)^{3/2}}\hat{\mathbf{z}} = \frac{\rho_l a z}{2\varepsilon_0 (a^2+z^2)^{3/2}}\hat{\mathbf{z}}$$

$$r^2 = a^2 + z^2$$

$$cos\theta = z \Big/ (a^2+z^2)^{\frac{1}{2}}$$

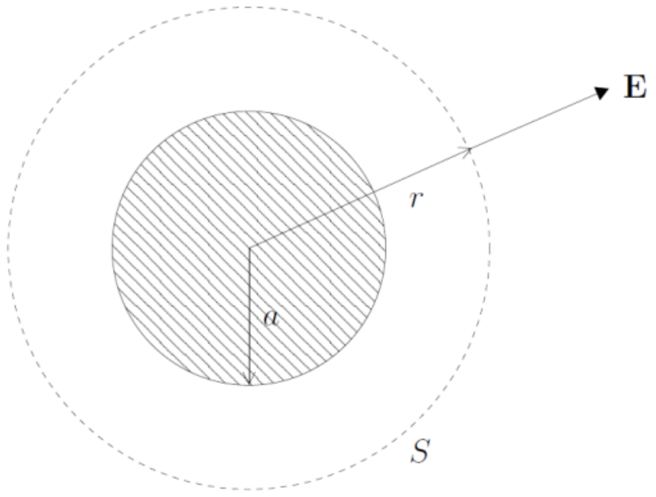
The resultant **E** at z-axis is also in the direction of z-axis.

Example: electric field around a charged ball

What is the electric field outside a charged ball with total charge Q? (r>>a)

$$\vec{\mathbf{E}} = E\hat{\boldsymbol{r}}$$

$$\vec{\mathbf{E}} = \frac{\boldsymbol{Q}}{\boldsymbol{4\pi\varepsilon_0 r^2}}\hat{\boldsymbol{r}}, \quad \text{(r>>a)}$$



Poisson's equation

Inserting $\mathbf{E} = -\nabla V$ into Maxwell's equation $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ gives

$$-\nabla \cdot (\nabla V) = \frac{\rho}{\epsilon_0},$$

which gives

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}.$$

This is called Poisson's equation. Here

$$\nabla^2 = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Potential for point, line, surface and space charge

a) A point charge:

$$V = \int_R^\infty \frac{Q}{4\pi\epsilon_0 r^2}\mathrm{d}r = \frac{Q}{4\pi\epsilon_0 R},$$

b) A line charge (line charge density Q'):

$$V = \int_C \frac{Q'}{4\pi\epsilon_0 R}\mathrm{d}l,$$

c) A surface charge (surface charge density ρ_s):

$$V = \int_S \frac{\rho_s}{4\pi\epsilon_0 R}\mathrm{d}S,$$

d) A space charge (charge density ρ):

$$V = \int_V \frac{\rho}{4\pi\epsilon_0 R}\mathrm{d}V.$$

Superposition principle

Gauss's law

Electric flux flowing out of a closed surface = Enclosed total charges divided by the permittivity

Vector: $\vec{\mathbf{E}} = \dfrac{\boldsymbol{Q}}{\boldsymbol{4\pi r^2 \varepsilon_0}}\hat{\boldsymbol{r}}$ in vacuum, $\vec{\boldsymbol{D}} = \dfrac{\boldsymbol{Q}}{\boldsymbol{4\pi r^2}}\hat{\boldsymbol{r}}$

$$\oint \mathbf{E} ds = Q/\varepsilon$$

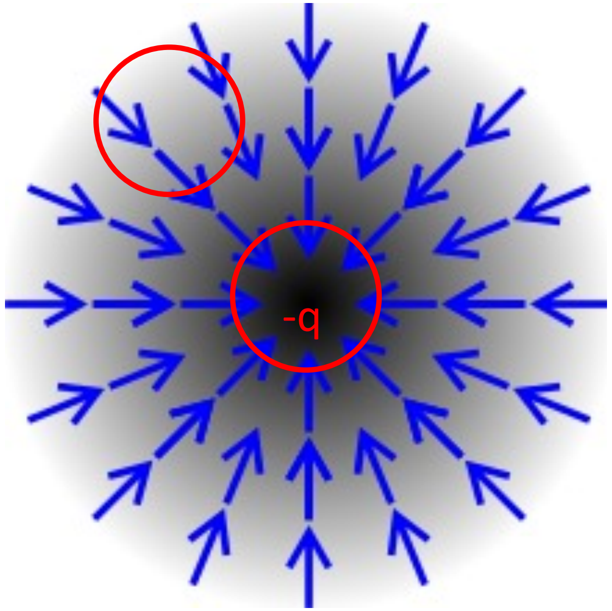
Electric displacement field, D = εE.

$$\boxed{\oint_S \mathbf{A}\cdot d\mathbf{S} = \int_v \nabla\cdot\mathbf{A}\,dv}$$

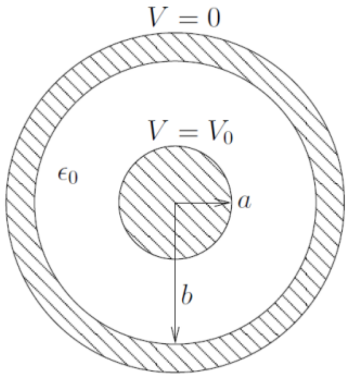
$$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon}$$

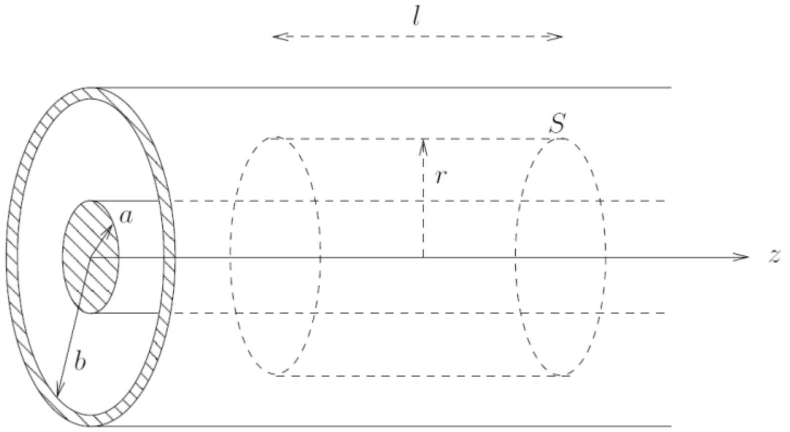
$$\nabla\cdot\mathbf{D} = \rho$$

$$\oint_S \mathbf{E}\cdot d\mathbf{S} = \sum_i \oint_{S_i} \mathbf{E}\cdot d\mathbf{S} = \sum_i \left(\frac{1}{\Delta V_i}\oint_{S_i} \mathbf{E}\cdot d\mathbf{S}_i\right)\Delta V_i \rightarrow \int \nabla\cdot\mathbf{E}\,dV.$$



Example: coaxial cable





Assuming an infinite long cable with an inner cable and an outer cable, information is shown in the left fig.
Calculate **E** between the out surface of inner cable and inner surface of the outer cable.

Assuming unit length and the surface charge density is ρ_s

$$\epsilon_0 \oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \epsilon_0 E 2\pi r l$$

$$\mathbf{E} = \begin{cases} \frac{Q'}{2\pi\epsilon_0 r}\hat{\mathbf{r}}, & \text{for } a < r < b \\ 0, & \text{ellers.} \end{cases}$$

$$V_0 = \int_a^b \mathbf{E} \cdot \mathrm{d}\mathbf{l} = \int_a^b E \mathrm{d}r = \frac{Q'}{2\pi\epsilon_0} \int_a^b \frac{\mathrm{d}r}{r} = \frac{Q'}{2\pi\epsilon_0} \ln \frac{b}{a}.$$

$$\mathbf{E} = \begin{cases} \frac{V_0}{r \ln \frac{b}{a}}\hat{\mathbf{r}}, & \text{for } a < r < b \\ 0, & \text{ellers.} \end{cases}$$

Electric field in dielectric

In metals charges **are free to move.**

Force in vacuum: $\boxed{\vec{F}_{tot} = \sum_{i=1}^{n} \frac{q q_{\mathrm{i}}}{4\pi\varepsilon_0 r_i^2} \hat{r}_i}$

Vector: $\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2} \hat{r}$

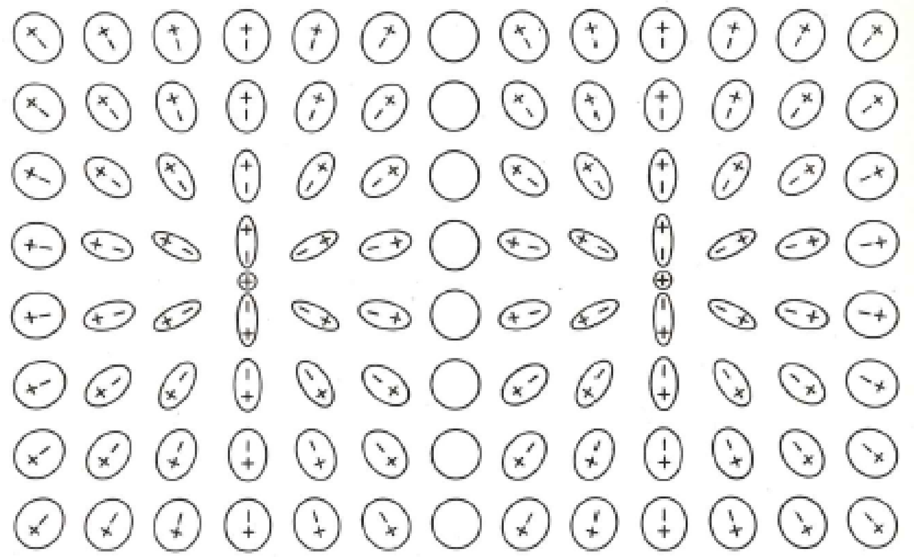


Fig. 1.3c Polarization of the atoms of a dielectric by a pair of equal positive charges.

In dielectrics **all the charges are attached to specific atoms and molecules**.

In dielectric media, the force between charges depends on the media/dielectric:

1) Once electric filed exists in dielectric, the atoms are polarized, similar for molecules.
2) Polarization induced by the electric field depends on material properties.
3) Polarization influences the force between charges.

Electric polarization and material permittivity

The influence of electric polarization: $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$

$$\boldsymbol{P} = \varepsilon_0 \chi_e \boldsymbol{E}$$

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \varepsilon_0 \chi_e \boldsymbol{E} = (1 + \chi_e)\varepsilon_0 \boldsymbol{E}$$

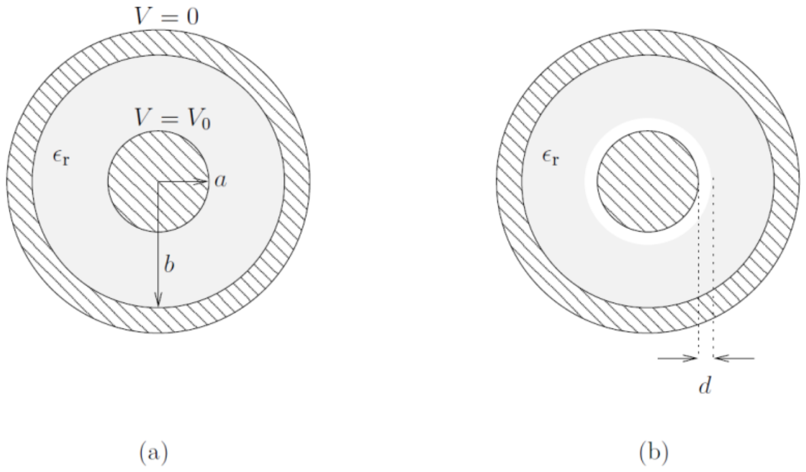
- χ_e is ***electric susceptibility***
- $1 + \chi_e = \varepsilon_r$, ***relative permittivity***
- $\varepsilon = \varepsilon_r \varepsilon_0$ electric permittivity (dielectric material property)

The polarization reduces the electric field E in the dielectric compared to vacuum.

The resultant E becomes less since ε_r > 1

$$\vec{\boldsymbol{E}} = \frac{\boldsymbol{D}}{\boldsymbol{\varepsilon}} = \frac{\boldsymbol{D}}{\boldsymbol{\varepsilon_r \varepsilon_0}}$$

Example: coaxial cable with dielectric material and airgap



Assuming: infinite long cable shown in the figure. Calculating electric field **E** between the out surface of inner cable and inner surface of the outer cable.

a) The permittivity is $\epsilon = \epsilon_r \epsilon_0$

b) when $$\epsilon(r) = \begin{cases} \epsilon_0, & \text{for } a < r < a+d, \\ \epsilon_r \epsilon_0, & \text{for } a+d < r < b. \end{cases}$$

a)

$$\mathbf{E} = \frac{Q'}{2\pi\epsilon r}\hat{\mathbf{r}} \quad \text{for } a < r < b.$$

$$\mathbf{E} = \frac{V_0}{r \ln \frac{b}{a}}\hat{\mathbf{r}} \quad \text{for } a < r < b.$$

b)

$$\epsilon(r) = \begin{cases} \epsilon_0, & \text{for } a < r < a+d, \\ \epsilon_r \epsilon_0, & \text{for } a+d < r < b. \end{cases}$$

$$V_0 = \int_a^b E(r)\mathrm{d}r = \frac{Q'}{2\pi}\int_a^b \frac{\mathrm{d}r}{\epsilon(r) r} = \frac{Q'}{2\pi\epsilon_0}\left(\ln\frac{a+d}{a} + \frac{1}{\epsilon_r}\ln\frac{b}{a+d}\right)$$

$$\mathbf{E} = \begin{cases} \frac{\epsilon_r V_0}{r\left(\epsilon_r \ln \frac{a+d}{a} + \ln \frac{b}{a+d}\right)}\hat{\mathbf{r}}, & \text{for } a < r \leq a+d \\ \frac{V_0}{r\left(\epsilon_r \ln \frac{a+d}{a} + \ln \frac{b}{a+d}\right)}\hat{\mathbf{r}}, & \text{for } a+d < r < b. \end{cases}$$