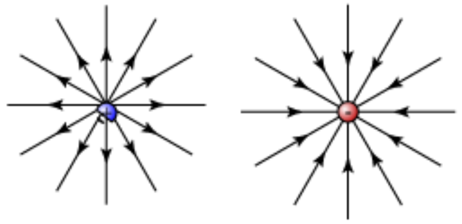
Lecture 2: Stationary electric field (Electrostatic)

- Electric field, electric displacement field, and electric potential
- Conservative vector field: **E** in stationary electric field
- Gauss's law

Divergence and Stokes' theorem

Gradient: fastest rate of increase in spatial

$$\nabla f = \frac{\partial f}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}}$$

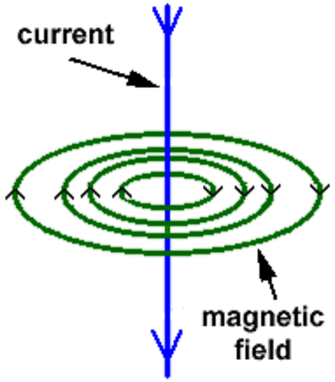


Divergence: Flux out of a point

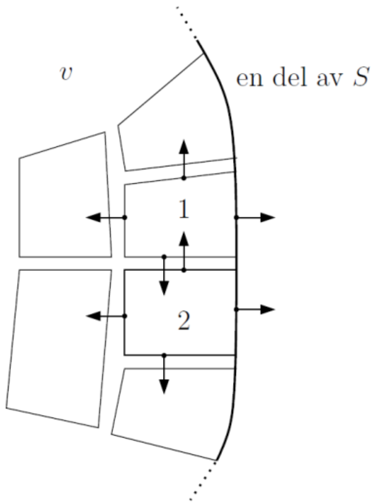
$$\nabla \cdot \boldsymbol{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

How much does a field circulate around a point

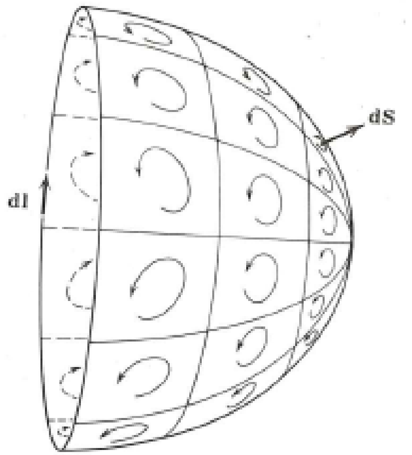
$$\nabla \times \boldsymbol{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\boldsymbol{z}}$$



$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{A} dv$$

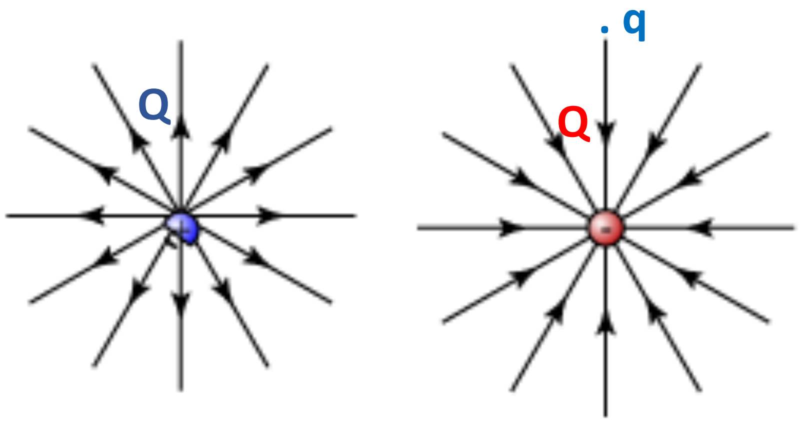


$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{A} \cdot d\mathbf{S}.$$



Electric field and density (vector)

A stationary distribution of charges produces an electric field **E** in vacuum



Field pattern of
a pointed electrode

Vector: $\vec{F} = \frac{Qq}{4\pi\varepsilon_0 r^2}\hat{r}$ $\quad$ ***Coulomb's law***

Vector: $\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$

$\vec{E} = \vec{F}/q$

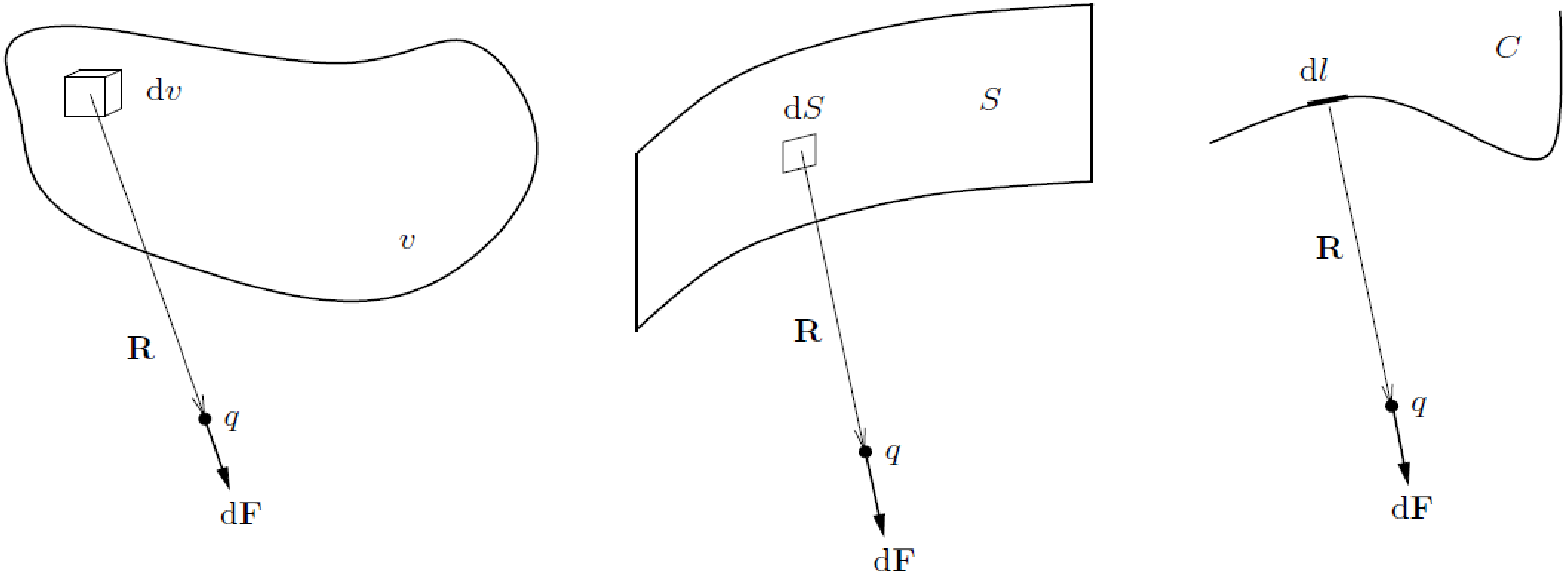
ε_0 is vacuum permittivity (physical property) $\qquad$ ε_0 = 8.85e^{-12} F/m

Electric displacement field: $\mathrm{D} = \varepsilon_0 \mathbf{E}$

Vector: $\vec{D} = \frac{Q}{4\pi r^2}\hat{r}$

Electric displacement field: the equation is material independent.

Superposition: Vector E



$$\mathbf{E} = \int_v \frac{\hat{\mathbf{R}}\rho \mathrm{d}v}{4\pi\epsilon_0 R^2}.$$

$$\mathbf{E} = \int_S \frac{\hat{\mathbf{R}}\rho_{\mathrm{s}} \mathrm{d}S}{4\pi\epsilon_0 R^2}.$$

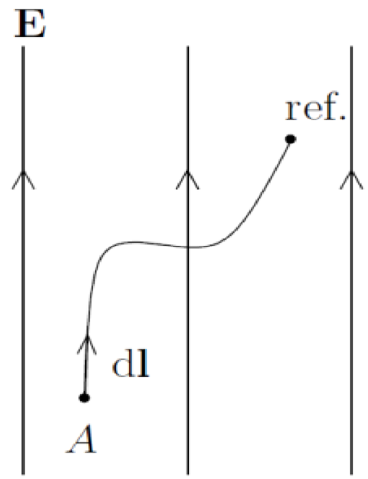
$$\mathbf{E} = \int_C \frac{\hat{\mathbf{R}}Q' \mathrm{d}l}{4\pi\epsilon_0 R^2}.$$

Electric potential

Electric potential definition $V = \frac{W}{q} = \frac{\int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{l}}{q} = -\int \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l}$

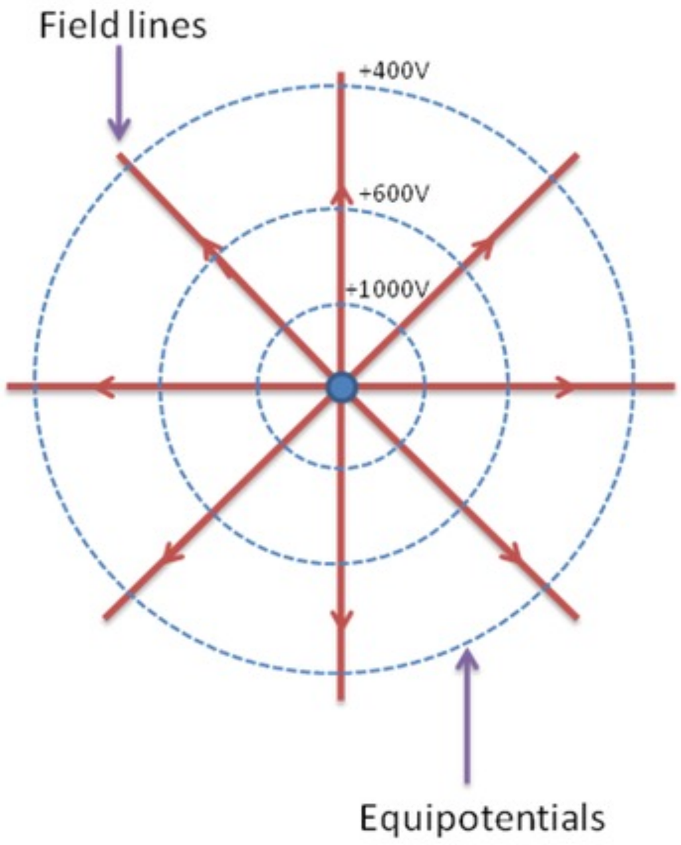
Vector: $\vec{\mathbf{E}} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$ Potential: $V = \frac{Q}{4\pi\varepsilon_0 r}$

$$V_A = \int_A^{\text{ref.}} \mathbf{E} \cdot \mathrm{d}\mathbf{l}.$$



The potential difference between two points A and B:

$$V_{AB} = -\int_{\boldsymbol{rA}}^{\boldsymbol{rB}} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\varepsilon_0 {r_i}^2} dr_i = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$



Electric field and potential

- Potential is a scalar
- $V = -\int \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l}$

$$\nabla f = \frac{\partial f}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}}$$

What happens to V when a positive and negative charge moves, respectively?

$$\boxed{\mathbf{E} = -\nabla V, \ \mathrm{V/m}}$$

$$V = \int_{\boldsymbol{r}}^{\infty} \boldsymbol{E} d\boldsymbol{l} = \int_{\boldsymbol{r}}^{\infty} \frac{Q}{4\pi\varepsilon_0 l^2} d\boldsymbol{l} = \frac{Q}{4\pi\varepsilon_0 r}$$

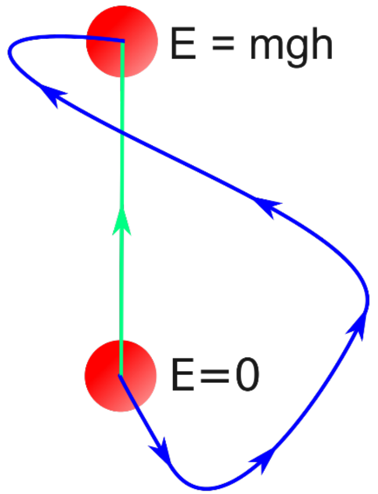
$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}}$$

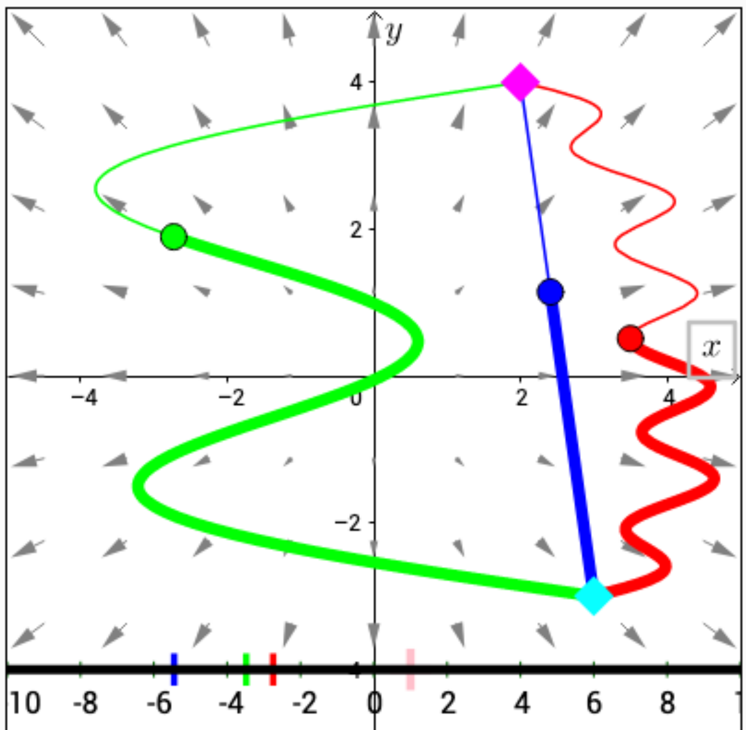
Conservative vector field

Stationary Electric field is a conservative vector.

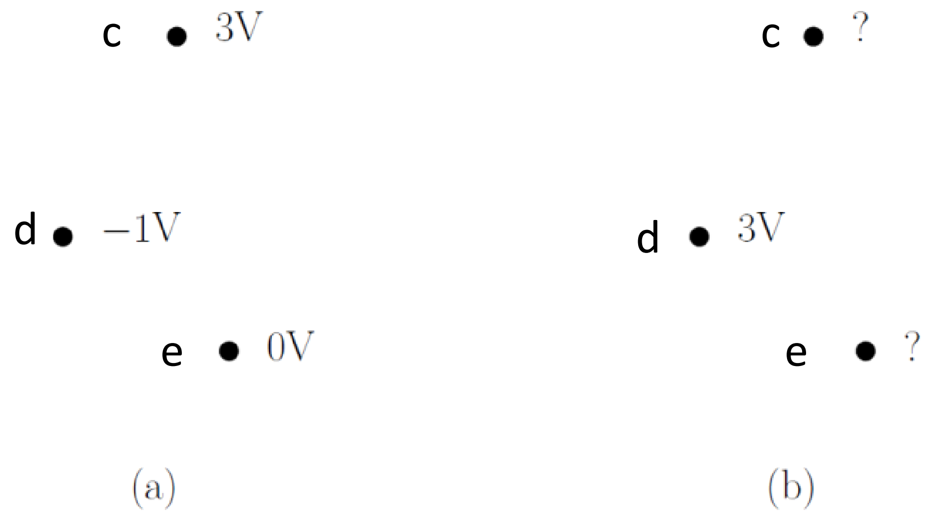
* Conservative vector fields have the property that the line integral is path independent.

* A conservative vector field is also irrotational. In three dimensions , it has vanishing curl. $\nabla \times \mathbf{E} = 0$





Example: find out the unknown potentials

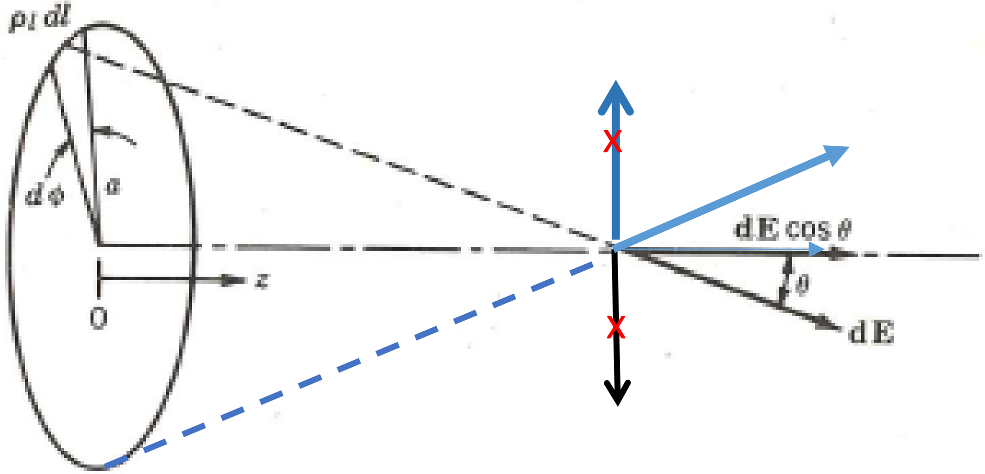


In a stationaly electric field, the potential at each point is listed in (a). In the same stationary field and the potential at point d is 3V with another reference point.

What are the potential that are missed in b)?

Example: field of a ring of charge

Charge ring with radius *a* and line charge density: ρ_l
What is **E** at a point on the z axis?



$$\vec{E} = \sum_{i=1}^{n} \frac{q_i}{4\pi\varepsilon_0 r_i^2}\hat{r}_i$$

$$\sum_{i=1}^{n} q_i = \oint \rho_l dl = \int_0^{2\pi a} \rho_l a d\phi$$

$$dl = a d\phi$$

$$\mathbf{E} = \int_0^{2\pi a} \rho_l a d\phi \frac{\cos\theta}{4\pi\varepsilon_0 r^2}\hat{\mathbf{z}} = \int_0^{2\pi a} \frac{\rho_l z a \;\; d\phi}{4\pi\varepsilon_0 (a^2 + z^2)^{3/2}}\hat{\mathbf{z}} = \frac{\rho_l a z}{2\varepsilon_0 (a^2 + z^2)^{3/2}}\hat{\mathbf{z}}$$

$$r^2 = a^2 + z^2$$

$$\cos\theta = z / (a^2 + z^2)^{\frac{1}{2}}$$

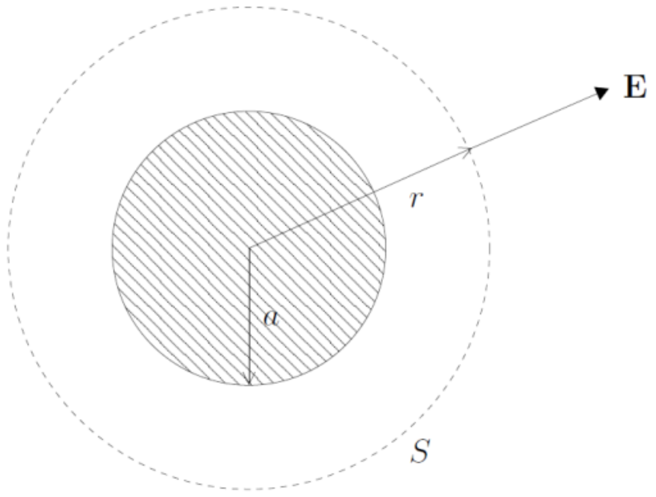
The resultant **E** at z-axis is also in the direction of z-axis.

Example: electric field around a charged ball

What is the electric field outside a charged ball with total charge Q? (r>>a)

$$\vec{\mathbf{E}} = E\hat{\boldsymbol{r}}$$

$$\vec{\mathbf{E}} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}}, \quad \text{(r>>a)}$$



Poisson equation

Inserting $\mathbf{E} = -\nabla V$ into Maxwell's equation $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ gives

$$-\nabla \cdot (\nabla V) = \frac{\rho}{\epsilon_0},$$

which gives

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}.$$

This is called Poisson's equation. Here

$$\nabla^2 =< \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2} >$$

Potential from point, line, surface and space charge

a) A point charge:

$$V = \int_R^\infty \frac{Q}{4\pi\epsilon_0 r^2}\mathrm{d}r = \frac{Q}{4\pi\epsilon_0 R},$$

b) A line charge (line charge density Q'):

$$V = \int_C \frac{Q'}{4\pi\epsilon_0 R}\mathrm{d}l,$$

c) A surface charge (surface charge density ρ_s):

$$V = \int_S \frac{\rho_s}{4\pi\epsilon_0 R}\mathrm{d}S,$$

d) A space charge (charge density ρ):

$$V = \int_V \frac{\rho}{4\pi\epsilon_0 R}\mathrm{d}V.$$

The last three is found by superposition from the point charge expression.

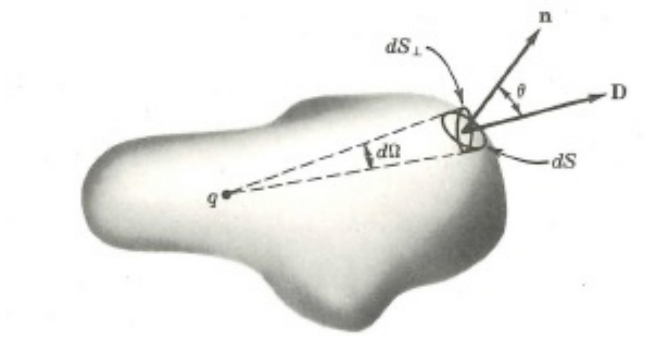
Gauss's law

Electric flux flowing out of a closed surface = Enclosed total charges divided by the permittivity $\oint \mathbf{E} d\boldsymbol{s} = Q/\varepsilon$ **Vector:** $\vec{\mathbf{E}} = \frac{Q}{4\pi r^2 \varepsilon_0}\hat{\boldsymbol{r}}$ in vacuum, $\vec{\boldsymbol{D}} = \frac{Q}{4\pi r^2}\hat{\boldsymbol{r}}$

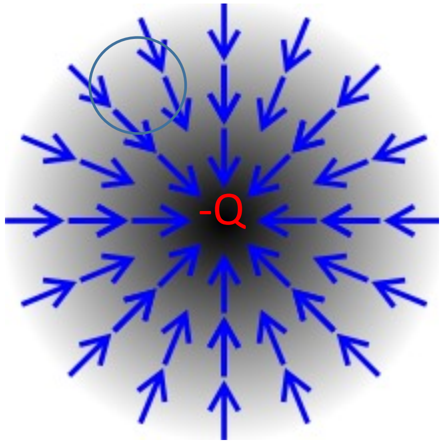
Electric displacement field, D = εE.

- $\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon}$
- $\nabla \cdot \mathbf{D} = \rho$

$$\oint_S \mathbf{A} \cdot \mathrm{d}\mathbf{S} = \int_v \nabla \cdot \mathbf{A} \mathrm{d}v$$

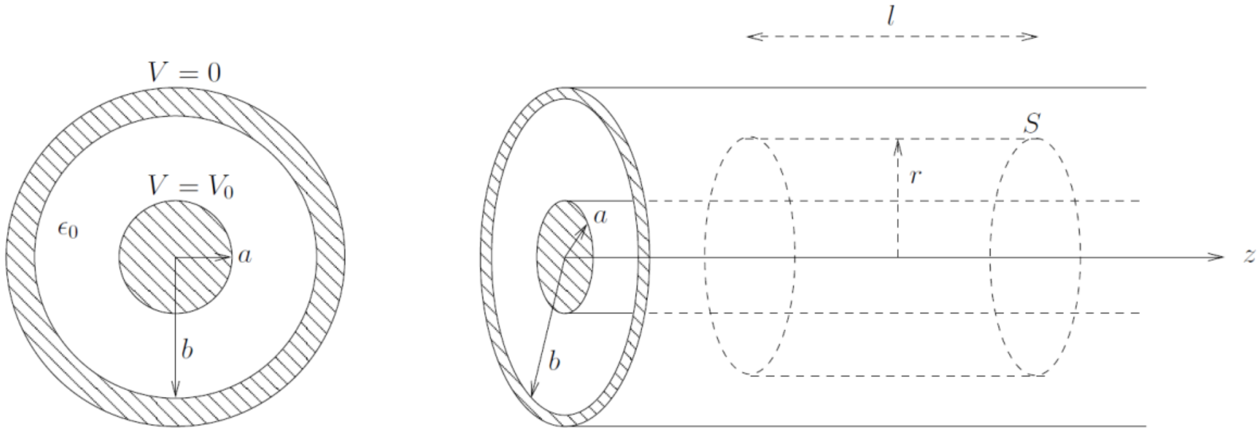


$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \sum_i \oint_{S_i} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \sum_i \left(\frac{1}{\Delta V_i} \oint_{S_i} \mathbf{E} \cdot \mathrm{d}\mathbf{S}_i\right) \Delta V_i \to \int \nabla \cdot \mathbf{E} \mathrm{d}V.$$



$$\epsilon_0 \oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = Q_{\text{total i } S}$$

Example: coaxial cable



Assuming an infinite long cable
Calculate **E** between the out surface of inner cable and inner surface of the outer cable.

Assuming unit length and the surface charge density is ρ_s

$$\epsilon_0 \oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \epsilon_0 E 2\pi r l$$

$$\mathbf{E} = \begin{cases} \frac{Q'}{2\pi\epsilon_0 r}\hat{\mathbf{r}}, & \text{for } a < r < b \\ 0, & \text{ellers.} \end{cases}$$

$$V_0 = \int_a^b \mathbf{E} \cdot \mathrm{d}\mathbf{l} = \int_a^b E \mathrm{d}r = \frac{Q'}{2\pi\epsilon_0} \int_a^b \frac{\mathrm{d}r}{r} = \frac{Q'}{2\pi\epsilon_0} \ln \frac{b}{a}.$$

$$\mathbf{E} = \begin{cases} \frac{V_0}{r \ln \frac{b}{a}}\hat{\mathbf{r}}, & \text{for } a < r < b \\ 0, & \text{ellers.} \end{cases}$$

Electric field in dielectric

Force in vacuum: $$\vec{F}_{tot} = \sum_{i=1}^{n} \frac{q q_i}{4\pi\varepsilon_0 r_i^2} \hat{r}_i$$

Vector: $$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

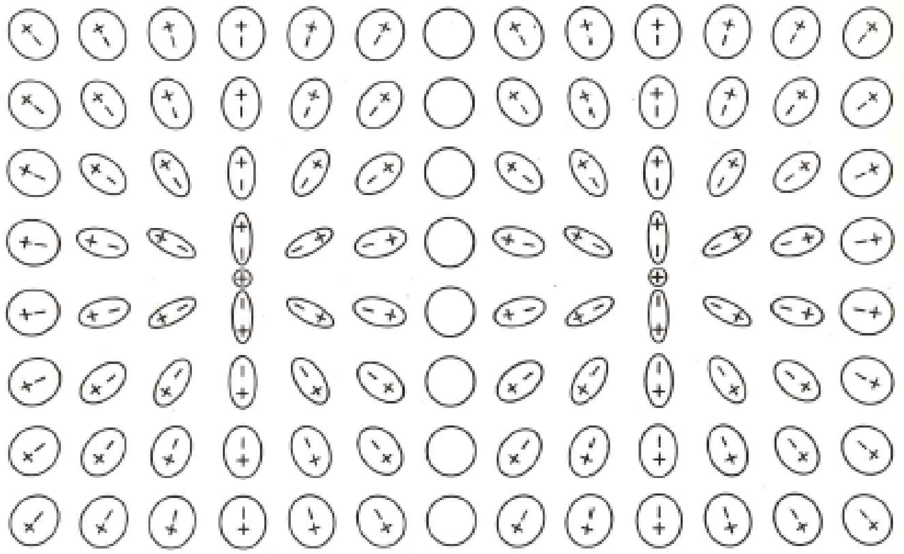


Fig. 1.3c Polarization of the atoms of a dielectric by a pair of equal positive charges.

In dielectric media, the force between charges depends upon the presence of media/dielectric:

1) Once electric filed exists in dielectric media, the atoms are polarized, similar distortions occur in molecules.
2) Electric polarization induced by the electric field in a material depends upon material properties.
3) Polarization influence the force between charges.

Electric polarization and material permittivity

The influence of electric polarization: $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$

- $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$
- $\boldsymbol{P} = \varepsilon_0 \chi_e \boldsymbol{E}$
- $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \varepsilon_0 \chi_e \boldsymbol{E} = (1 + \chi_e)\varepsilon_0 \boldsymbol{E}$, χ_e is called ***electric susceptibility***

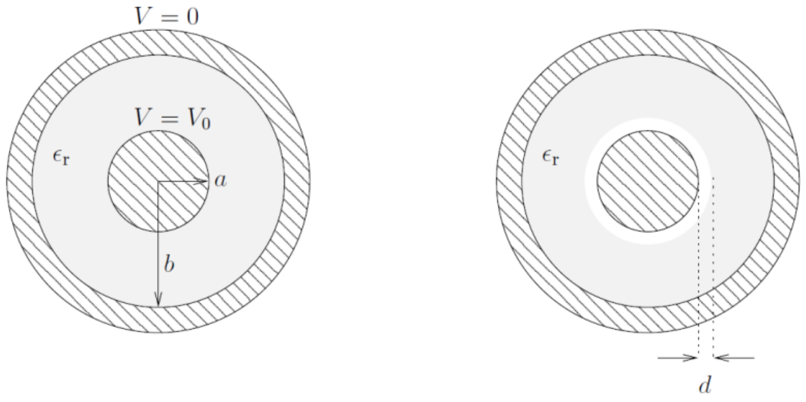
- $(1 + \chi_e) = \varepsilon_r$, ***relative permittivity***
- $\varepsilon = \varepsilon_r \varepsilon_0$ electric permittivity (dielectric material property)

The polarization reduces the electric field E in the dielectric material compared to vacuum.

The resultant E becomes less. ε_r > 1

$$\vec{\boldsymbol{E}} = \frac{\boldsymbol{D}}{\boldsymbol{\varepsilon}} = \frac{\boldsymbol{D}}{\boldsymbol{\varepsilon_r \varepsilon_0}}$$

Example: coaxial cable with dielectric material and airgap



(a) (b)

Assuming: infinite long cable
Calculating electric field **E** between the out surface of inner cable and inner surface of the outer cable.

a) The permittivity is $\epsilon = \epsilon_r \epsilon_0$

b) when
$$\epsilon(r) = \begin{cases} \epsilon_0, & \text{for } a < r < a+d, \\ \epsilon_r \epsilon_0, & \text{for } a+d < r < b. \end{cases}$$

a)

$$\mathbf{E} = \frac{Q'}{2\pi\epsilon r}\hat{\mathbf{r}} \quad \text{for } a < r < b.$$

$$\mathbf{E} = \frac{V_0}{r \ln \frac{b}{a}}\hat{\mathbf{r}} \quad \text{for } a < r < b.$$

b)

$$\epsilon(r) = \begin{cases} \epsilon_0, & \text{for } a < r < a+d, \\ \epsilon_r \epsilon_0, & \text{for } a+d < r < b. \end{cases}$$

$$V_0 = \int_a^b E(r)\mathrm{d}r = \frac{Q'}{2\pi}\int_a^b \frac{\mathrm{d}r}{\epsilon(r) r} = \frac{Q'}{2\pi\epsilon_0}\left(\ln\frac{a+d}{a} + \frac{1}{\epsilon_r}\ln\frac{b}{a+d}\right)$$

$$\mathbf{E} = \begin{cases} \frac{\epsilon_r V_0}{r\left(\epsilon_r \ln\frac{a+d}{a} + \ln\frac{b}{a+d}\right)}\hat{\mathbf{r}}, & \text{for } a < r \le a+d \\ \frac{V_0}{r\left(\epsilon_r \ln\frac{a+d}{a} + \ln\frac{b}{a+d}\right)}\hat{\mathbf{r}}, & \text{for } a+d < r < b. \end{cases}$$