

# Lecture 2: Stationary electric field (Electrostatics)

- Electric field, electric displacement field
- Electric potential
- Conservative vector field
- Gauss's law

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# Divergence and Stokes' theorem

Gradient: fastest rate of increase in spatial

$$\nabla f = \frac{\partial f}{\partial x} \hat{\mathbf{x}} + \frac{\partial f}{\partial y} \hat{\mathbf{y}} + \frac{\partial f}{\partial z} \hat{\mathbf{z}}$$

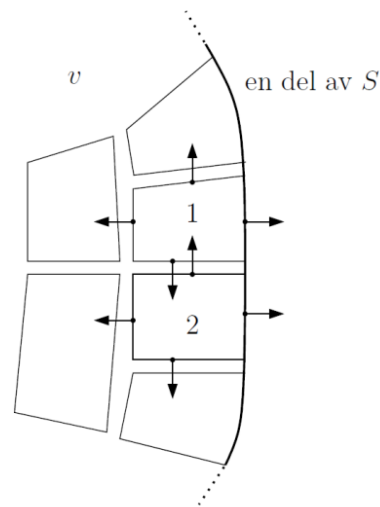
Divergence: Flux out of a point

$$\nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

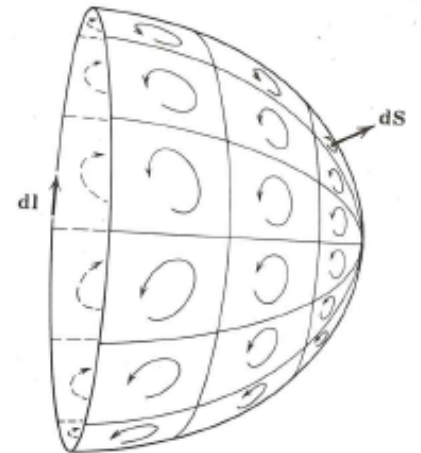
How much does a field circulate around a point

$$\nabla \times \mathbf{A} = \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{\mathbf{x}} + \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{\mathbf{y}} + \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{\mathbf{z}}$$

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{A} dv$$

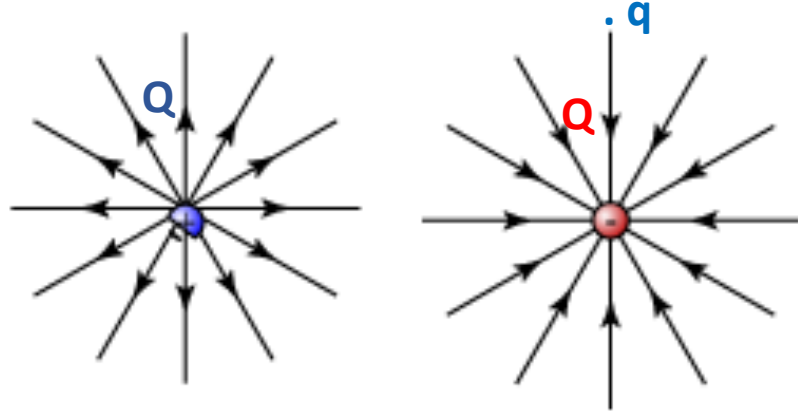


$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{A} \cdot d\mathbf{S}$$



# Electric field and electric displacement field

A stationary distribution of charges produces an electric field  $\mathbf{E}$  in vacuum



Field pattern of  
a pointed electrode

$$\text{Vector: } \vec{F} = \frac{Qq}{4\pi\epsilon_0 r^2} \hat{r} \quad \text{Coulomb's law}$$

$$\text{Vector: } \vec{E} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{r}$$

$$\vec{E} = \vec{F}/q$$

$\epsilon_0$  is vacuum permittivity (physical property)

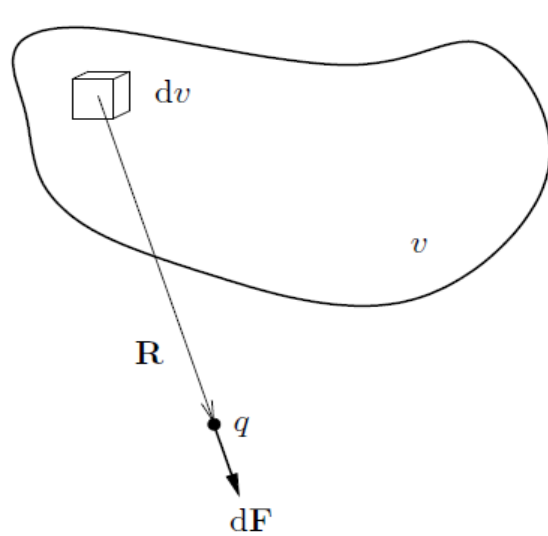
$$\epsilon_0 = 8.85e^{-12} \text{ F/m}$$

Electric displacement field:  $\mathbf{D} = \epsilon_0 \mathbf{E}$

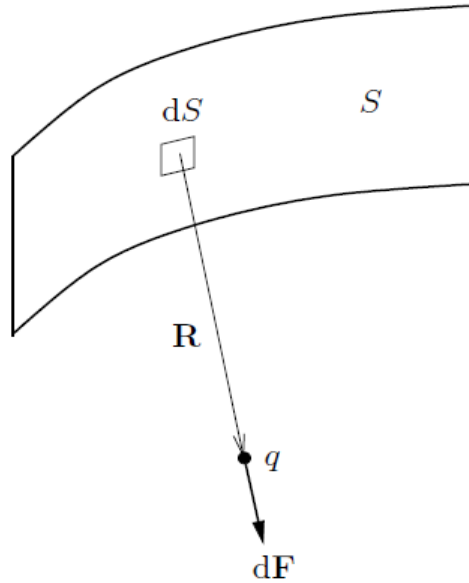
$$\text{Vector: } \vec{D} = \frac{Q}{4\pi r^2} \hat{r}$$

Electric displacement field: the equation is material independent.

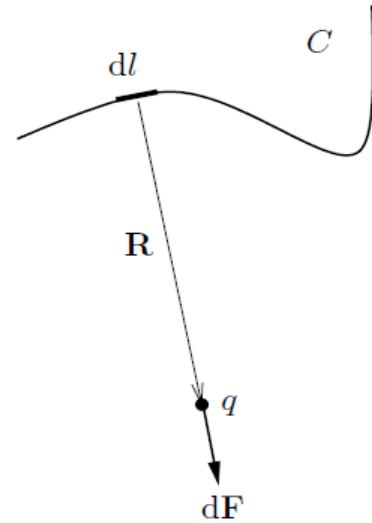
# Superposition: Vector $\mathbf{E}$



$$\mathbf{E} = \int_v \frac{\hat{\mathbf{R}}\rho dv}{4\pi\epsilon_0 R^2}.$$



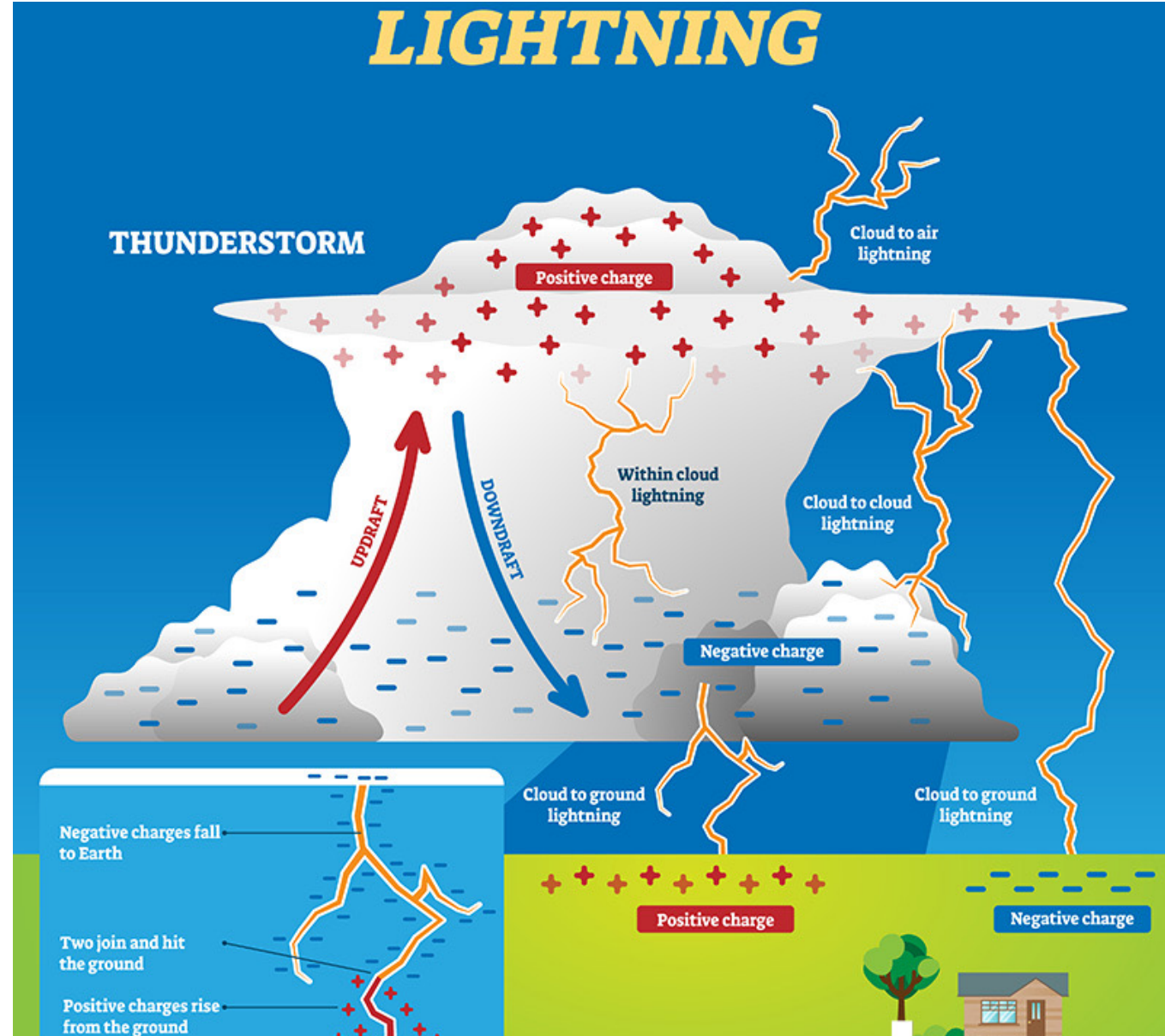
$$\mathbf{E} = \int_S \frac{\hat{\mathbf{R}}\rho_s dS}{4\pi\epsilon_0 R^2}.$$



$$\mathbf{E} = \int_C \frac{\hat{\mathbf{R}}Q' dl}{4\pi\epsilon_0 R^2}.$$

# Electric potential

Work needed per unit of charge

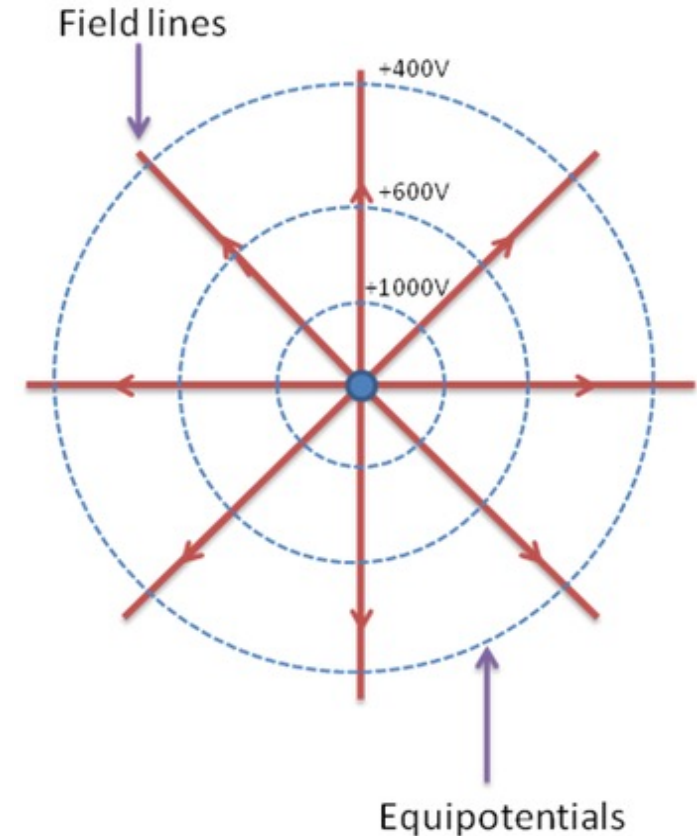
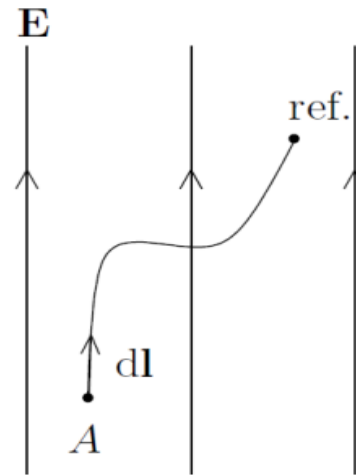


# Electric potential

Electric potential definition  $V = \frac{W}{q} = \frac{\int \mathbf{F} \cdot d\mathbf{l}}{q} = - \int \mathbf{E} \cdot d\mathbf{l}$

Vector:  $\vec{\mathbf{E}} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}}$       Potential:  $V = \frac{Q}{4\pi\epsilon_0 r}$

$$V = \int_r^\infty \mathbf{E} d\mathbf{l} = \int_r^\infty \frac{Q}{4\pi\epsilon_0 l^2} dl = \frac{Q}{4\pi\epsilon_0 r}$$



The potential difference between two points A and B:

$$V_{AB} = - \int_{r_A}^{r_B} \mathbf{E} \cdot d\mathbf{l} = - \int_{r_A}^{r_B} \frac{Q}{4\pi\epsilon_0 r_i^2} dr_i = \frac{Q}{4\pi\epsilon_0} \left( \frac{1}{r_B} - \frac{1}{r_A} \right)$$

# Electric field and potential

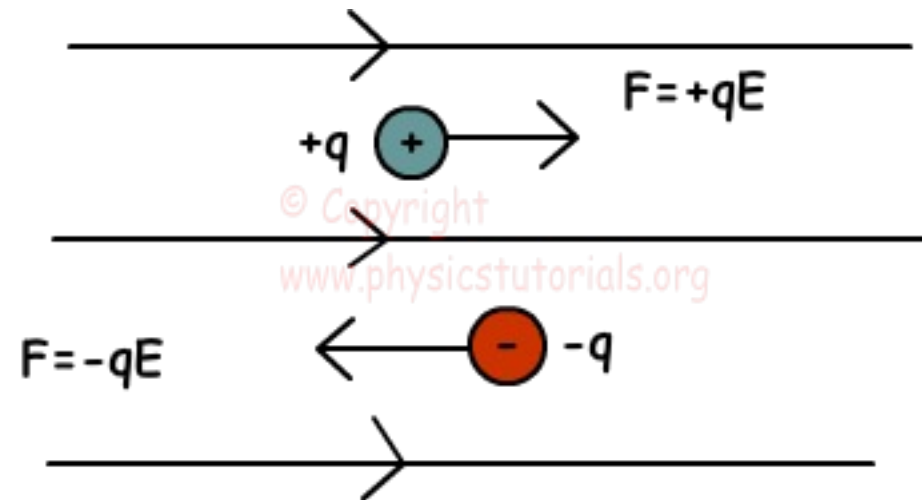
- Potential is a scalar
- $V = - \int \mathbf{E} \cdot d\mathbf{l}$

What happens to  $V$  when a positive and negative charge moves, respectively?

$$\mathbf{E} = -\nabla V, \text{ V/m}$$

$$V = \frac{1}{4\pi\epsilon_0} \frac{Q}{r}$$

$$\nabla f = \frac{\partial f}{\partial x} \hat{\mathbf{x}} + \frac{\partial f}{\partial y} \hat{\mathbf{y}} + \frac{\partial f}{\partial z} \hat{\mathbf{z}}$$

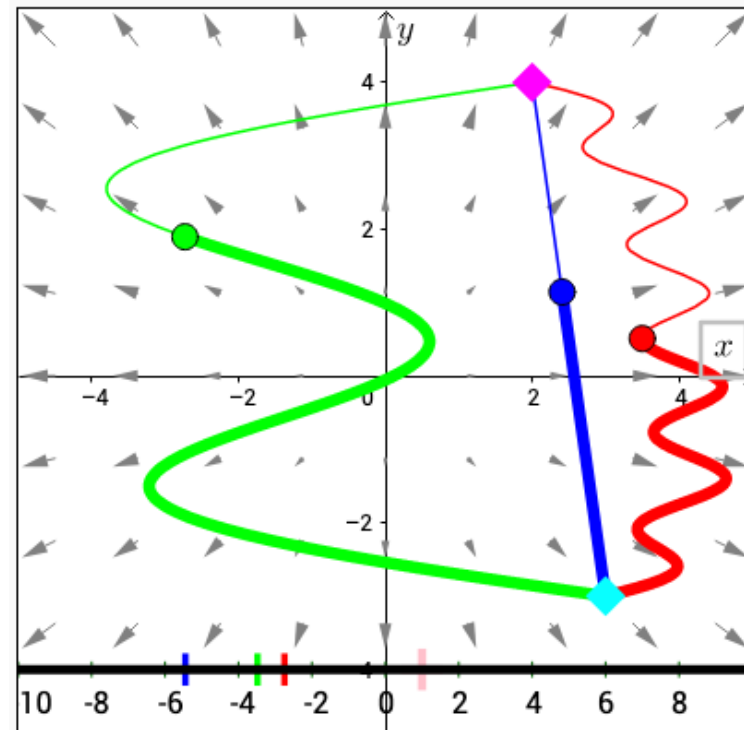
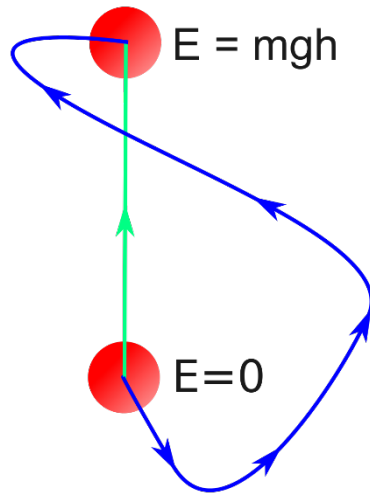


# Conservative vector field

Stationary electric field is a conservative vector.

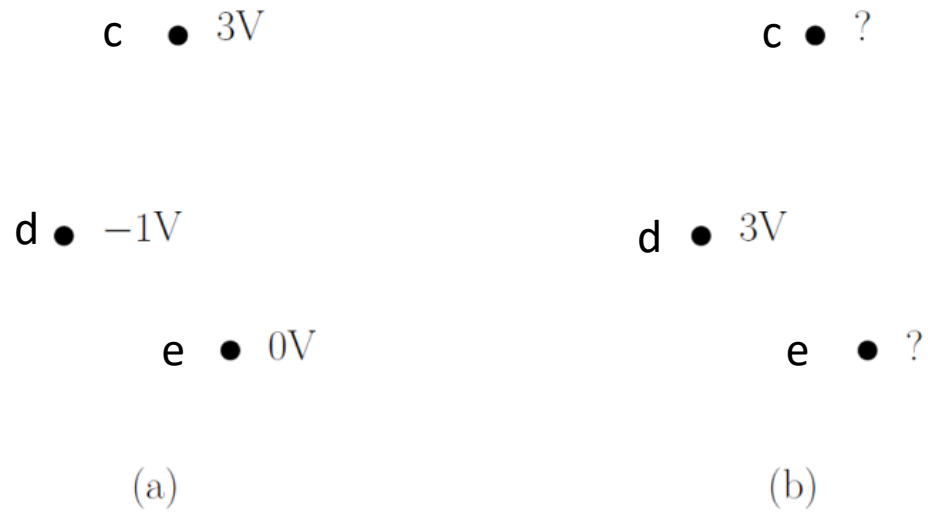
\* Conservative vector fields have the property that the line integral is path independent.

\* A conservative vector field is also irrotational. In three dimensions, it has vanishing curl.  $\nabla \times \mathbf{E} = 0$





# Example: find out the unknown potentials



In a stationary electric field, the potential at each point is listed in (a). In the same stationary field and the potential at point d is 3V with another reference point.

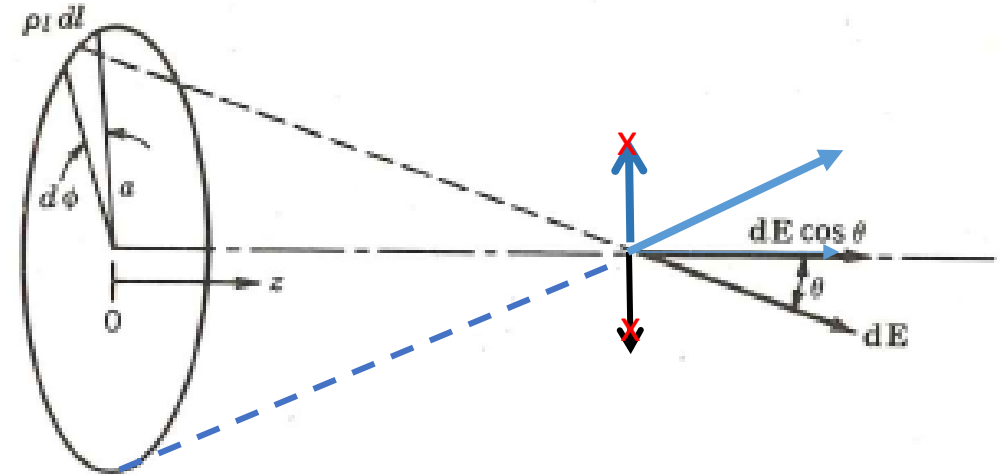
What are the potential that are missed in b)?

# Example: field of a ring of charge

Charge ring with radius  $a$  and line charge density:  $\rho_l$   
 What is  $\mathbf{E}$  at a point on the  $z$  axis?

$$\vec{\mathbf{E}} = \sum_{i=1}^n \frac{q_i}{4\pi\epsilon_0 r_i^2} \hat{\mathbf{r}}_i$$

$$\hat{\mathbf{r}}_i = \hat{\mathbf{z}}_i \cos\theta$$



$$\sum_{i=1}^n q_i = \oint \rho_l dl = \int_0^{2\pi a} \rho_l dl = \int_0^{2\pi} \rho_l a d\phi$$

$$dl = a d\phi$$

$$\mathbf{E} = \int_0^{2\pi} \rho_l a d\phi \frac{\cos\theta}{4\pi\epsilon_0 r^2} \hat{\mathbf{z}} = \int_0^{2\pi} \frac{\rho_l z a d\phi}{4\pi\epsilon_0 (a^2 + z^2)^{3/2}} \hat{\mathbf{z}} = \frac{\rho_l a z}{2\epsilon_0 (a^2 + z^2)^{3/2}} \hat{\mathbf{z}}$$

$$r^2 = a^2 + z^2$$

$$\cos\theta = \frac{z}{(a^2 + z^2)^{1/2}}$$

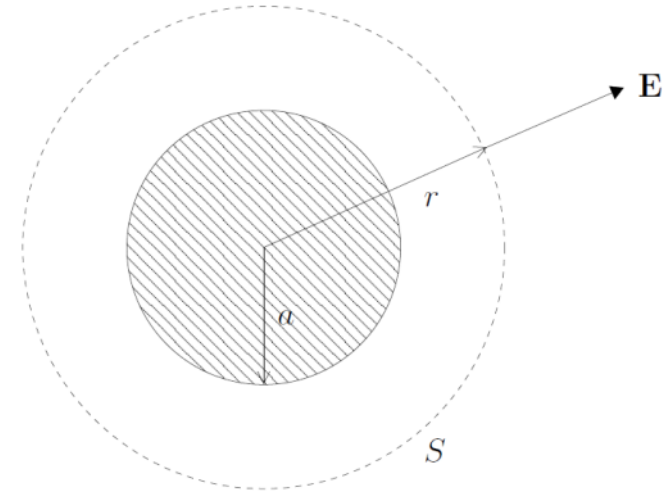
The resultant  $\mathbf{E}$  at  $z$ -axis is also in the direction of  $z$ -axis.

# Example: electric field around a charged ball

What is the electric field outside a charged ball with total charge  $Q$ ? ( $r \gg a$ )

$$\vec{\mathbf{E}} = E \hat{\mathbf{r}}$$

$$\vec{\mathbf{E}} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}}, \quad (r \gg a)$$



# Poisson's equation

Inserting  $\mathbf{E} = -\nabla V$  into Maxwell's equation  $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$  gives

$$-\nabla \cdot (\nabla V) = \frac{\rho}{\epsilon_0},$$

which gives

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}.$$

This is called Poisson's equation. Here

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

# Potential for point, line, surface and space charge

a) A point charge:

$$V = \int_R^\infty \frac{Q}{4\pi\epsilon_0 r^2} dr = \frac{Q}{4\pi\epsilon_0 R},$$

b) A line charge (line charge density  $Q'$ ):

$$V = \int_C \frac{Q'}{4\pi\epsilon_0 R} dl,$$

c) A surface charge (surface charge density  $\rho_s$ ):

$$V = \int_S \frac{\rho_s}{4\pi\epsilon_0 R} dS,$$

d) A space charge (charge density  $\rho$ ):

$$V = \int_V \frac{\rho}{4\pi\epsilon_0 R} dV.$$

Superposition  
principle

# Gauss's law

Electric flux flowing out of a closed surface = Enclosed total charges **divided**  
**by the permittivity**

$$\oint \mathbf{E} d\mathbf{s} = Q / \epsilon$$

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{A} dv$$

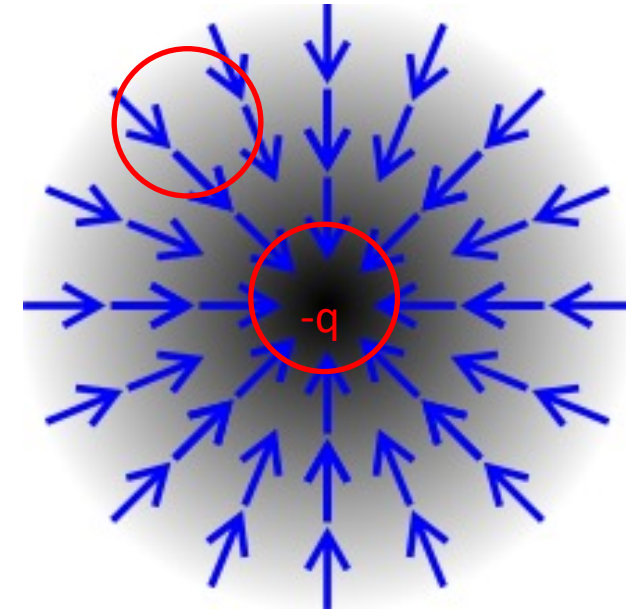
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon}$$

$$\nabla \cdot \mathbf{D} = \rho$$

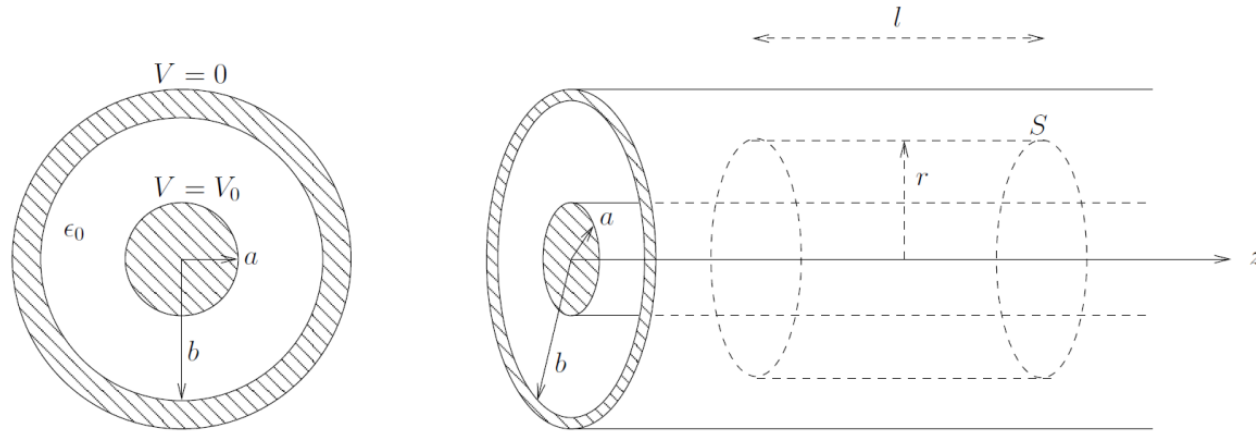
Vector:  $\vec{\mathbf{E}} = \frac{Q}{4\pi r^2 \epsilon_0} \hat{\mathbf{r}}$  in vacuum,  $\vec{\mathbf{D}} = \frac{Q}{4\pi r^2} \hat{\mathbf{r}}$

Electric displacement field,  $\mathbf{D} = \epsilon \mathbf{E}$ .

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \sum_i \oint_{S_i} \mathbf{E} \cdot d\mathbf{S} = \sum_i \left( \frac{1}{\Delta V_i} \oint_{S_i} \mathbf{E} \cdot d\mathbf{S}_i \right) \Delta V_i \rightarrow \int \nabla \cdot \mathbf{E} dV.$$



# Example: coaxial cable



Assuming an infinite long cable with an inner cable and an outer cable, information is shown in the left fig.

Calculate  $\mathbf{E}$  between the out surface of inner cable and inner surface of the outer cable.

Assuming unit length and the surface charge density is  $\rho_s$

$$\epsilon_0 \oint_S \mathbf{E} \cdot d\mathbf{S} = \epsilon_0 E 2\pi r l$$

$$\mathbf{E} = \begin{cases} \frac{Q'}{2\pi\epsilon_0 r} \hat{\mathbf{r}}, & \text{for } a < r < b \\ 0, & \text{ellers.} \end{cases}$$

$$V_0 = \int_a^b \mathbf{E} \cdot d\mathbf{l} = \int_a^b E dr = \frac{Q'}{2\pi\epsilon_0} \int_a^b \frac{dr}{r} = \frac{Q'}{2\pi\epsilon_0} \ln \frac{b}{a}.$$

$$\mathbf{E} = \begin{cases} \frac{V_0}{r \ln \frac{b}{a}} \hat{\mathbf{r}}, & \text{for } a < r < b \\ 0, & \text{ellers.} \end{cases}$$

# Electric field in dielectric

In metals charges **are free to move**.

Force in vacuum:

$$\vec{F}_{tot} = \sum_{i=1}^n \frac{qq_i}{4\pi\epsilon_0 r_i^2} \hat{r}_i$$

$$\text{Vector: } \vec{E} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{r}$$

In dielectrics **all the charges are attached to specific atoms and molecules**.

In dielectric media, the force between charges depends on the media/dielectric:

- 1) Once electric field exists in dielectric, the atoms are polarized, similar for molecules.
- 2) Polarization induced by the electric field depends on material properties.
- 3) Polarization influences the force between charges.

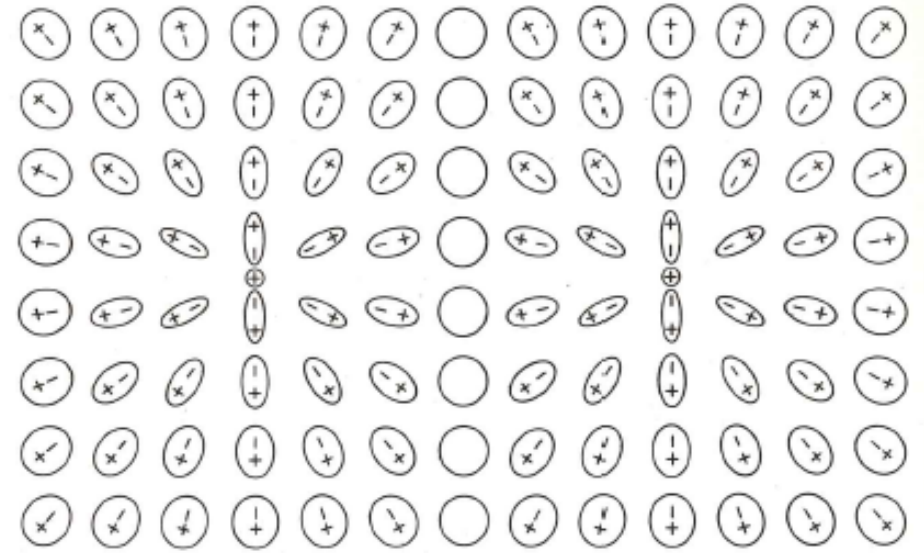


Fig. 1.3c Polarization of the atoms of a dielectric by a pair of equal positive charges.



# Electric polarization and material permittivity

The influence of electric polarization:  $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \varepsilon_0 \chi_e \mathbf{E} = (1 + \chi_e) \varepsilon_0 \mathbf{E}$$

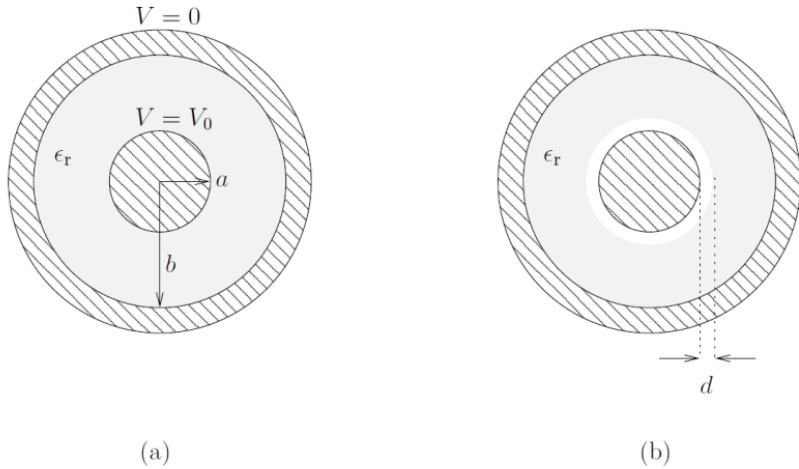
- $\chi_e$  is *electric susceptibility*
- $1 + \chi_e = \varepsilon_r$ , *relative permittivity*
- $\varepsilon = \varepsilon_r \varepsilon_0$  electric permittivity (dielectric material property)

**The polarization reduces the electric field  $\mathbf{E}$  in the dielectric compared to vacuum.**

**The resultant  $\mathbf{E}$  becomes less since  $\varepsilon_r > 1$**

$$\vec{\mathbf{E}} = \frac{\mathbf{D}}{\varepsilon} = \frac{\mathbf{D}}{\varepsilon_r \varepsilon_0}$$

# Example: coaxial cable with dielectric material and airgap



Assuming: infinite long cable shown in the figure.  
Calculating electric field  $\mathbf{E}$  between the out surface of inner cable and inner surface of the outer cable.

a) The permittivity is  $\epsilon = \epsilon_r \epsilon_0$

b) when 
$$\epsilon(r) = \begin{cases} \epsilon_0, & \text{for } a < r < a + d, \\ \epsilon_r \epsilon_0, & \text{for } a + d < r < b. \end{cases}$$

a)

$$\mathbf{E} = \frac{Q'}{2\pi\epsilon r} \hat{\mathbf{r}} \quad \text{for } a < r < b.$$

$$\mathbf{E} = \frac{V_0}{r \ln \frac{b}{a}} \hat{\mathbf{r}} \quad \text{for } a < r < b.$$

b)

$$\epsilon(r) = \begin{cases} \epsilon_0, & \text{for } a < r < a + d, \\ \epsilon_r \epsilon_0, & \text{for } a + d < r < b. \end{cases}$$

$$V_0 = \int_a^b E(r) dr = \frac{Q'}{2\pi} \int_a^b \frac{dr}{\epsilon(r)r} = \frac{Q'}{2\pi\epsilon_0} \left( \ln \frac{a+d}{a} + \frac{1}{\epsilon_r} \ln \frac{b}{a+d} \right)$$

$$\mathbf{E} = \begin{cases} \frac{\epsilon_r V_0}{r(\epsilon_r \ln \frac{a+d}{a} + \ln \frac{b}{a+d})} \hat{\mathbf{r}}, & \text{for } a < r \leq a + d \\ \frac{V_0}{r(\epsilon_r \ln \frac{a+d}{a} + \ln \frac{b}{a+d})} \hat{\mathbf{r}}, & \text{for } a + d < r < b. \end{cases}$$