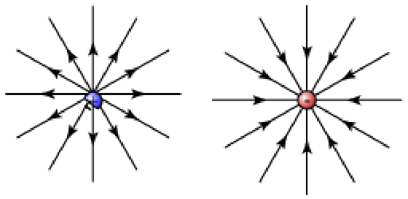
# Lecture 2: Stationary electric field (Electrostatic)

- Coulomb’s Law
- Electric field , electric displacement field, and electric potential
- Conservative vector field: E in stationary electric field
- Gauss’s law

# Divergence and Stoke’s theorem

Gradient: fastest rate of increase in spatial. $\nabla \boldsymbol{f} = \frac{\partial f}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}}$

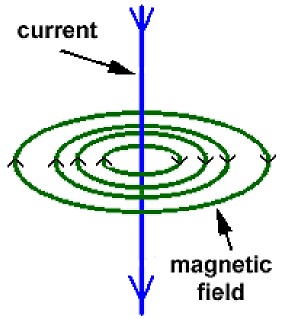



Divergence: Flux out of a point

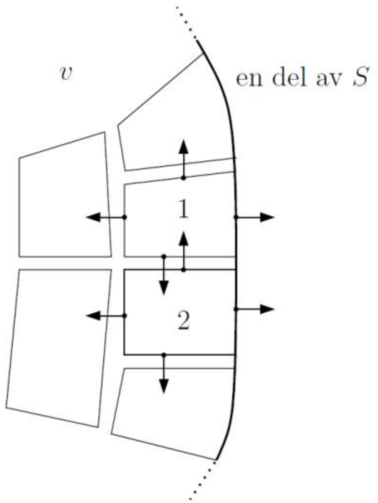
$$\nabla \cdot E = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

Curl: field circulating around a point

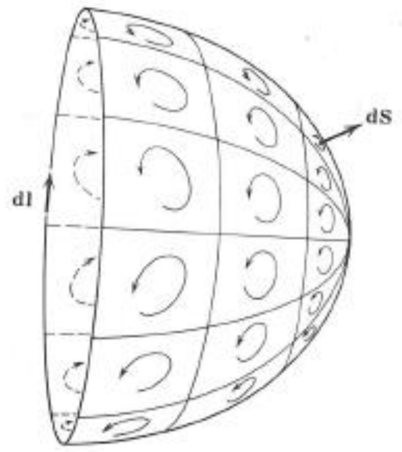
$$\nabla \times A = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}$$




$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{A} dv$$

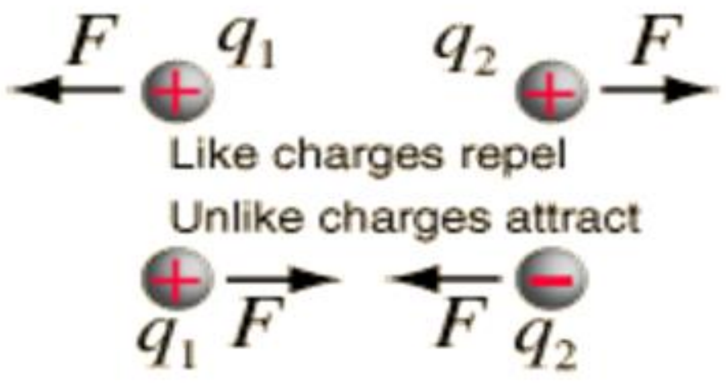



$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{A} \cdot d\mathbf{S}.$$



# Coulomb's law: force between electrostatic charges

Published in 1785 by French physicist Charles-Augustin de Coulomb and was essential to the development of the theory of electromagnetism




**Scalar:** $F = k\frac{q_1 q_2}{r_{12}^2} = \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}^2}$

**Vector:** $\vec{F} = \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}^2}\widehat{r_{12}}$

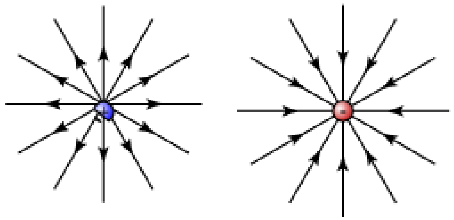
$\widehat{r_{12}}$ is just for direction, its absolut value is 1.

$$k = \frac{1}{4\pi\varepsilon_0} \approx 9x10^9 N \cdot m^2/C^2 = \text{Coulomb's constant}$$

The electrostatic force has the same functional form as Newton's law of gravity

The electrostatic force between two electric charges:

1) directly proportional to the product of the magnitudes of charges
2) inversely proportional to the square of the distance between them
3) The force is along the straight line joining them.


Field pattern of a pointed electrode

# Vector force: superposition

**Electric forces follow the law of superposition.**

**If more than one charge is causing a force on object 1, then the net force acting on object 1 is just the sum of all the individual forces acting on 1.**

$$\vec{F}_{tot} = \sum_{i=1}^{n} \frac{q q_i}{4\pi\varepsilon_0 r_i^2} \hat{r}_i$$

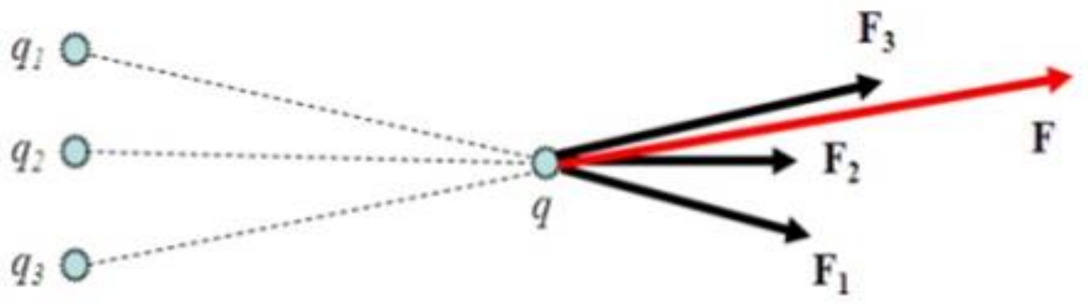
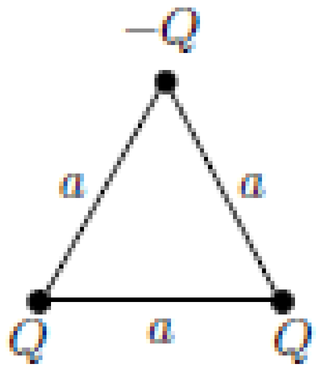


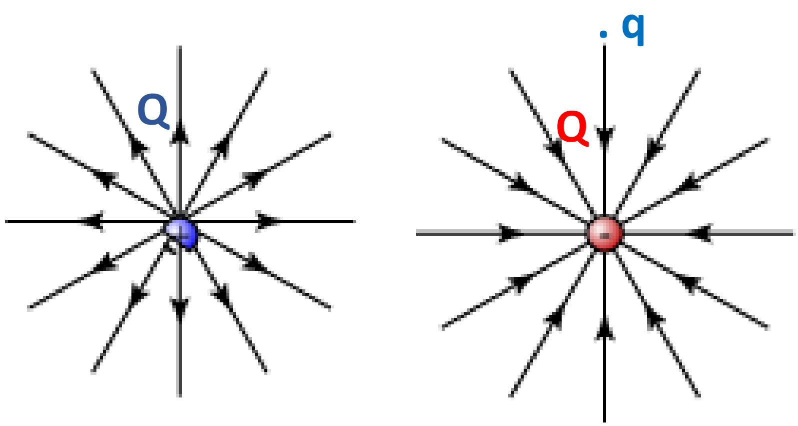

Fig. Superposition Law

**Net force on *q*: F = $F_1$ + $F_2$ + $F_3$**



# Electric field (vector)

A stationary distribution of charges produces an electric field **E** in vacuum



Field pattern of
a pointed electrode

**Vector:** $\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$

**Vector:** $\vec{F} = \frac{qQ}{4\pi\varepsilon_0 r^2}\hat{r}$ : ***Coulomb's law***

$\vec{E}=\vec{F}/q$

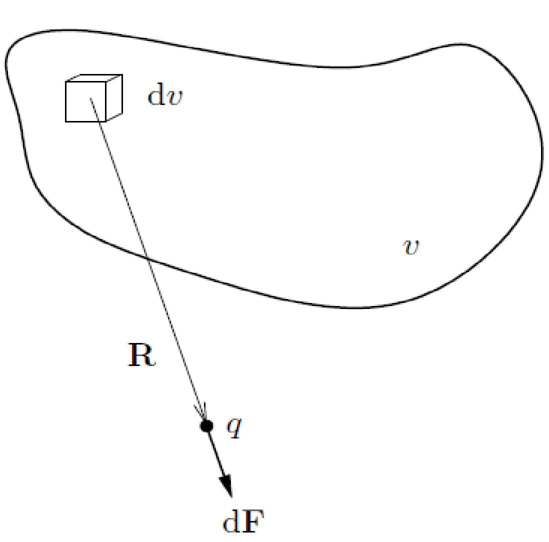
$\boldsymbol{\varepsilon_0}$ is vacuum permittivity (physical Property) $\varepsilon_0$ = 8.85e-12 F/m

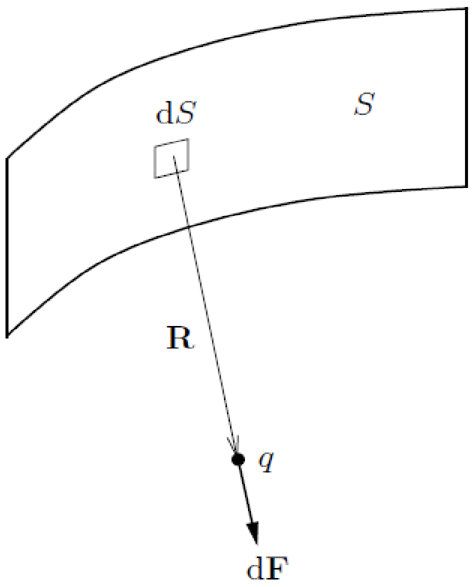
**Electric displacement field D and D= $\varepsilon_0$E.**

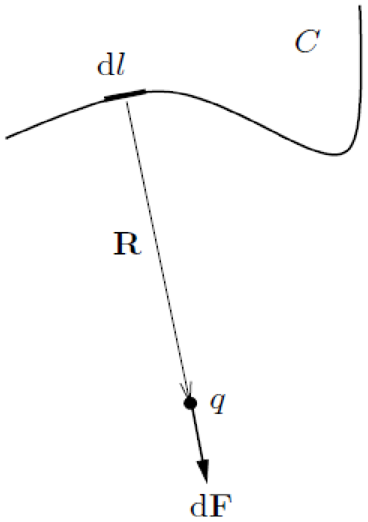
**Vector:** $\vec{D} = \frac{Q}{4\pi r^2}\hat{r}$

Electric displacement field: the equation is material independant.

# Superposition: Vector **E**




$$\mathbf{E} = \int_v \frac{\hat{\mathbf{R}}\rho \mathrm{d}v}{4\pi\epsilon_0 R^2}.$$

$$\mathbf{E} = \int_S \frac{\hat{\mathbf{R}}\rho_{\mathrm{s}} \mathrm{d}S}{4\pi\epsilon_0 R^2}.$$

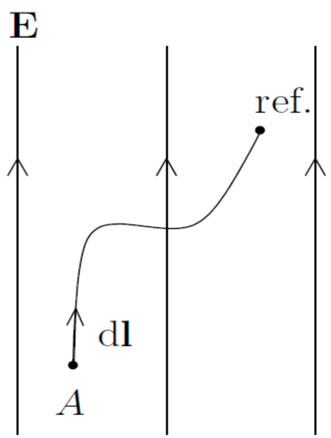
$$\mathbf{E} = \int_C \frac{\hat{\mathbf{R}}Q' \mathrm{d}l}{4\pi\epsilon_0 R^2}.$$

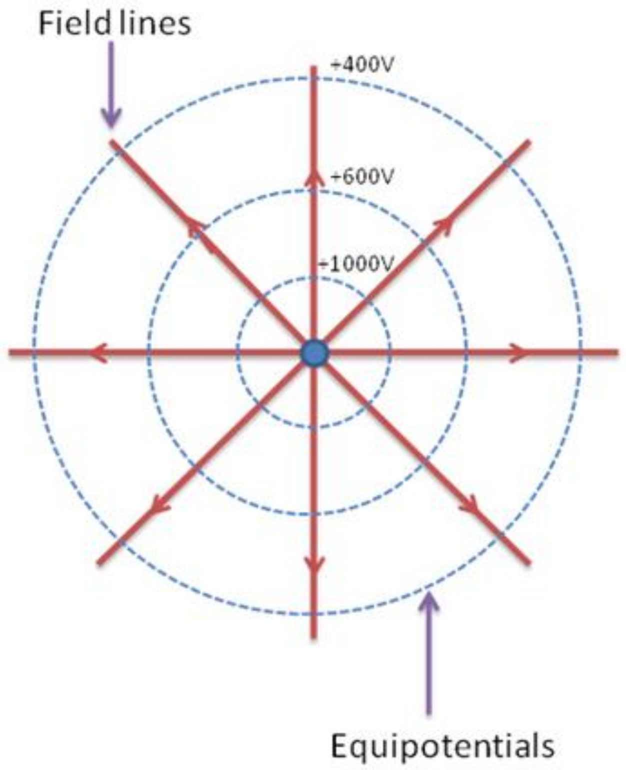
# Electric potential ( Voltage)

Electric potential definition $V = \frac{W}{q} = \frac{\int \boldsymbol{F.dl}}{q} = -\int \boldsymbol{E.dl}$

**Vector:** $\vec{\boldsymbol{F}} = \frac{qQ}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}}$ : $\boldsymbol{Coulomb's\ law}$

**Vector:** $\vec{\boldsymbol{E}} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}}$




$V_A = \int_A^{\text{ref.}} \mathbf{E} \cdot d\mathbf{l}.$ The potential difference between two points A and B:

$$V_{AB} = -\int_{rA}^{rB} \boldsymbol{E.dl} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\varepsilon_0 {r_i}^2} dr_i = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$

# Electric field and potential ( voltage)

- Potential is a scalar.
- $V = -\int \boldsymbol{E}.\boldsymbol{dl}$

$$\nabla \boldsymbol{f} = \frac{\partial f}{\partial x}\hat{\boldsymbol{x}}+\frac{\partial f}{\partial y}\hat{\boldsymbol{y}}+\frac{\partial f}{\partial z}\hat{\boldsymbol{z}}$$

$$\boxed{\mathbf{E} = -\nabla V,\ \mathrm{V/m}}$$

$$\nabla V \equiv \mathrm{grad}\, V$$

$$\mathrm{V}= \int_{\boldsymbol{r}}^{\infty} \boldsymbol{E} d\boldsymbol{l}=\int_{\boldsymbol{r}}^{\infty} \frac{Q}{4\pi\varepsilon_0 l^2} d\boldsymbol{l}=\frac{Q}{4\pi\varepsilon_0 r}$$

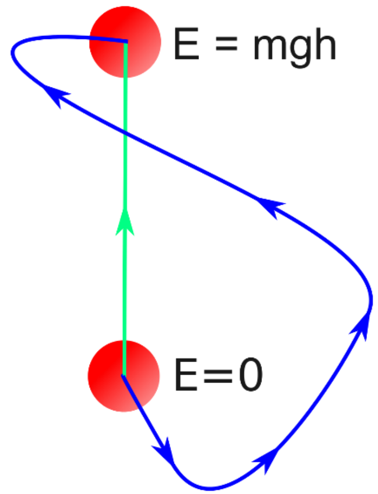
$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}}$$

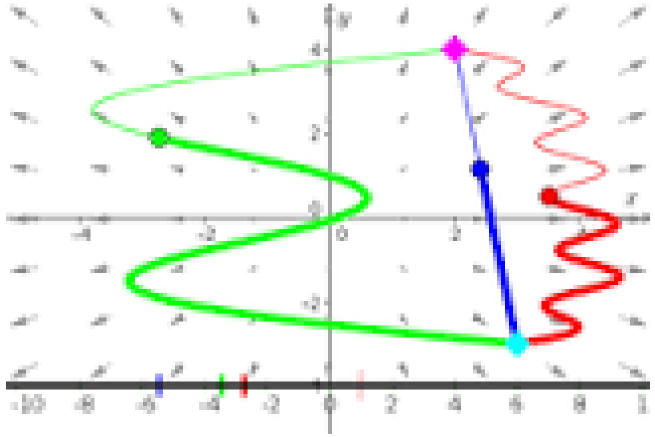
# Conservative vector field: stationary electric field

Stationary Electric field is a conservative vector.

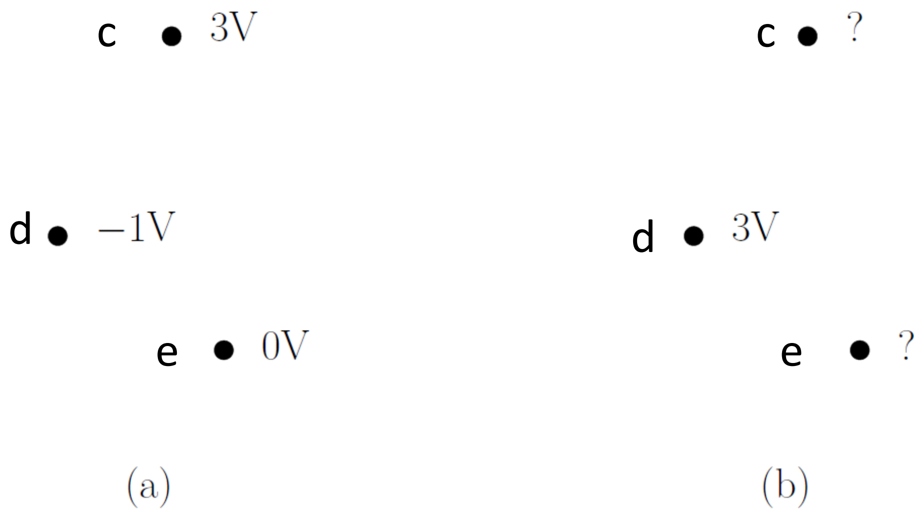
. Conservative vector fields have the property that the line integral is path independent.

. A conservative vector field is also Ir-rotational. In three dimensions , it has vanishing curl. $\nabla \times f=0$.



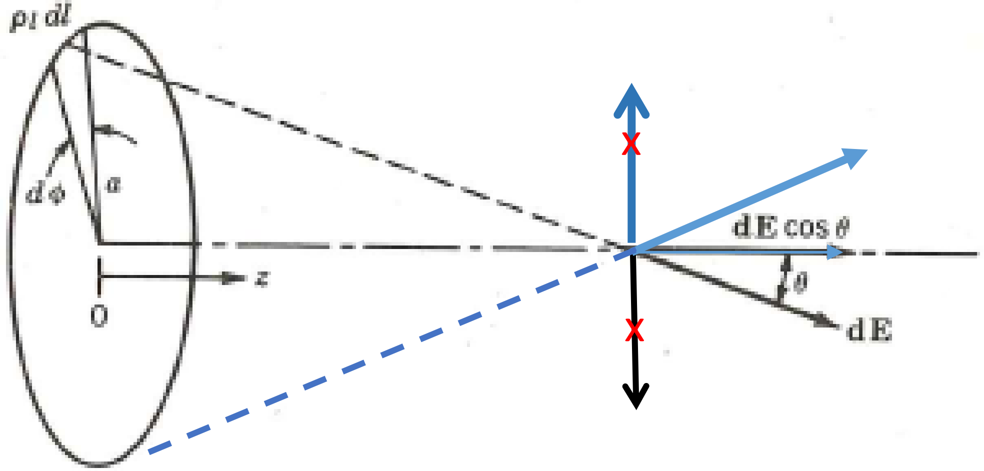
# Example: find out the uknown potentials




in a stationaly electric field,  the potential at each point is listed in a)
in the same stationary field  and the potential at point d is 3V with another reference point. what are the potential that are missed in b).

# Example: field of a ring of charge

Charge ring with radius *a* and line charge density: ρl
Let calculate E at pont on the z axis.




$$\vec{E} = \sum_{i=1}^{n} \frac{q_i}{4\pi\varepsilon_0 r_i^2} \hat{r}_i$$

$$\sum_{i=1}^{n} q_i = \oint \rho_l dl = \int_0^{2\pi a} \rho_l a d\phi$$

$dl = a d\phi$

$$E = \int_0^{2\pi a} \rho_l a d\phi \frac{.\cos\theta}{4\pi\varepsilon r^2} = \int_0^{2\pi a} \frac{\rho_l z a \;\; d\phi}{4\pi\varepsilon (a^2+z^2)^{3/2}} = \frac{\rho_l a z}{2\varepsilon (a^2+z^2)^{3/2}}$$

$r^2 = a^2 + z^2$

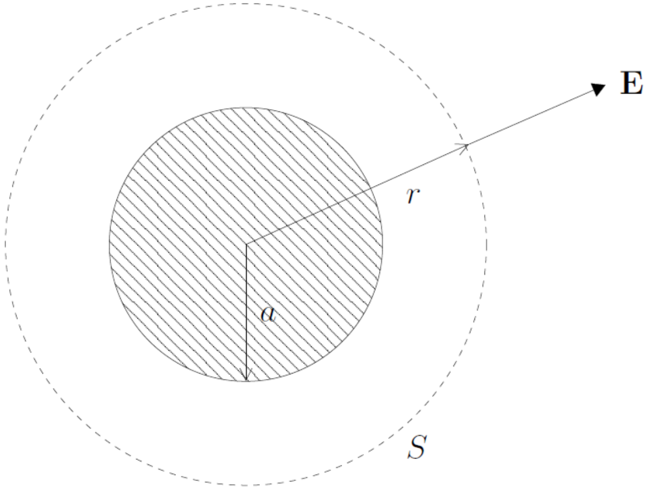
$\cos\theta = z \big/ (a^2+z^2)^{\frac{1}{2}}$

The resultant E at z-axis is also in the direction of z-axis.

# Example: electric field around a charged ball

What is the electric field outside a charged ball with total charge Q? ( r>>a )

$$\text{dvs. } \mathbf{E} = E\hat{\mathbf{r}}$$



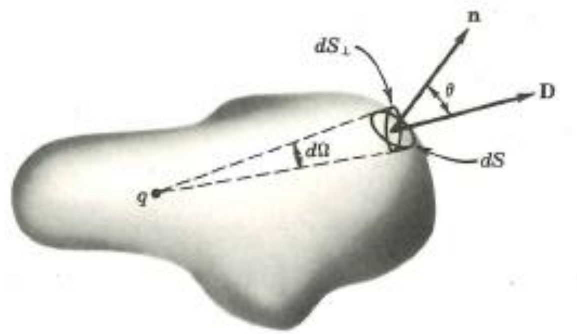

**Vector:** $\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r},\quad$ ( r>>a)

# Gauss’ law

**Electric flux flowing out of a closed surface = Total charge enclosed** divided by the permittivity of the medium

$$\oint E d\boldsymbol{s} = \frac{Q}{\varepsilon}$$

**Vector:** $\vec{\boldsymbol{E}} = \frac{\boldsymbol{Q}}{\boldsymbol{4\pi r^2 \varepsilon_0}}$ in vacuum, $\vec{\boldsymbol{D}} = \frac{\boldsymbol{Q}}{\boldsymbol{4\pi r^2}}$



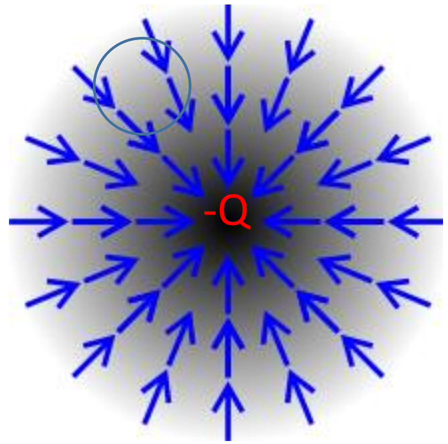

$$\nabla \cdot E = \frac{\rho}{\varepsilon}$$

**Electric displacement field D = εE.**

D is independent of medium:

$$\epsilon_0 \oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = Q_{\text{total i } S}$$

- $\nabla \cdot D = \rho$

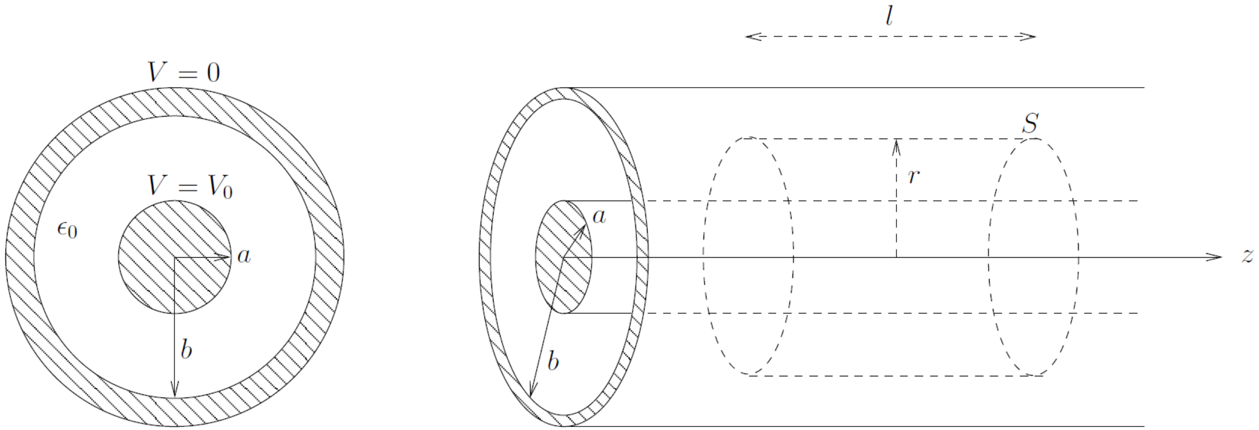



$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \int_V \nabla \cdot \mathbf{E} dV.$$

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \sum_i \oint_{S_i} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \sum_i \left(\frac{1}{\Delta V_i} \oint_{S_i} \mathbf{E} \cdot \mathrm{d}\mathbf{S}_i\right) \Delta V_i \to \int \nabla \cdot \mathbf{E} dV.$$

$$\oint \boldsymbol{E} d\boldsymbol{s} = \oint \frac{-\boldsymbol{Q}}{\boldsymbol{4\pi r^2 \varepsilon_0}} d\boldsymbol{s} = \frac{-\boldsymbol{Q}}{\boldsymbol{\varepsilon_0}}$$

# Example:coaxial cable




Assumming : infinite long cable
Calculating electric field E between the out surface of inner cable and inner surface of the outer cable.

Assuming Unit length and the surface charge density is $\rho_s$

$$\epsilon_0 \oint_S \mathbf{E}\cdot d\mathbf{S} = \epsilon_0 E 2\pi r l$$

Circumference $\rho_s 2\pi a l = Q' l = Q$

$$\mathbf{E} = \begin{cases} \frac{Q'}{2\pi\epsilon_0 r}\hat{\mathbf{r}}, & \text{for } a < r < b \\ 0, & \text{ellers.} \end{cases}$$

$$V_0 = \int_a^b \mathbf{E}\cdot d\mathbf{l} = \int_a^b E dr = \frac{Q'}{2\pi\epsilon_0}\int_a^b \frac{dr}{r} = \frac{Q'}{2\pi\epsilon_0}\ln\frac{b}{a}.$$

$$\mathbf{E} = \begin{cases} \frac{V_0}{r\ln\frac{b}{a}}\hat{\mathbf{r}}, & \text{for } a < r < b \\ 0, & \text{ellers.} \end{cases}$$

# Poisson equation

Inserting $\mathbf{E} = -\nabla V$ into Maxwell's equation $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ gives

$$-\nabla \cdot (\nabla V) = \frac{\rho}{\epsilon_0},$$

which gives

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}.$$

This is called Poisson's equation. Here

$$\nabla^2 =< \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2} >$$

# Point, line, surface and space charge

## Potential from point/line/surface/space charge

We let $\mathbf{r}_{\text{ref}} \to \infty$. We list the potentials from

a) A point charge:

$$V = \int_R^\infty \frac{Q}{4\pi\epsilon_0 r^2}\mathrm{d}r = \frac{Q}{4\pi\epsilon_0 R},$$

b) A line charge (line charge density $Q'$):

$$V = \int_C \frac{Q'}{4\pi\epsilon_0 R}\mathrm{d}l,$$

c) A surface charge (surface charge density $\rho_s$):

$$V = \int_S \frac{\rho_s}{4\pi\epsilon_0 R}\mathrm{d}S,$$

d) A space charge (charge density $\rho$):

$$V = \int_V \frac{\rho}{4\pi\epsilon_0 R}\mathrm{d}V.$$

The last three is found by superposition from the point charge expression.

# Electric field in dielectric

Force in vaccum: $\vec{F}_{tot} = \sum_{i=1}^{n} \frac{q q_i}{4\pi\varepsilon_0 r_i^2} \hat{r}_i$

**Vector:** $\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2} \hat{r}$

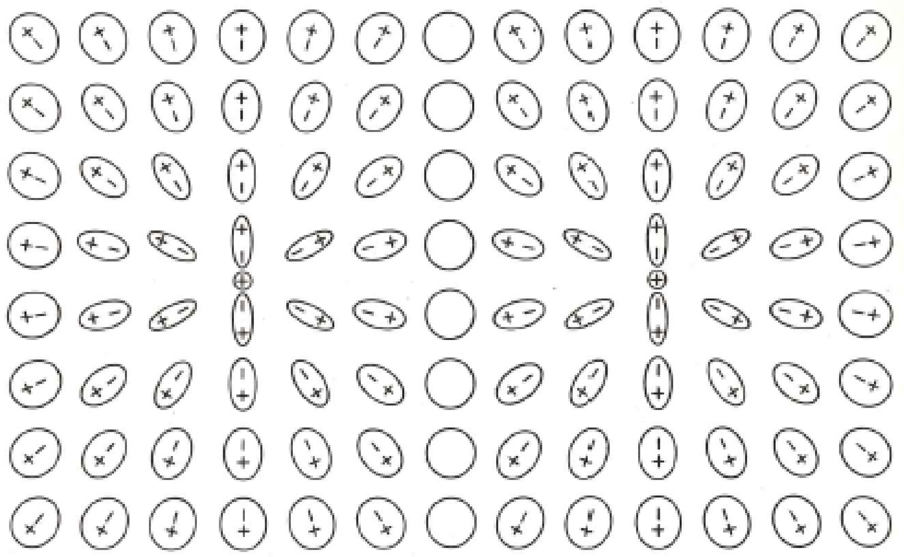


Fig. 1.3c Polarization of the atoms of a dielectric by a pair of equal positive charges.

When a dielectric material is placed in an electric field:

1: electric charges do not flow through the material, but instead they shift slightly from their average equilibrium positions, causing dielectric polarization:

2: positive charges are displaced in the direction of the field and negative charges shift in the direction opposite to the field. This creates an internal electric field that reduces the overall field within the dielectric itself

Electric polarization produced by the electric field in a material dependent.

Polarization reduces the overall field within the dielectric material.

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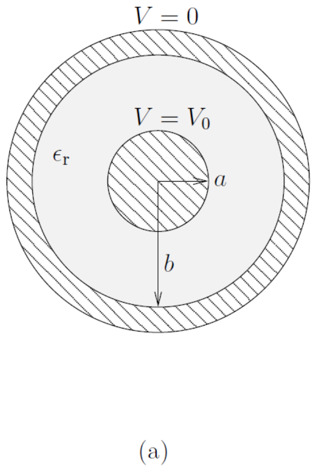
# Electric polarization and material permittivity

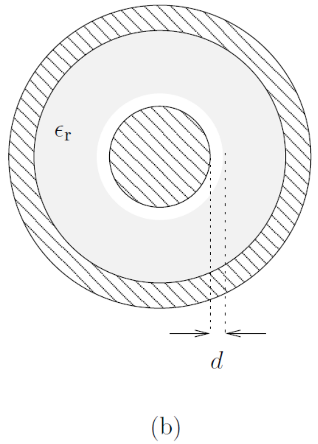
The influence of electric polarization: $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$

- $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$
- $\boldsymbol{P} = \varepsilon_0 \chi_e \boldsymbol{E}$
- $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \varepsilon_0 \chi_e \boldsymbol{E} = (1 + \chi_e)\varepsilon_0 \boldsymbol{E}$ **:** $\chi_e$ *called* ***electric susceptibility***

- $(1 + \chi_e) = \varepsilon_r$ relative permittivity:
- $\varepsilon = \varepsilon_r \varepsilon_0$ electric permittivity .( dielectric material property)

# Example: coaxial cable with dielectric material and airgap




Assumming : infinite long cable

Calculating electric field E between the out surface of inner cable and inner surface of the outer cable.

a) The permittivity is $\epsilon = \epsilon_r \epsilon_0$

b) when $\epsilon(r) = \begin{cases} \epsilon_0, & \text{for } a < r < a+d, \\ \epsilon_r \epsilon_0, & \text{for } a+d < r < b. \end{cases}$

a) D is material independant: $\mathbf{D} = \frac{Q'}{2\pi r}\hat{\mathbf{r}} \quad \text{for } a < r < b,$

$$\mathbf{E} = \frac{Q'}{2\pi \epsilon r}\hat{\mathbf{r}} \quad \text{for } a < r < b.$$

$$\mathbf{E} = \frac{V_0}{r \ln \frac{b}{a}}\hat{\mathbf{r}} \quad \text{for } a < r < b.$$

b) D is material independant:

$$\epsilon(r) = \begin{cases} \epsilon_0, & \text{for } a < r < a+d, \\ \epsilon_r \epsilon_0, & \text{for } a+d < r < b. \end{cases}$$

$$V_0 = \int_a^b E(r)\mathrm{d}r = \frac{Q'}{2\pi}\int_a^b \frac{\mathrm{d}r}{\epsilon(r) r} = \frac{Q'}{2\pi\epsilon_0}\left(\ln\frac{a+d}{a} + \frac{1}{\epsilon_r}\ln\frac{b}{a+d}\right)$$

$$\mathbf{E} = \begin{cases} \frac{\epsilon_r V_0}{r\left(\epsilon_r \ln\frac{a+d}{a} + \ln\frac{b}{a+d}\right)}\hat{\mathbf{r}}, & \text{for } a < r \leq a+d \\ \frac{V_0}{r\left(\epsilon_r \ln\frac{a+d}{a} + \ln\frac{b}{a+d}\right)}\hat{\mathbf{r}}, & \text{for } a+d < r < b. \end{cases}$$