TFE4120 Electromagnetism: crash course

Intensive course: Two-weeks

Lecturer: Shunguo Wang, shunguo.wang@ntnu.no

Assistant: Sandra Yuste Munoz, Sandra.y.munoz@ntnu.no

Participants: should have Bsc in electronic, electrical/ power engineering, etc.

Aim of the course: Give students a minimum pre-requisite to follow a 2-year master program in electronics or electrical/power engineering.

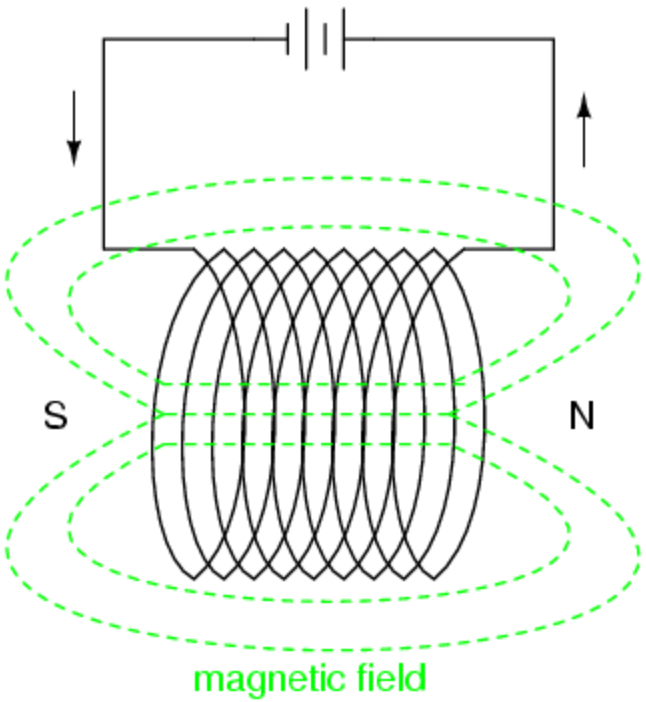
Information is posted there:
https://www.ntnu.no/wiki/display/tfe4120/Crash+Course+in+Electromagnetism+2023

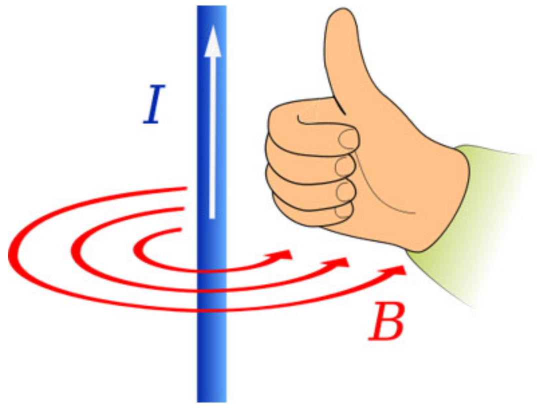
Content for lectures

- Lecture 1: Introduction and vector calculus
- Lecture 2: Electro-statics
- Lecture 3: Electro-statics
- Lecture 4: Magneto-statics
- Lecture 5: Electro-dynamics
- Lecture 6: Electro-magnetics

Lecture1: Electro-magnetism introduction and vector calculus

1) What does electro-magnetism describe?
2) Brief induction about Maxwell equations
3) Electric force: Coulomb's law
4) Vector calculus



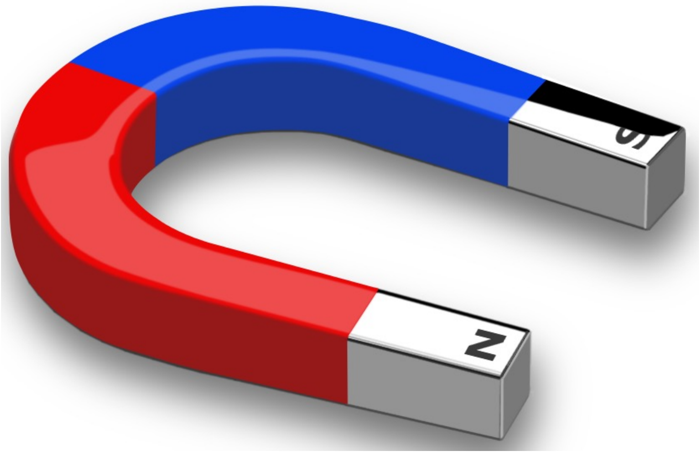


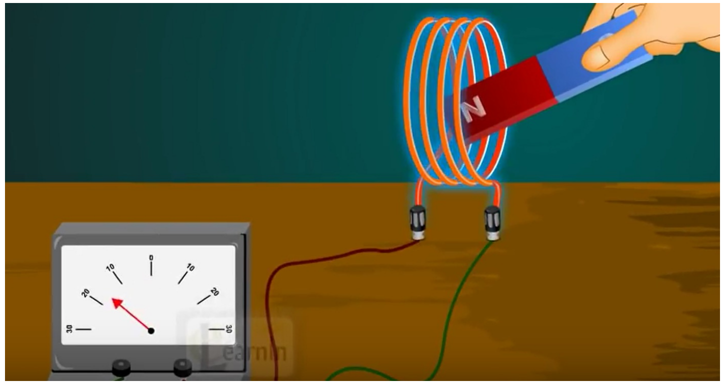
Electro-magnetism

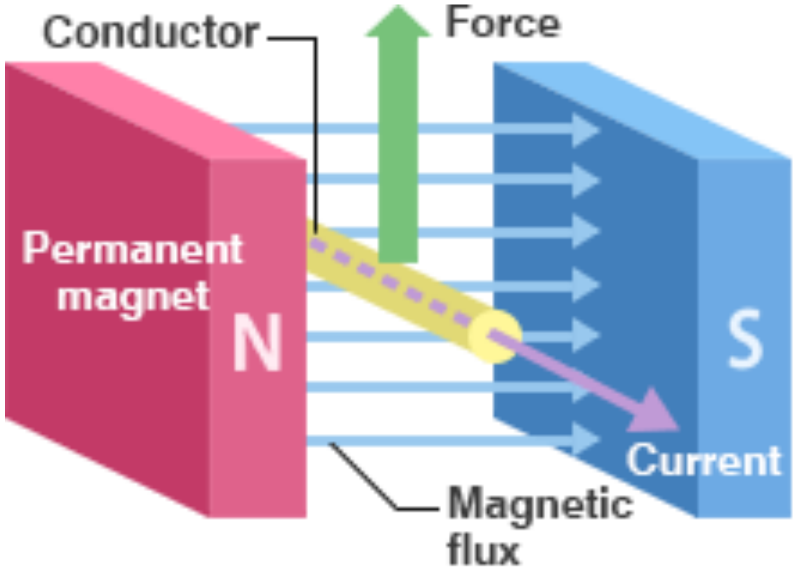
Originally, electricity and magnetism were considered to be two separate phenomena.

Electro-magnetism: Physical interaction between electricity and magnetism.

Electromagnetic force: One of the four fundamental interactions in the nature (gravitation, electromagnetism, the strong and weak forces).







History

Carl Friedrich Gauss (1777-1855): German mathematician and physicist

The electric flux out of a closed surface = total enclosed charge divided by the permittivity of free space

Electrostatics

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{H} = 0$$

Carl Friedrich Gauss

Andre-Marie Ampere (1775-1836): French physicist and mathematician

The magnetic field produced by an electric current is proportional to the magnitude of the current times a constant equal to the permeability of free space (μ_0)

Magnetostatics

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

André-Marie Ampère

Michael Faraday (1791-1867): English Scientist

In 1831 Faraday observed that a moving magnet could induce a current in a circuit and a changing current could, through its magnetic effects, induce a current to flow in another circuit.

Magneto-dynamics

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Michael Faraday
FRS

Founder of electromagnetism

James Clerk Maxwell: (1839-1879)

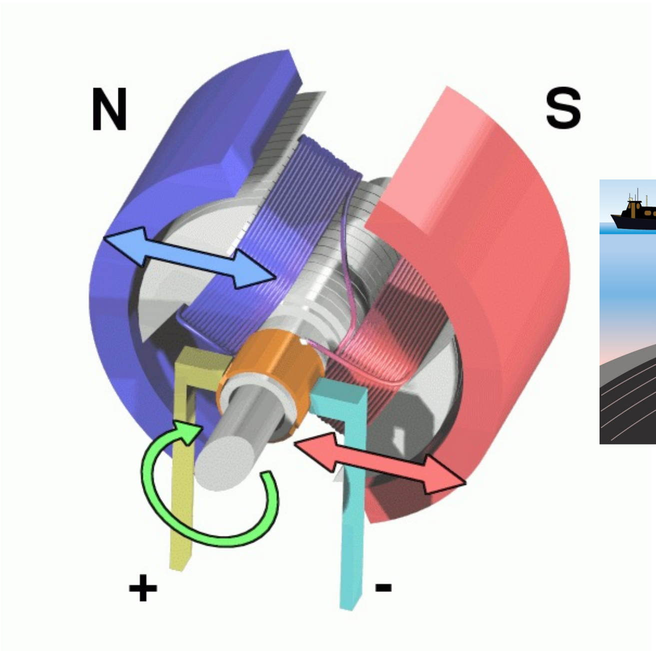
Scottish mathematician

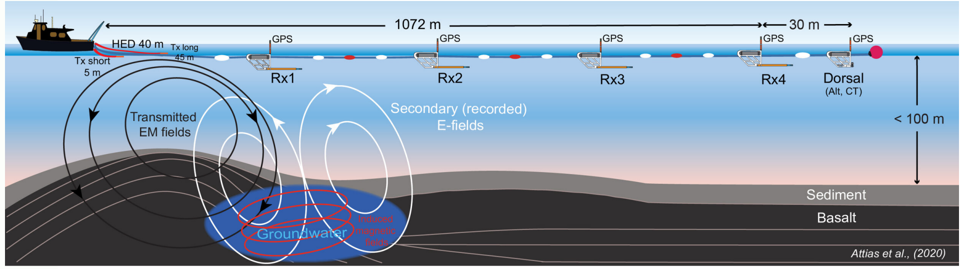
- Developed a scientific theory to explain electromagnetic fields and waves
- Coupled the electrical fields and magnetic fields together
- Established the foundations of electricity and magnetism as **electromagnetism**
- Maxwell's equations

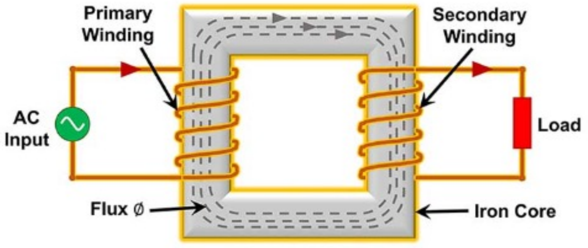
Daily life applications

For example:

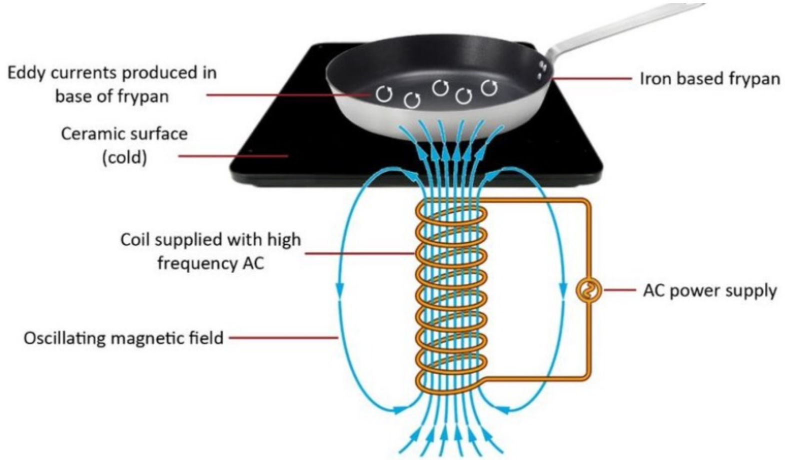
- Electric motor/generator
- Water/minerals exploration
- Transformer
- Induction oven
- ……





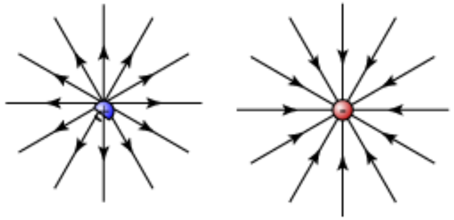


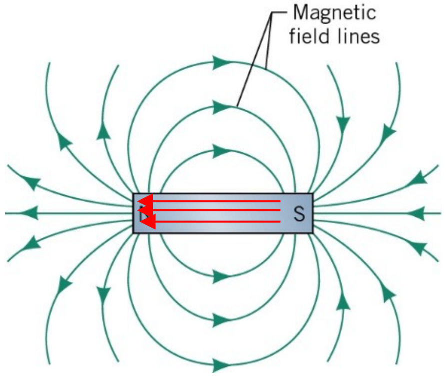
Basic Construction of Transformer



Electromagnetism: Maxwell's equations

- A static electric charges produces an electric field
- There is no magnetic charge (monopole)
- A *changing* magnetic field produces an electric field
- Charges in *motion* (an electrical current) produce a magnetic field
- A changing electric field produces a magnetic field





Electric and magnetic fields can produce forces on charges

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0},$$
$$\nabla \cdot \mathbf{B} = 0,$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$
$$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}.$$

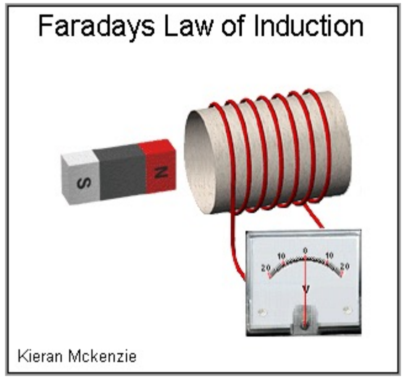
E: electric field, vector

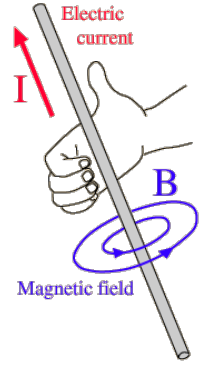
D: electric flux density, vector

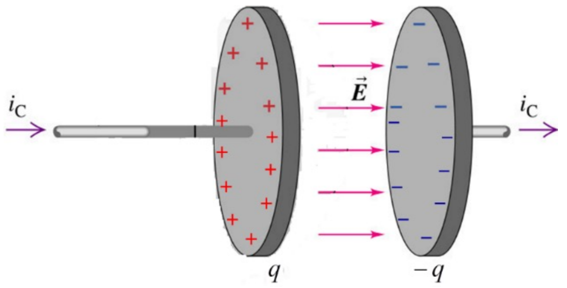
H: magnetic field, vector

B: magnetic flux density, vector

J: current density, vector
ρ: static charge density



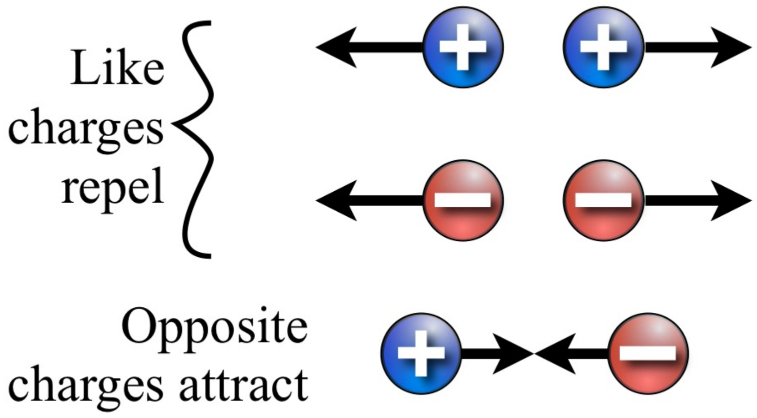




Electricity and magnetism had been unified into electromagnetism!

Coulomb's law: force between electrostatic charges

Published in 1785 by French physicist Charles-Augustin de Coulomb and was essential to the development of the theory of electromagnetism



Scalar: $F = k\frac{q_1 q_2}{r_{12}^2} = \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}^2}$

Vector: $\vec{F} = \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}^2}\widehat{r_{12}}$

$\widehat{r_{12}}$ is for direction, its absolute value is 1

$k = 9\times10^9 \mathrm{Nm^2/C^2}$, Coulomb's constant

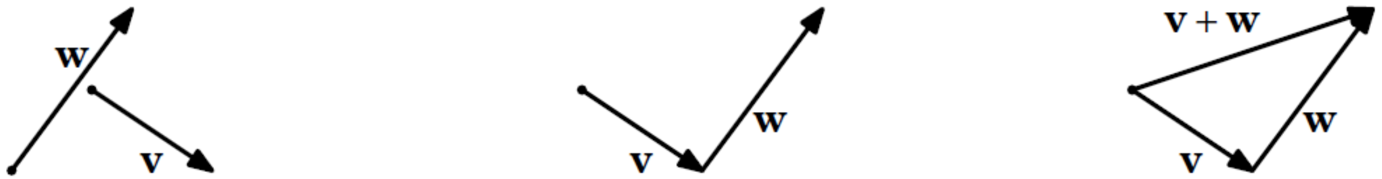
The electrostatic force had the same functional form as Newton's law of gravity

The magnitude of the electrostatic force between two-point charges:

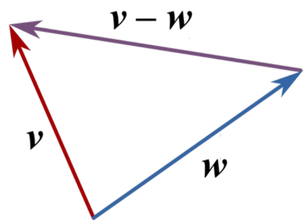
1) Directly proportional to the product of the magnitudes of charges
2) Inversely proportional to the square of the distance between them
3) The force is along the straight line joining them

Vector calculus:

$\mathbf{v} + \mathbf{w}$

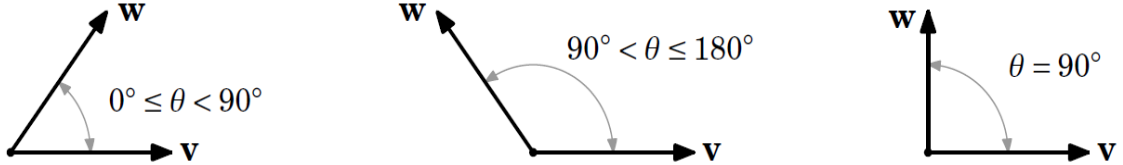


(a) Vectors **v** and **w** (b) Translate **w** to the end of **v** (c) The sum **v** + **w**



Dot product

$$\mathbf{v} \cdot \mathbf{w} = \cos\theta \, \|\mathbf{v}\| \, \|\mathbf{w}\|$$

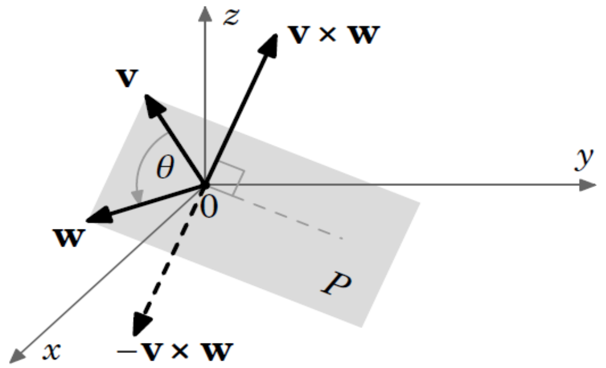


(a) $\mathbf{v} \cdot \mathbf{w} > 0$ (b) $\mathbf{v} \cdot \mathbf{w} < 0$ (c) $\mathbf{v} \cdot \mathbf{w} = 0$

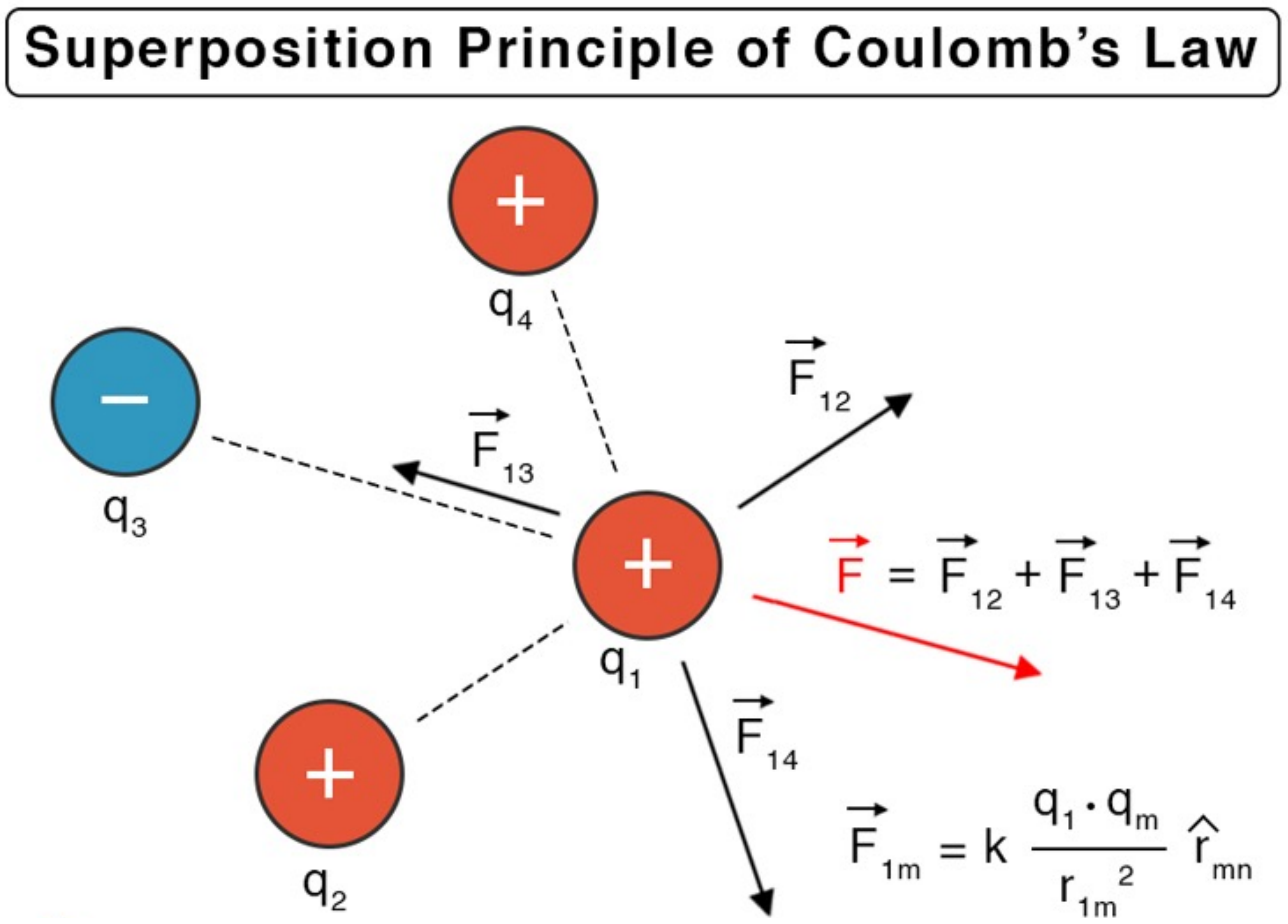
V = (1, 0)
W = (0, 1)

Cross product

$$\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\| \, \|\mathbf{w}\| \sin\theta$$



Vector force



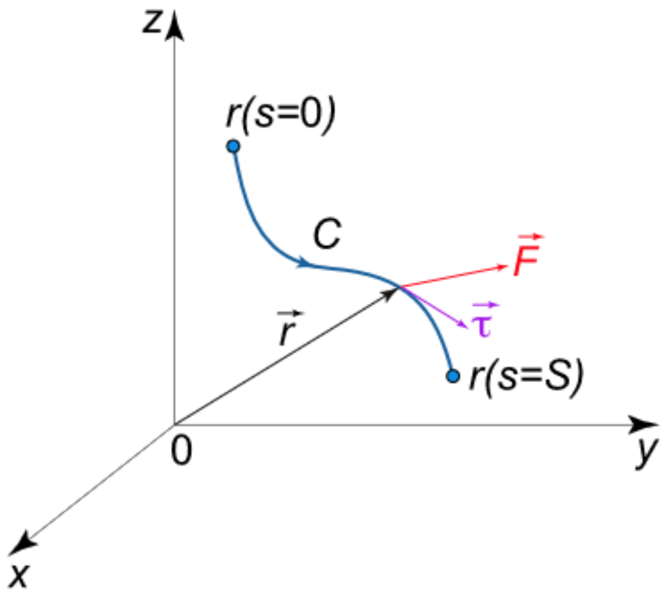
$$\vec{F}_{tot} = \sum_{i=1}^{n} \frac{q q_i}{4\pi\varepsilon_0 r_i^2} \hat{r}_i$$

$$\vec{F}_{tot} = \sum_{i=2}^{4} \frac{q_1 q_i}{4\pi\varepsilon_0 r_i^2} \hat{r}_i$$

Line integral of vector

Line integral of vector force **F** along the curve **C**.
Suppose that a curve **C** is defined by the vector function r=r(s), 0≤s≤S, where s is the arc length of the curve. Then the derivative of the vector function

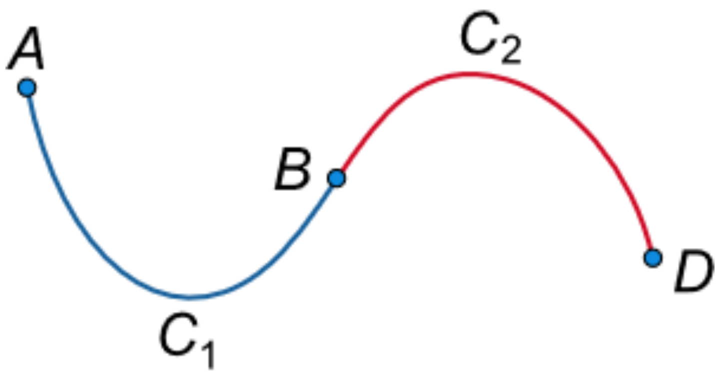
$$\frac{d\boldsymbol{r}}{ds} = \boldsymbol{\tau}$$ **(Tangent direction at each point of the curve)**



Curve direction is important.

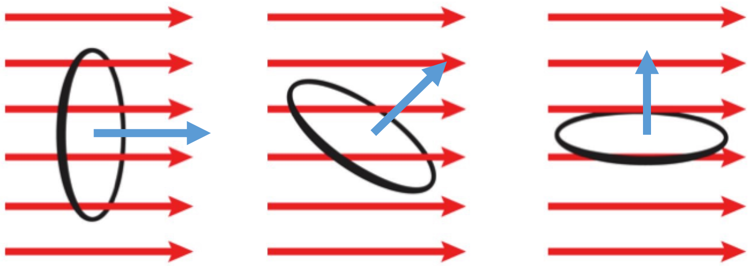
$$\int_C (\mathbf{F}\cdot d\mathbf{r}) = \int_0^S (\mathbf{F}(\mathbf{r}(s))\cdot\boldsymbol{\tau})\, ds,$$

$$\int_C (\mathbf{F}\cdot d\mathbf{r}) = \int_{C_1\cup C_2} (\mathbf{F}\cdot d\mathbf{r}) = \int_{C_1} (\mathbf{F}\cdot d\mathbf{r}) + \int_{C_2} (\mathbf{F}\cdot d\mathbf{r});$$



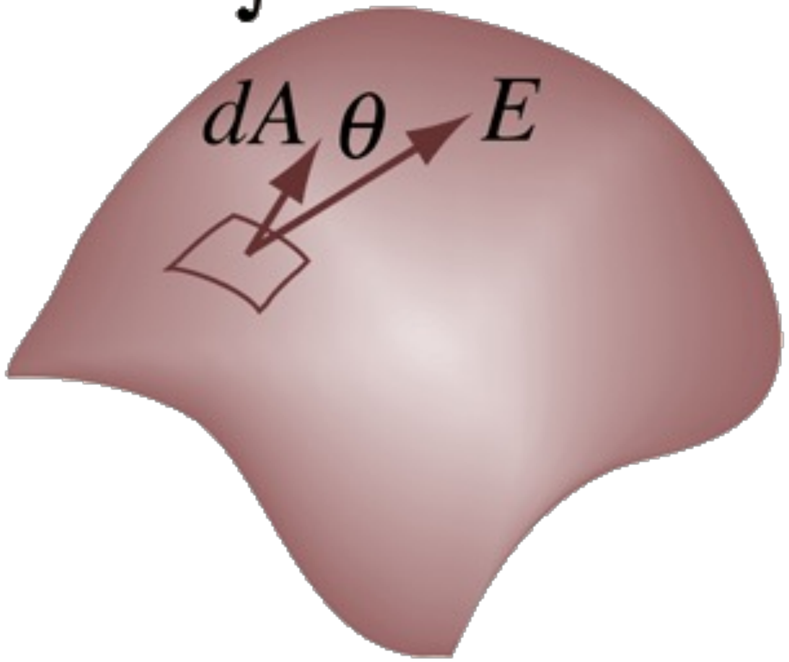
Surface integral of vector

d**A** direction is perpendicular to the tangent plane to that surface at A



Max flux Less flux No flux given.

$$\int \vec{E} \cdot d\vec{A} = \int E \cos\theta \, dA$$



$$\iint_S \mathbf{E} \cdot d\mathbf{A} = \iint_S \mathbf{E} \cdot \hat{\mathbf{n}} \, dA$$

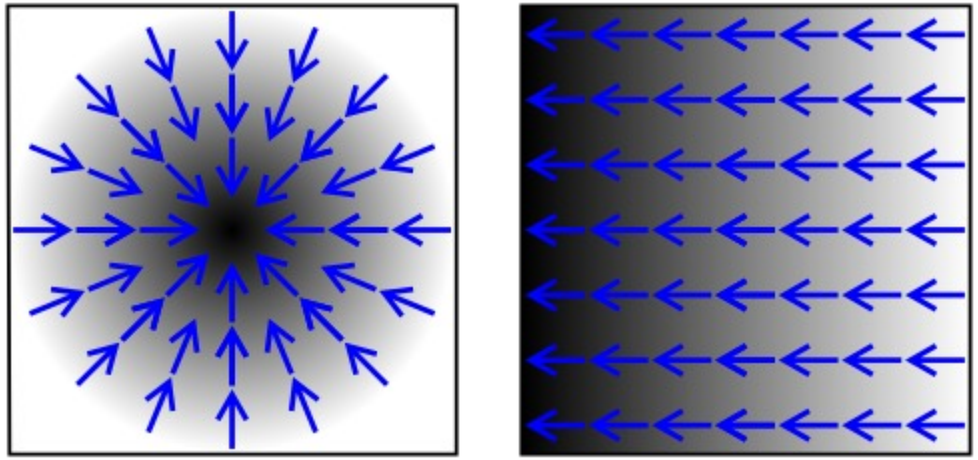
Gradient

Gradient: 3-dimension derivative of a scalar function

Showing the **direction and rate of of fastest increase** of the scalar function *f* at a point space

How quickly something changes from one point to another

$$\nabla f = \frac{\partial f}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}}$$



f: Scalar function

∇f (Gradient): Vector function

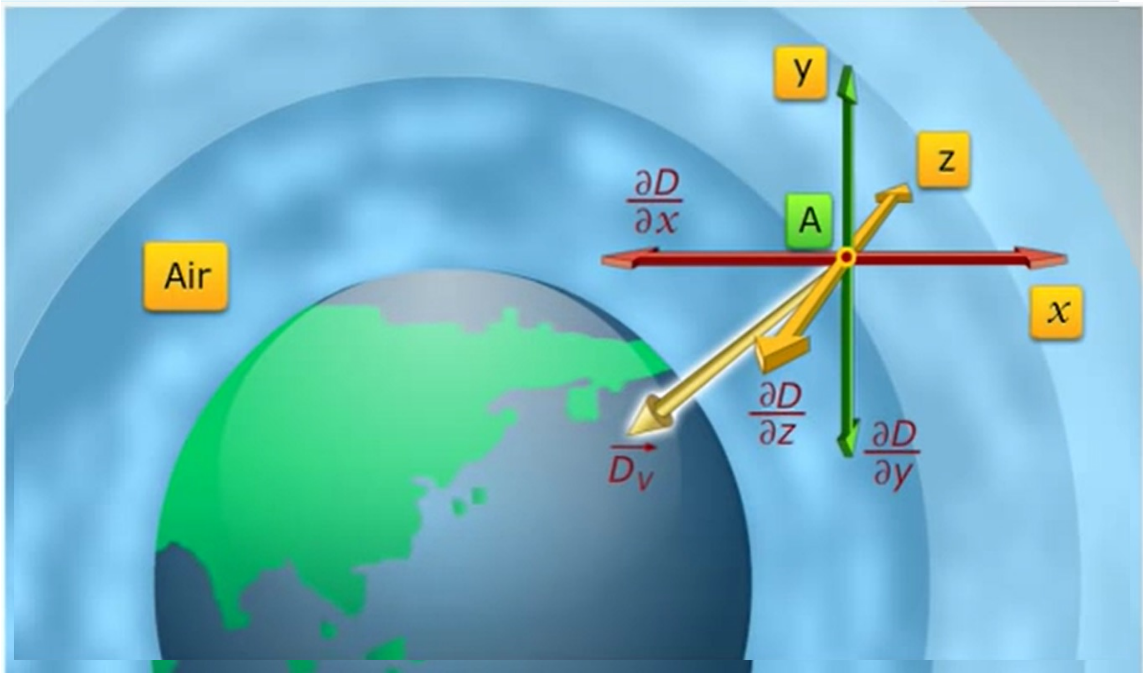
Direction: Rate of the fastest increase

Example: air density in the space

$$D = f(x, y, z)$$

$$\nabla = \frac{\partial}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial}{\partial z}\hat{\boldsymbol{z}}$$

$$\nabla D = \frac{\partial D}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial D}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial D}{\partial z}\hat{\boldsymbol{z}}$$



Maximum rate at which the Density Increases

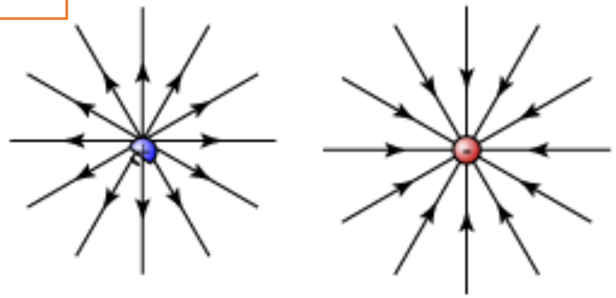
Divergence: Flux/field out of a point

Divergence represents the volume density of the outward flux of a vector field from an infinitesimal volume around a given point

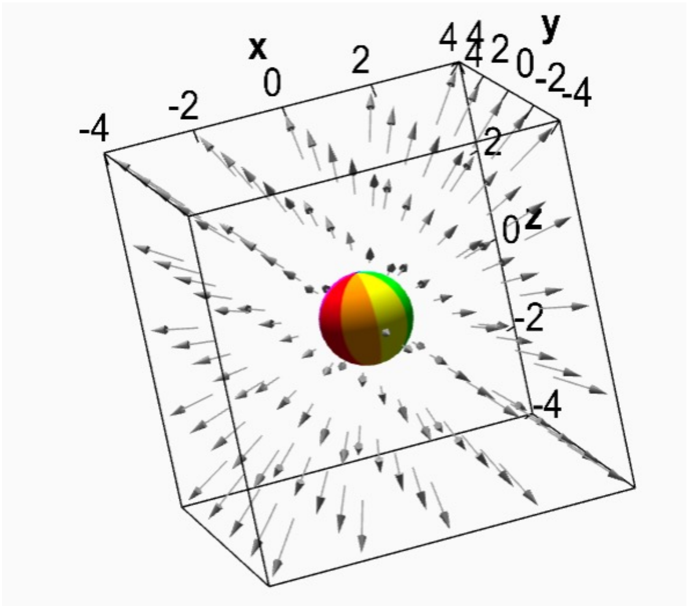
$$\operatorname{div}\mathbf{F}\big|_{\mathbf{x_0}} = \lim_{V\to 0}\frac{1}{|V|}\oiint_{S(V)}\mathbf{F}\cdot\hat{\mathbf{n}}\,dS$$

$$\text{div}\mathbf{F} = \boldsymbol{\nabla}\cdot\mathbf{F} = \frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}$$

$\mathbf{F}$ is a vector
$\boldsymbol{\nabla}\cdot\mathbf{F}$ is a scalar

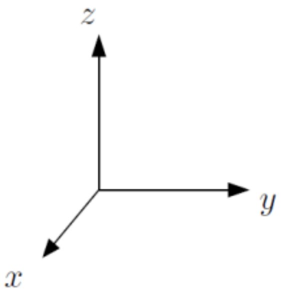


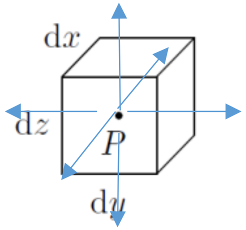
Field pattern of
a pointed electrode



Divergence: mathematical calculation

$$\boxed{\operatorname{div}\mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A}\cdot \mathrm{d}\mathbf{S}}{\Delta v}.}$$





$\mathbf{A} = (A_x, A_y, A_z)$

$P = (x_0, y_0, z_0)$

$$\oint_S \mathbf{A}\cdot \mathrm{d}\mathbf{S} = \oint_S \mathbf{A}\cdot \hat{\mathbf{n}}\mathrm{d}S$$

dydz *is total area*

$$\oint_S \mathbf{A}\cdot \mathrm{d}\mathbf{S} = \int_{\text{foran}} A_x(\text{foran})\mathrm{d}y\mathrm{d}z - \int_{\text{bak}} A_x(\text{bak})\mathrm{d}y\mathrm{d}z$$
$$- \int_{\text{venstre}} A_y(\text{venstre})\mathrm{d}x\mathrm{d}z + \int_{\text{høyre}} A_y(\text{høyre})\mathrm{d}x\mathrm{d}z$$
$$+ \int_{\text{topp}} A_z(\text{topp})\mathrm{d}x\mathrm{d}y - \int_{\text{bunn}} A_z(\text{bunn})\mathrm{d}x\mathrm{d}y.$$

$$A_x(\text{foran}) - A_x(\text{bak}) = A_x(x_0 + \mathrm{d}x/2, y_0, z_0) - A_x(x_0 - \mathrm{d}x/2, y_0, z_0) = \frac{\partial A_x}{\partial x}\mathrm{d}x$$

$$A_y(\text{høyre}) - A_y(\text{venstre}) = \frac{\partial A_y}{\partial u}\mathrm{d}y$$

$$A_z(\text{topp}) - A_z(\text{bunn}) = \frac{\partial A_z}{\partial z}\mathrm{d}z.$$

⇩

$$\oint_S \mathbf{A}\cdot \mathrm{d}\mathbf{S} = \frac{\partial A_x}{\partial x}\mathrm{d}x\mathrm{d}y\mathrm{d}z + \frac{\partial A_y}{\partial y}\mathrm{d}x\mathrm{d}y\mathrm{d}z + \frac{\partial A_z}{\partial z}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$\boxed{\operatorname{div}\mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A}\cdot \mathrm{d}\mathbf{S}}{\Delta v}.}$$

$\Delta v = \mathrm{d}x\mathrm{d}y\mathrm{d}z$

⇨

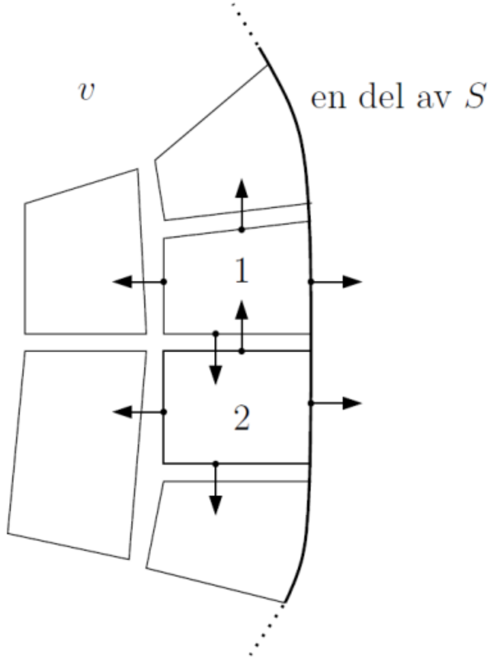
$$\boxed{\operatorname{div}\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \nabla\cdot\mathbf{A}.}$$

Divergence theorem

It states that the surface integral of a vector field over a closed surface is equal to the volume integral of the divergence over the region inside the surface.

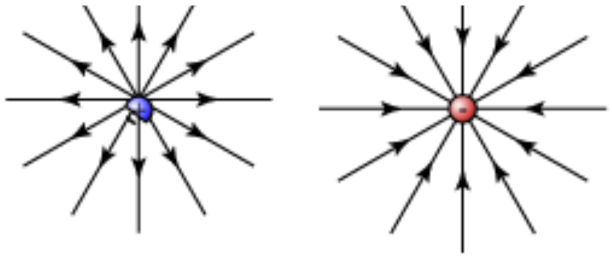
$$\oint_S \mathbf{A} \cdot \mathrm{d}\mathbf{S} = \int_v \nabla \cdot \mathbf{A} \mathrm{d}v$$

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \sum_i \oint_{S_i} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \sum_i \left(\frac{1}{\Delta V_i} \oint_{S_i} \mathbf{E} \cdot \mathrm{d}\mathbf{S}_i\right) \Delta V_i \rightarrow \int \nabla \cdot \mathbf{E} \mathrm{d}V$$



$$\nabla \cdot E = \frac{\rho}{\varepsilon_0}$$

Gauss's Law



Field pattern of
a pointed electrode

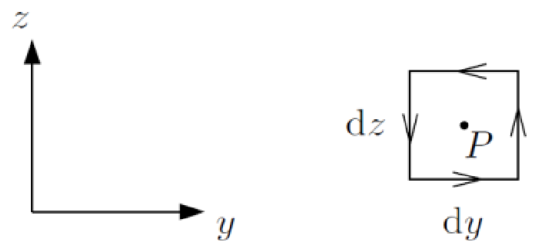
Curl

The curl of a field presents the infinitesimal circulation density at each point of the field

How much does a field circulate around a point

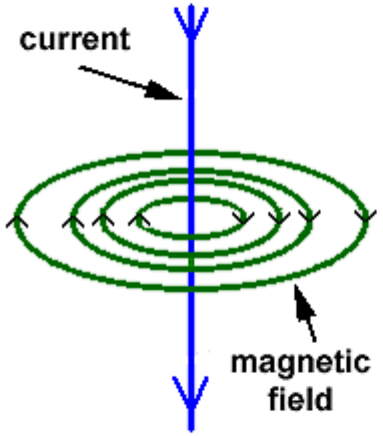
One dimension x

$$(\text{curl}\,\mathbf{A})_x = \lim_{\Delta S \to 0} \frac{\oint_C \mathbf{A} \cdot \mathrm{d}\mathbf{l}}{\Delta S}$$



Three dimensions x, y and z

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\boldsymbol{z}}$$



Curl

The curl around x-axis, in yz plane

$$\oint_C \mathbf{A}\cdot \mathrm{d}\mathbf{l} = \int_{\text{nede}} A_y(\text{nede})\mathrm{d}y - \int_{\text{oppe}} A_y(\text{oppe})\mathrm{d}y - \int_{\text{venstre}} A_z(\text{venstre})\mathrm{d}z + \int_{\text{høyre}} A_z(\text{høyre})\mathrm{d}z$$

$$A_y(\text{nede}) - A_y(\text{oppe}) = A_y(x_0, y_0, z_0 - \mathrm{d}z/2) - A_y(x_0, y_0, z_0 + \mathrm{d}z/2) = -\frac{\partial A_y}{\partial z}\mathrm{d}z,$$

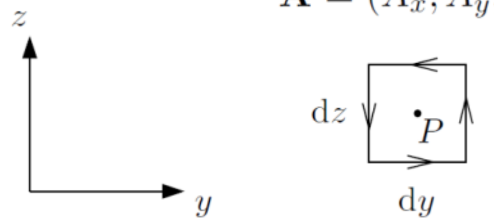
$$A_z(\text{høyre}) - A_z(\text{venstre}) = A_z(x_0, y_0 + \mathrm{d}y/2, z_0) - A_z(x_0, y_0 - \mathrm{d}y/2, z_0) = \frac{\partial A_z}{\partial y}\mathrm{d}y,$$

$$\oint_C \mathbf{A}\cdot \mathrm{d}\mathbf{l} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathrm{d}y\mathrm{d}z,$$

$$(\text{curl}\,\mathbf{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}.$$

$P = (x_0, y_0, z_0)$

$\mathbf{A} = (A_x, A_y, A_z)$



$\Delta S = \mathrm{d}y\mathrm{d}z$

Similar to the curl around y and z-axis

$$\nabla\times\mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\boldsymbol{z}}$$

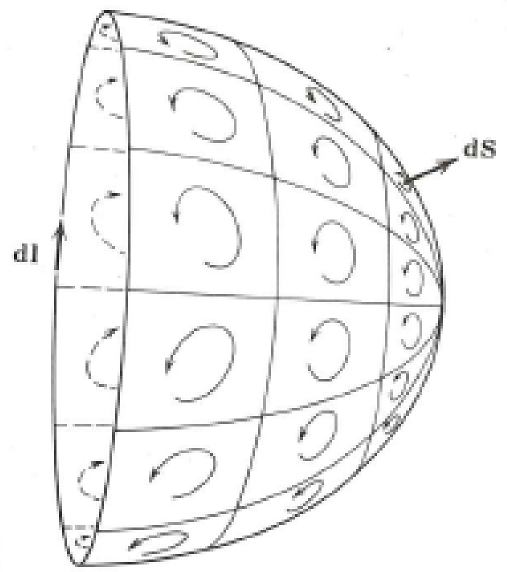
Stokes' theorem

Given a vector field, the theorem relates the integral of the curl of the vector field over a surface, to the line integral of the vector field around the boundary of the surface.

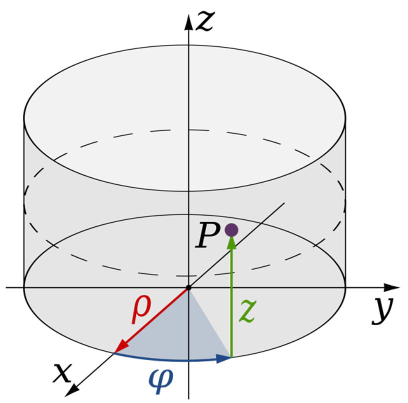
$$\oint_C \mathbf{A} \cdot \mathrm{d}\mathbf{l} = \int_S \nabla \times \mathbf{A} \cdot \mathrm{d}\mathbf{S}.$$

$$\int_S \nabla \times \mathbf{A} \cdot \mathrm{d}\mathbf{S} = \sum_i \oint_{C_i} \mathbf{A} \cdot \mathrm{d}\mathbf{l}.$$

$$\oint_C \mathbf{E} \cdot \mathrm{d}\mathbf{l} = \sum_i \oint_{C_i} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = \sum_i \left(\frac{\oint_{S_i} \mathbf{E} \cdot \mathrm{d}\mathbf{S}_i}{\Delta S_i}\right)\Delta S_i \rightarrow \int \nabla \times \mathbf{E} \cdot \mathrm{d}\mathbf{S}$$

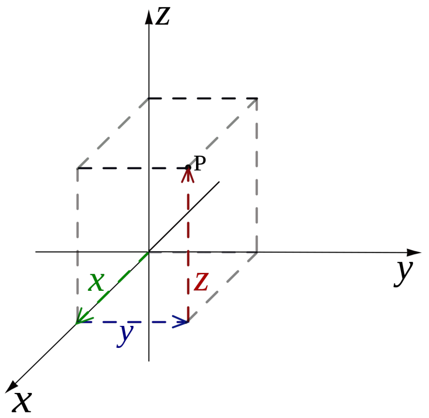


Different coordinates



Cylindrical coordinate

$$x = \rho\cos\varphi$$
$$y = \rho\sin\varphi$$
$$z = z$$



Cartesian Coordinate

$$\nabla = \frac{\partial}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial}{\partial z}\hat{\boldsymbol{z}}$$

$$\nabla D = \frac{\partial D}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial D}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial D}{\partial z}\hat{\boldsymbol{z}}$$

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

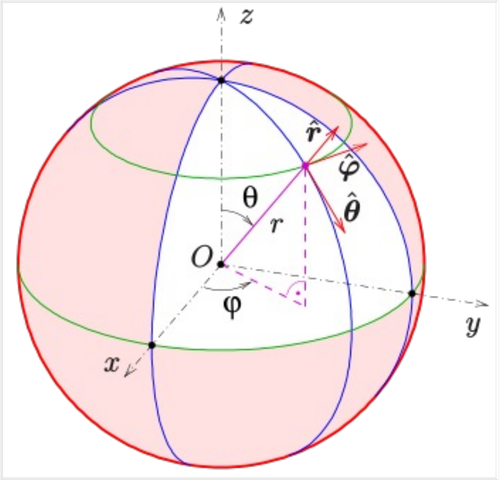
$$\nabla f = \frac{\partial f}{\partial \rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}}$$

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\boldsymbol{z}}$$

$$\nabla \cdot \boldsymbol{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \boldsymbol{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\boldsymbol{\varphi}} + \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}(\rho A_\varphi) - \frac{\partial A_\rho}{\partial \varphi}\right)\hat{\boldsymbol{z}}$$

Different coordinates



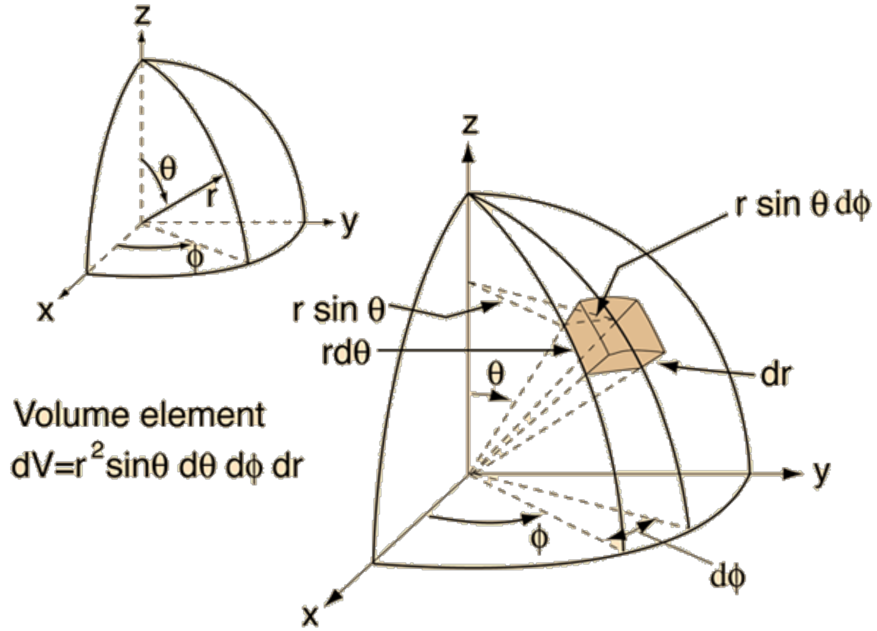
Spherical coordinate

$$x = r\sin\theta\,\cos\varphi,$$
$$y = r\sin\theta\,\sin\varphi,$$
$$z = r\cos\theta.$$

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}},$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi},$$

$$\nabla\times\mathbf{A} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\left(A_\varphi\sin\theta\right) - \frac{\partial A_\theta}{\partial\varphi}\right)\hat{\mathbf{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial}{\partial r}\left(rA_\varphi\right)\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{\varphi}},$$



$$x^2 + y^2 + z^2 = a^2 \quad \Rightarrow \quad r = a$$

Example 1:

Calculate the integral

$$I = \int_V (\nabla \cdot \mathbf{F}) \mathrm{d}V \tag{1}$$

where $\mathbf{F} = r\hat{\mathbf{r}} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$, and the volume V is a sphere with radius R placed in the origin.

a) Calculate the integral directly.

b) Calculate the integral using the divergence theorem.

$$\mathrm{div}\,\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \nabla \cdot \mathbf{A}.$$

$$\oint_S \mathbf{A} \cdot \mathrm{d}\mathbf{S} = \int_v \nabla \cdot \mathbf{A} \mathrm{d}v$$

Solution for a

Calculate the integral

$$I = \int_V (\nabla \cdot \mathbf{F}) \mathrm{d}V \tag{1}$$

where $\mathbf{F} = r\hat{\mathbf{r}} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$, and the volume V is a sphere with radius R placed in the origin.

a) Calculate the integral directly.

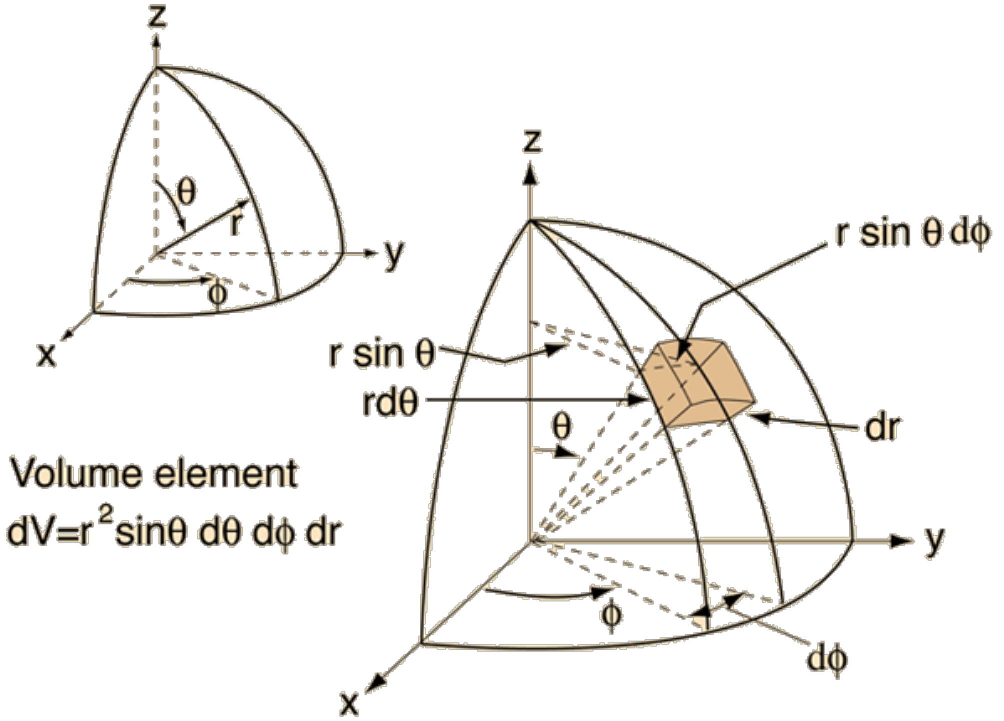
b) Calculate the integral using the divergence theorem.

$$\nabla \cdot \mathbf{F} = 3.$$

$$\mathrm{div}\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \nabla \cdot \mathbf{A}.$$

$$\int_v (\nabla \cdot \mathbf{F}) \mathrm{d}V = \int_0^R \int_0^{2\pi} \int_0^{\pi} 3r^2 \sin\theta \mathrm{d}\varphi \mathrm{d}\theta \mathrm{d}r$$
$$= 3 \int_0^R r^2 \mathrm{d}r \int_0^{\pi} \sin\theta \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi$$
$$= \underline{\underline{4\pi R^3}}.$$

$$\underline{\underline{\int_V (\nabla \cdot \mathbf{F}) \mathrm{d}V = 3 \int_V \mathrm{d}V = 4\pi R^3.}}$$



Solution for b

$$\mathbf{F} = r\hat{\mathbf{r}} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

$$\oint_S \mathbf{A} \cdot \mathrm{d}\mathbf{S} = \int_v \nabla \cdot \mathbf{A} \mathrm{d}v$$

$$\begin{aligned}
\int_v (\nabla \cdot \mathbf{F}) \mathrm{d}v &= \oint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} \\
&= \int_0^{2\pi} \int_0^{\pi} (R\hat{\mathbf{r}}) \cdot (R^2 \sin\theta \mathrm{d}\theta \mathrm{d}\varphi \hat{\mathbf{r}}) \\
&= R^3 \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \sin\theta \mathrm{d}\theta \\
&= \underline{\underline{4\pi R^3}}.
\end{aligned}$$

Example 2:

Calculate the line integral

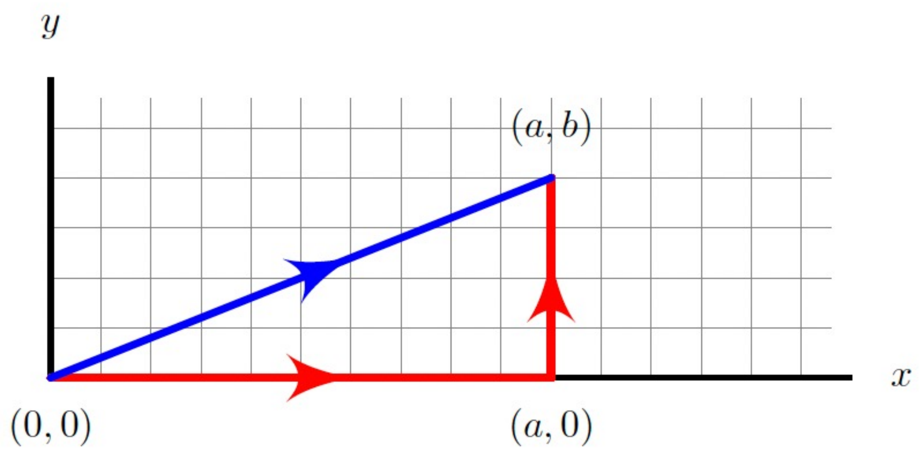
$$I = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{l}, \qquad (2)$$

Where $\mathbf{F} = (xy^2 + 2y)\hat{\mathbf{x}} + (x^2y + 2x)\hat{\mathbf{y}}$,

i) Along the curve C_1 which consists of two straight lines connecting the points $(0,0)$, $(a,0)$ and (a,b), see figure below.

ii) Along the curve C_2 which consists of one straight line connecting the points $(0,0)$ and (a,b), see figure below.

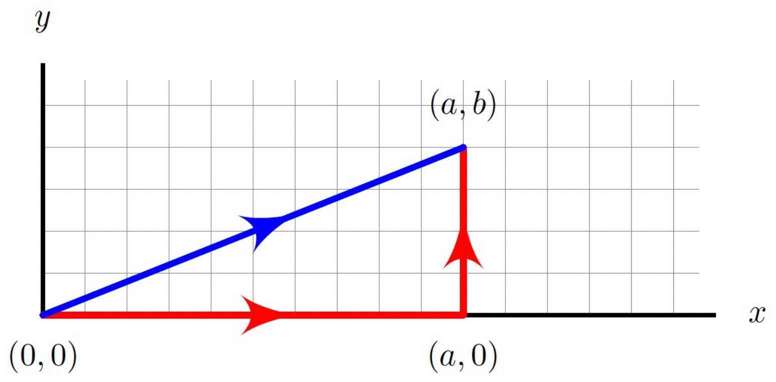
iii) Why do these calculations produce the same answer? Explain using Stoke's theorem.



$$\oint_C \mathbf{A} \cdot \mathrm{d}\mathbf{l} = \int_S \nabla \times \mathbf{A} \cdot \mathrm{d}\mathbf{S}.$$

$$\nabla \times A = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}$$

Solution for i) and ii)



$$I = \int_C \mathbf{F}\cdot \mathrm{d}\mathbf{l}$$

$$\mathbf{F} = (xy^2 + 2y)\hat{\mathbf{x}} + (x^2 y + 2x)\hat{\mathbf{y}}$$

$$\mathrm{d}\mathbf{l} = \mathrm{d}x\hat{\mathbf{x}} + \mathrm{d}y\hat{\mathbf{y}}.$$

$$\begin{aligned}\mathbf{F}\cdot\mathrm{d}\mathbf{l} &= F_x \mathrm{d}x + F_y \mathrm{d}y \\ &= (xy^2 + 2y)\mathrm{d}x + (x^2 y + 2x)\mathrm{d}y.\end{aligned}$$

$$y = \frac{bx}{a}$$

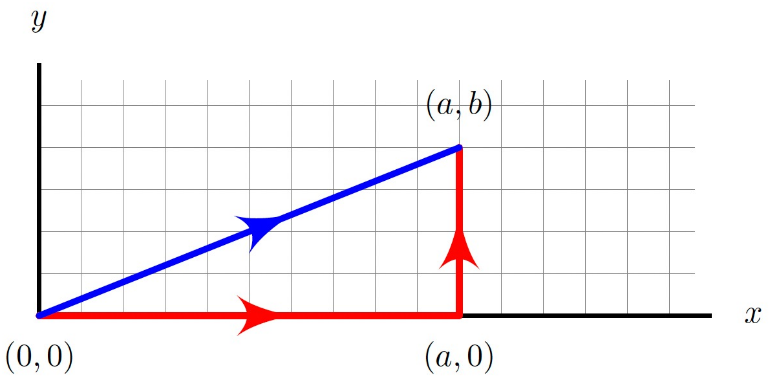
i) Along the curve C_1 which consists of two straight lines connecting the points $(0,0)$, $(a,0)$ and (a,b), see figure below.

$$\begin{aligned}I &= \int_C \mathbf{F}\cdot\mathrm{d}\mathbf{l} \\ &= \int_0^b (a^2 y + 2a)\mathrm{d}y \\ &= \underline{\underline{\frac{1}{2}a^2b^2 + 2ab.}}\end{aligned}$$

ii) Along the curve C_2 which consists of one straight line connecting the points $(0,0)$ and (a,b), see figure below.

$$\begin{aligned}I &= \int_C \mathbf{F}\cdot\mathrm{d}\mathbf{l} \\ &= \int_0^a \left[x\left(\frac{bx}{a}\right)^2 + 2\left(\frac{bx}{a}\right)\right]\mathrm{d}x + \int_0^b \left[\left(\frac{ay}{b}\right)^2 y + 2\left(\frac{ay}{b}\right)\right]\mathrm{d}y \\ &= \underline{\underline{\frac{1}{2}a^2b^2 + 2ab.}}\end{aligned}$$

Solution for iii)



$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\boldsymbol{z}}$$

iii) Why do these calculations produce the same answer? Explain using Stoke's theorem.

$$\mathbf{F} = (xy^2 + 2y)\hat{\mathbf{x}} + (x^2y + 2x)\hat{\mathbf{y}}$$

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{A} \cdot d\mathbf{S}.$$

These integrals have equal values since F is a conservative field:

$$\begin{aligned} \nabla \times \mathbf{F} &= \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\mathbf{z}} \\ &= (2xy - 2xy)\,\hat{\mathbf{z}} \\ &= 0. \end{aligned}$$