# TFE4120 Electromagnetism: crash course

Intensive course: Two-weeks.

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Exercises help:

Paticipants: should have Bsc in electronic, electrical/ power engineering.

Aim of the course: Give students a minimum pre-requisity to follow a 2-year master program in electronics or electrical /power engineering.

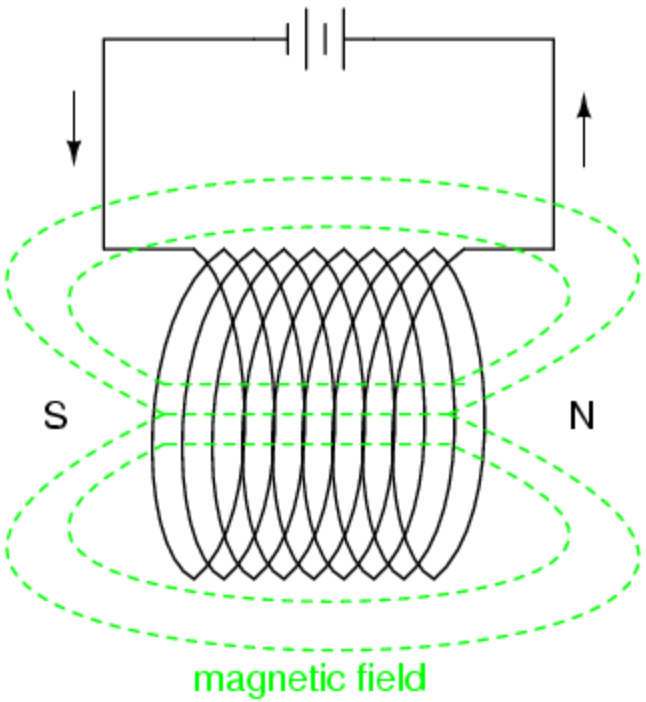
Webpage: All information is posted there .
https://www.ntnu.no/wiki/display/tfe4120/Crash+course+in+Electromagnetics+2022

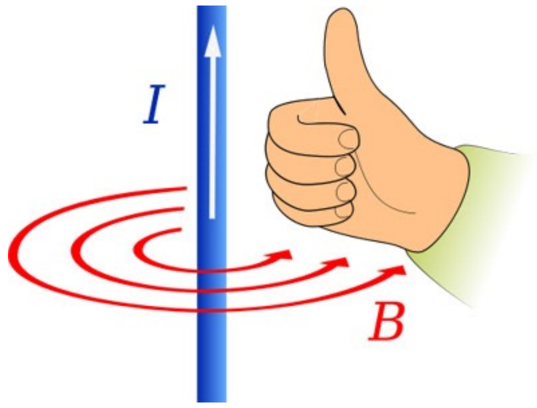
# Content for lectures:

- Lecture 1: Introduction and Vector calculus
- Lecture 2: Electro-static
- Lecture 3: Electro-dynamic
- Lectrue 4: Magnetic-static
- Lecture 5: Electro-magnetic
- Lecture 6: Electro-magnetic wave

# Lecture1: Electro-magnetism and vector calculus

1) What does electro-magnetism describe?
2) Brief induction about Maxwell equations
3) Electric force: Coulomb's law
4) Vector calculus



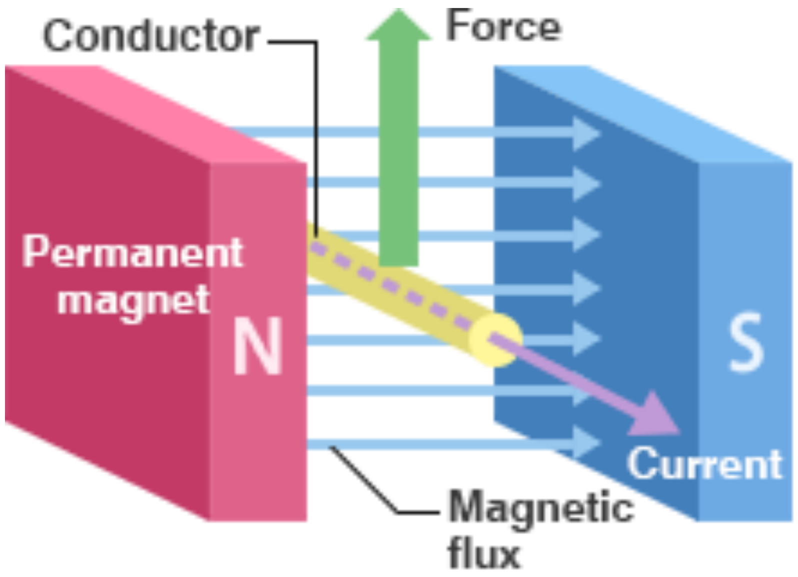



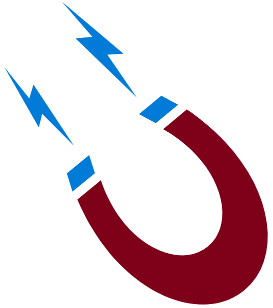
# Electro-magnetism

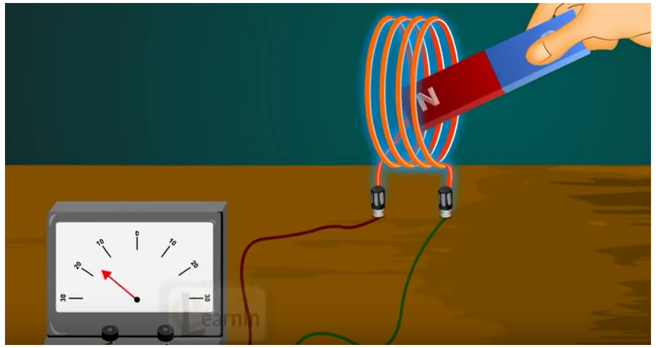
Originally, electricity and magnetism were considered to be two separate phenomena

Electro-magnetism: Physical interaction between electricity and magnetism.

Electro-magnetic force: one of the four fundamental interactions in the nature.  (gravitation, electromagnetism, the strong  and weak forces)





# History:

**Carl Friedrich Gauss (1777-1855): German mathematician and physicist**

The electric flux out of a closed surface = total enclosed charge divided by the permittivity of free space

Electrostatic

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{in}}{\varepsilon_0}$$

**Andre-Marie Ampere ( 1775-1836): French physicist and mathematician**

The magnetic field produced by an electric current is proportional to the magnitude of the current with a proportionality constant equal to the permeability of free space ($\mu_o$)

Magnetostatic

$$\oint \vec{B}.\vec{dl} = \mu_o I$$

**Michael Faraday (1791-1867): English Scientist**

- In 1831 Faraday observed that a moving magnet could induce a current in a circuit.
- He also observed that a changing current could, through its magnetic effects, induce a current to flow in another circuit.

*Magnetodynamic* $V = -\frac{d\emptyset}{dt}$

Carl Friedrich Gauss

André-Marie Ampère

Michael Faraday

FRS

# Founder of electromagnetism

**James Clerk Maxwell: (1839-1879)**

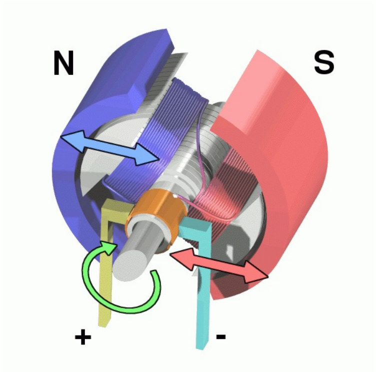
**Scottish Mathematician**

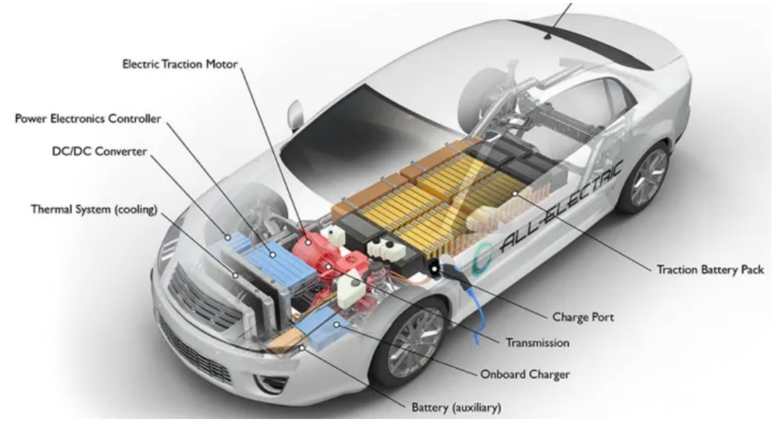
- Developed a scientific theory to explain electromagnetic waves.
- Coupled the electrical fields and magnetic fields together
- he established the foundations of electricity and magnetism as **electromagnetism.**
- Maxwell equations

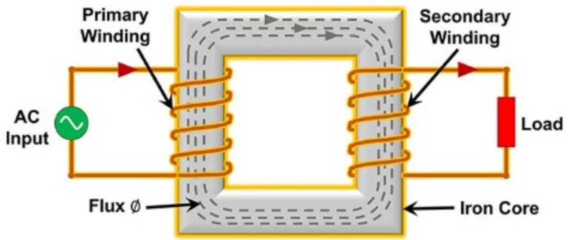
# Daily life applications

For example:

- Electric motor/generator:
- Battery charger
- Induction oven
- ……

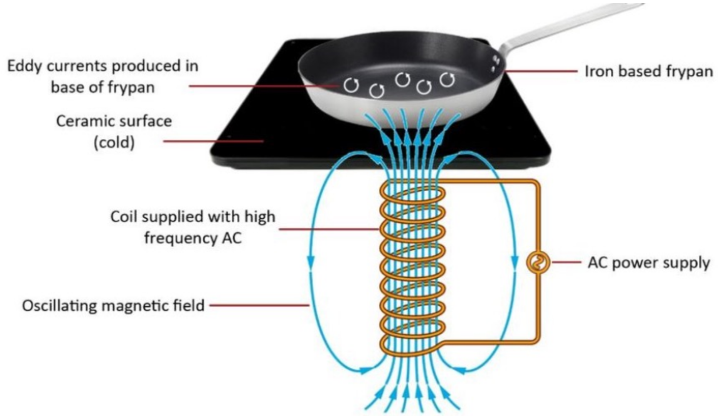


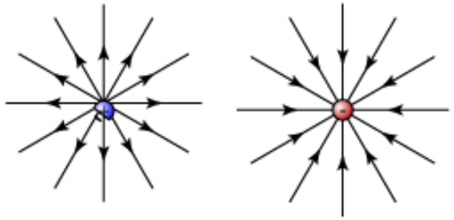

Basic Construction of Transformer

Iphone charger

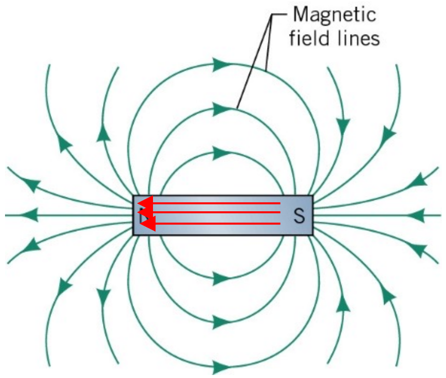


# Electromagnetism: Maxwell equations

- A static electric charges produces an electric field
- There is no magnetic charge  (monopole).
- A *changing* magnetic field produces an electric field,
- Charges in *motion* (an electrical current) produce a magnetic field
- A changing electric field produces a magnetic field.



Field pattern of
a pointed electrode




Electric and Magnetic fields can produce forces on charges

$$\nabla \cdot \mathbf{E} = \frac{\rho_v}{\varepsilon} \quad \text{(Gauss' Law)}$$

$$\nabla \cdot \mathbf{H} = 0 \quad \text{(Gauss' Law for Magnetism)}$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad \text{(Faraday's Law)}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \quad \text{(Ampere's Law)}$$

**E: electric field, Vector**

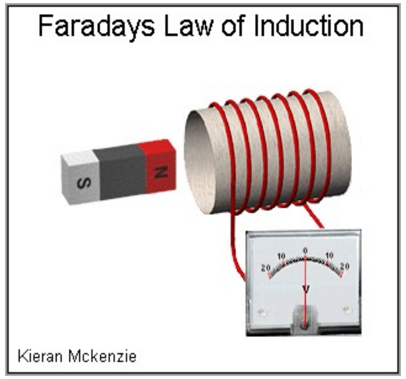
D: electric flux density, Vector

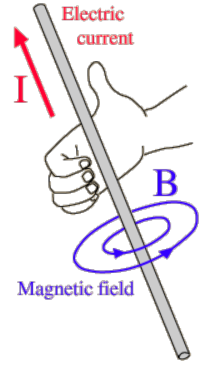
**H: magnetic field, Vector**

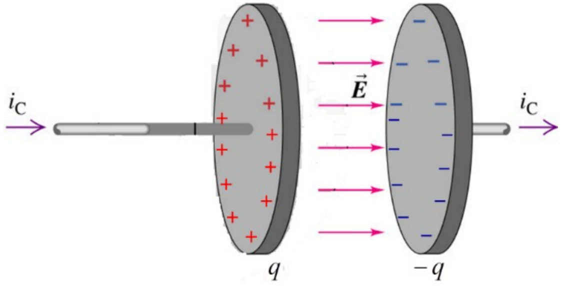
B: magnetic flux density, Vector

J: current density, Vector

ρ: Static charge density



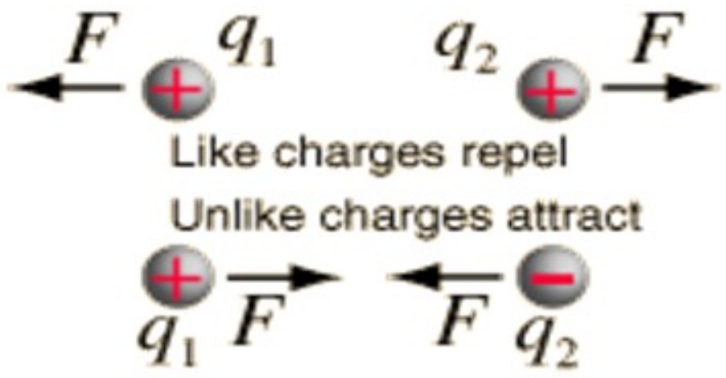




**Electricity and magnetism had been unified into electromagnetism!**

# Coulomb's law:  force between electrostatic charges

Published in 1785 by French physicist Charles-Augustin de Coulomb and was essential to the development of the theory of electromagnetism




**Scalar:** $F = k\frac{q_1 q_2}{r_{12}^2} = \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}^2}$

**Vector:** $\vec{F} = \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}^2}\widehat{r_{12}}$

$\widehat{r_{12}}$ is just for direction, its absolut value is 1.

$$k = \frac{1}{4\pi\varepsilon_0} \approx 9x10^9 N\cdot m^2/C^2 = \text{Coulomb's constant}$$

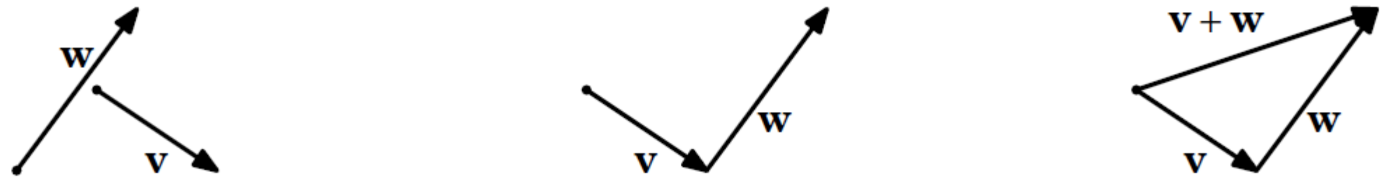
The electrostatic force had the same functional form as Newton's law of gravity

The magnitude of the electrostatic force between two-point charges:

1) directly proportional to the product of the magnitudes of charges
2) inversely proportional to the square of the distance between them
3) The force is along the straight line joining them.

# Vector calculus:

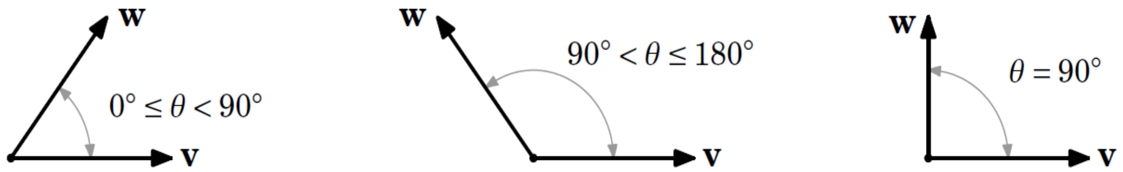
$\mathbf{v} + \mathbf{w}$




(a) Vectors **v** and **w** (b) Translate **w** to the end of **v** (c) The sum **v** + **w**

Dot product

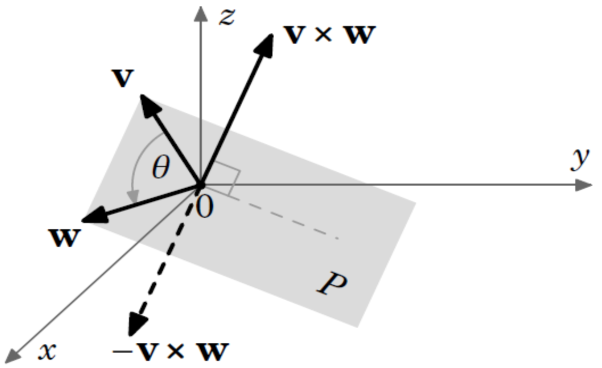
$$\mathbf{v} \cdot \mathbf{w} = \cos\theta \, \|\mathbf{v}\| \, \|\mathbf{w}\|$$




(a) $\mathbf{v} \cdot \mathbf{w} > 0$ (b) $\mathbf{v} \cdot \mathbf{w} < 0$ (c) $\mathbf{v} \cdot \mathbf{w} = 0$

cross product

$$\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\| \, \|\mathbf{w}\| \sin\theta$$



# Vector force:

**Electric forces follow the law of superposition.**

**If more than one charge is causing a force on object 1, then the net force acting on object 1 is just the sum of all the individual forces acting on 1.**

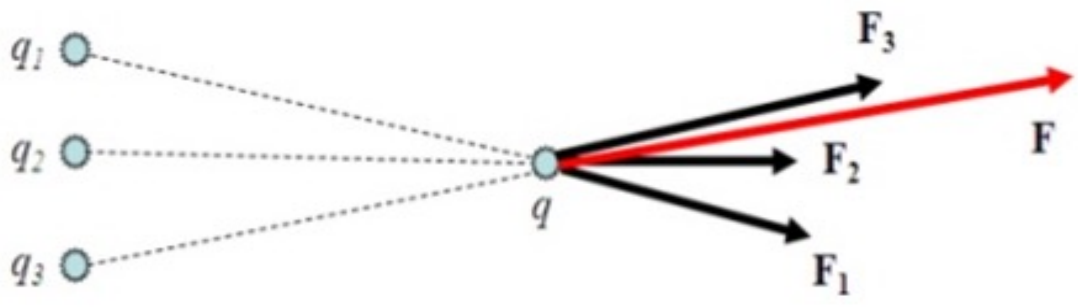
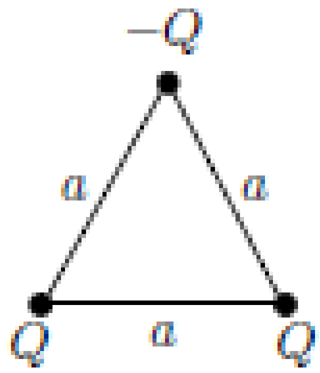



*Fig. Superposition Law*

**Net force on *q*: F = $F_1 + F_2 + F_3$**

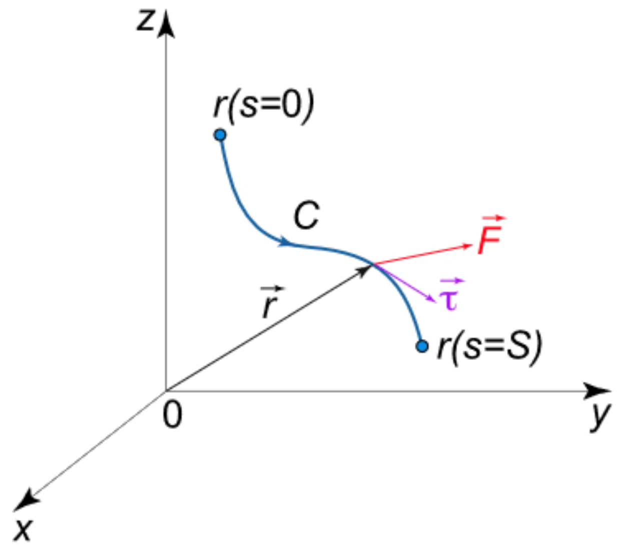
$$\vec{F}_{tot} = \sum_{i=1}^{n} \frac{q q_i}{4\pi\varepsilon_0 r_i^2} \hat{r}_i$$



# Line integral of vector

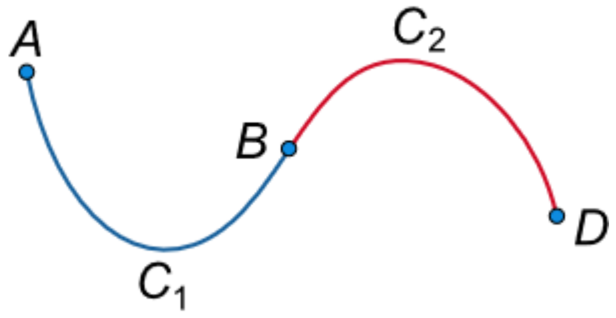
Line integral of vector force F along the curve C.
Suppose that a curve C is defined by the vector function r=r(s), 0≤s≤S, where s is the arc length of the curve. Then the derivative of the vector function

$$\frac{dr}{dS} = \boldsymbol{\tau} \quad \textit{( Tangent direction at each point of the curve )}$$




Curve direction is important.

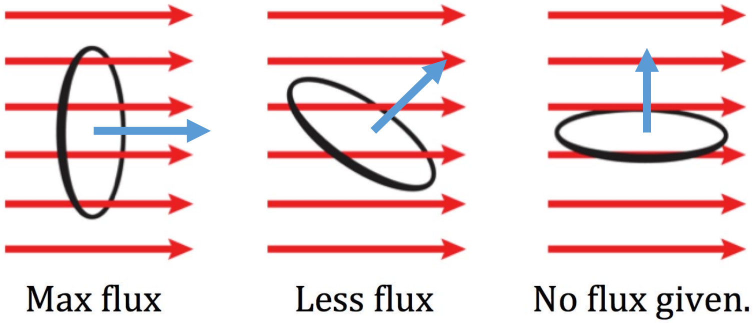
$$\int_C (\mathbf{F}\cdot d\mathbf{r}) = \int_0^S (\mathbf{F}(\mathbf{r}(s))\cdot\boldsymbol{\tau})\, ds,$$



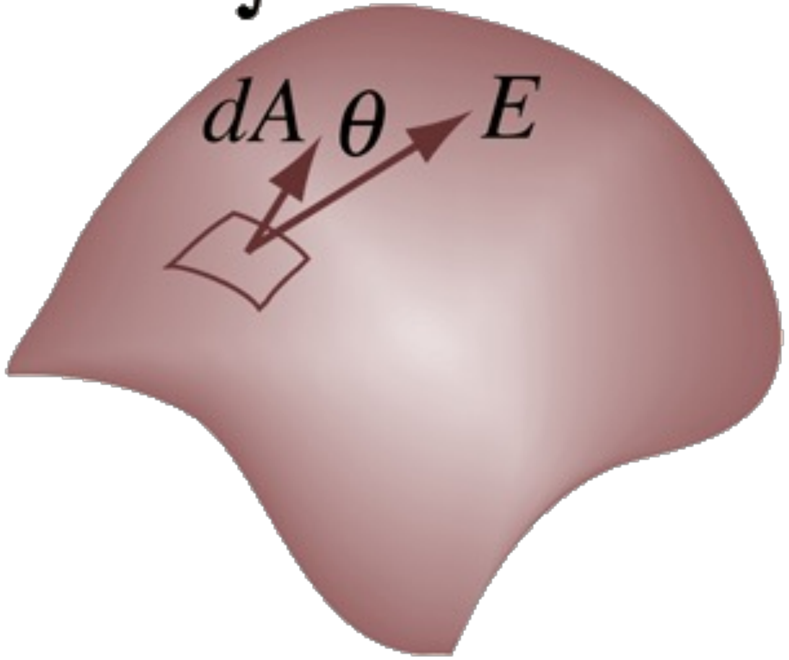

$$\int_C (\mathbf{F}\cdot d\mathbf{r}) = \int_{C_1\cup C_2} (\mathbf{F}\cdot d\mathbf{r}) = \int_{C_1} (\mathbf{F}\cdot d\mathbf{r}) + \int_{C_2} (\mathbf{F}\cdot d\mathbf{r});$$

# Surface integral of vector

dA direction is perpendicular to the tangent plane to that surface at A




$$\int \vec{E} \cdot d\vec{A} = \int E \cos\theta dA$$




$$\iint_S \vec{F} \bullet d\vec{S} = \iint_S \vec{F} \bullet \vec{n}\, dS$$

# Gradient:Greatest rate of increase

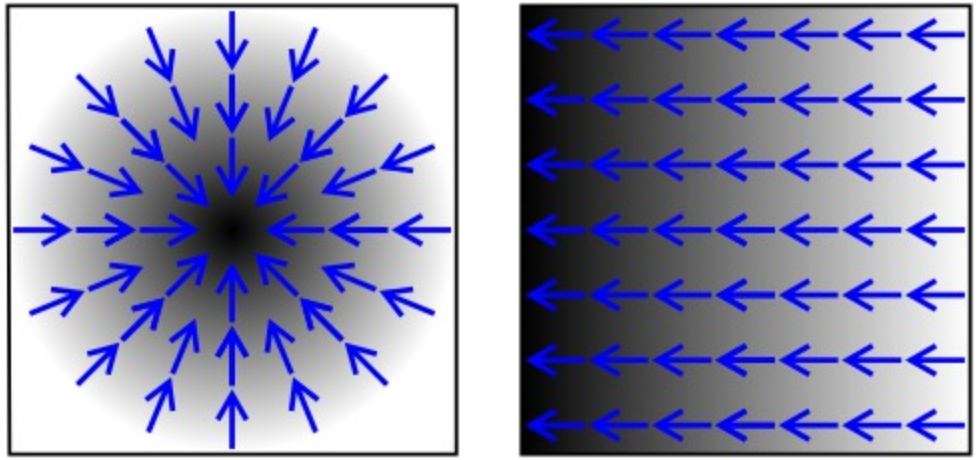
**Gradient: 3-dimension derivative of a scalar function**

showing the **direction and rate of fastest increase** of the scalar function *f* at a point space.

**How quickly something changes from one point to another**

$$\nabla f = \text{grad } f = < \frac{\partial f}{\partial x}(x,y,z), \frac{\partial f}{\partial y}(x,y,z), \frac{\partial f}{\partial z}(x,y,z) >$$

$$\nabla \boldsymbol{f} = \frac{\partial f}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}}$$

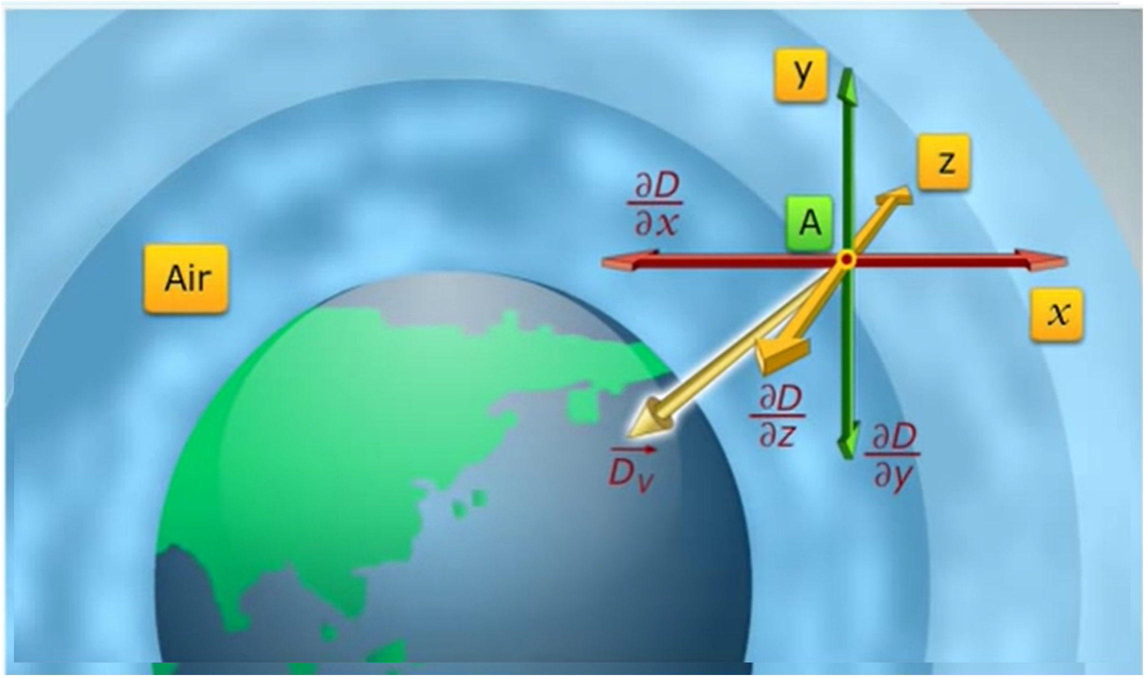


$f$ : Scalar function

$\nabla \boldsymbol{f}$ (Gradient): Vector function

Direction: fastest rate of increase

# Example: air density in the space




$$D = f(x, y, z)$$

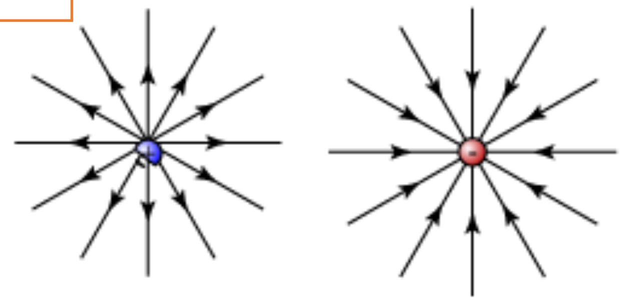
$$\overrightarrow{D_V} = \frac{\partial D}{\partial x}\hat{i} + \frac{\partial D}{\partial y}\hat{j} + \frac{\partial D}{\partial z}$$

$|\overrightarrow{D_V}|$ = **Maximum Rate** at which the **Density Increases**

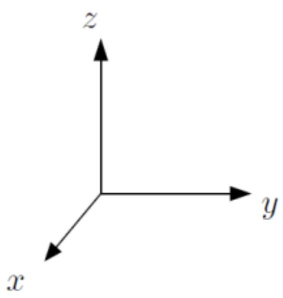
# Divergence: Flux/field out of a point

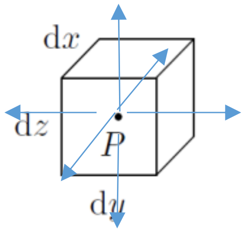
Divergence represents the volume density of the outward flux of a vector field from an infinitesimal volume around a given point

$$\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \lim_{V\to 0}\frac{1}{|V|}\oiint_{S(V)} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS$$



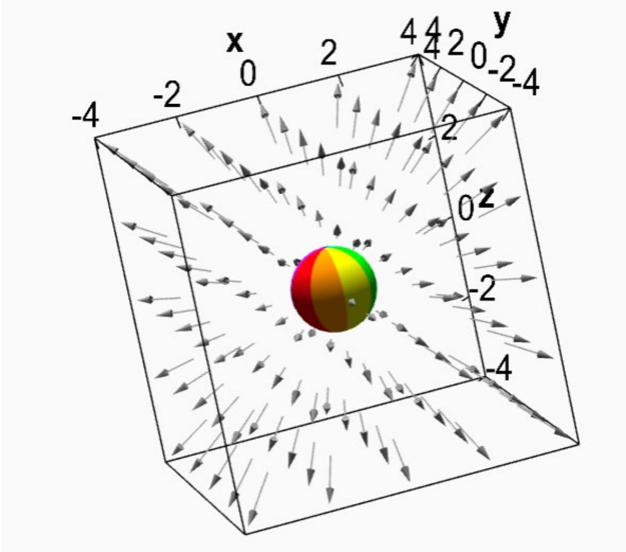
Field pattern of
a pointed electrode



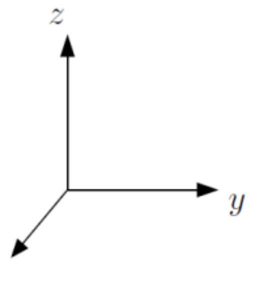

$$\nabla\cdot F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

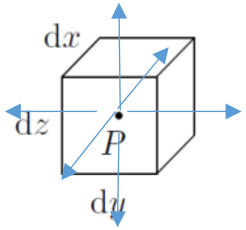
**F** is a vector
$\nabla\cdot F$ is a scalar.



# Divergence: Mathematical calculation

$$\boxed{\mathrm{div}\,\mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A}\cdot \mathrm{d}\mathbf{S}}{\Delta v}.}$$




$$\mathbf{A} = (A_x, A_y, A_z)$$

$$P = (x_0, y_0, z_0)$$

$$\oint_S \mathbf{A}\cdot \mathrm{d}\mathbf{S} = \oint_S \mathbf{A}\cdot \hat{\mathbf{n}}\mathrm{d}S$$

$$\oint_S \mathbf{A}\cdot \mathrm{d}\mathbf{S} = \int_{\mathrm{foran}} A_x(\mathrm{foran})\mathrm{d}y\mathrm{d}z - \int_{\mathrm{bak}} A_x(\mathrm{bak})\mathrm{d}y\mathrm{d}z$$
$$- \int_{\mathrm{venstre}} A_y(\mathrm{venstre})\mathrm{d}x\mathrm{d}z + \int_{\mathrm{høyre}} A_y(\mathrm{høyre})\mathrm{d}x\mathrm{d}z$$
$$+ \int_{\mathrm{topp}} A_z(\mathrm{topp})\mathrm{d}x\mathrm{d}y - \int_{\mathrm{bunn}} A_z(\mathrm{bunn})\mathrm{d}x\mathrm{d}y.$$

$$A_x(\mathrm{foran}) - A_x(\mathrm{bak}) = A_x(x_0 + \mathrm{d}x/2, y_0, z_0) - A_x(x_0 - \mathrm{d}x/2, y_0, z_0) = \frac{\partial A_x}{\partial x}\mathrm{d}x$$

$$A_y(\mathrm{høyre}) - A_y(\mathrm{venstre}) = \frac{\partial A_y}{\partial u}\mathrm{d}y$$

$$A_z(\mathrm{topp}) - A_z(\mathrm{bunn}) = \frac{\partial A_z}{\partial z}\mathrm{d}z.$$

⇩

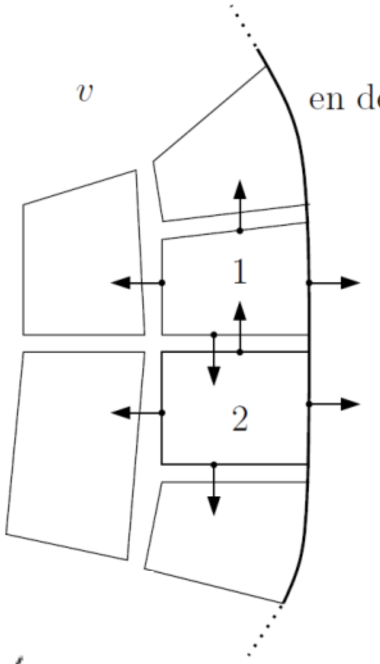
$$\oint_S \mathbf{A}\cdot \mathrm{d}\mathbf{S} = \frac{\partial A_x}{\partial x}\mathrm{d}x\mathrm{d}y\mathrm{d}z + \frac{\partial A_y}{\partial y}\mathrm{d}x\mathrm{d}y\mathrm{d}z + \frac{\partial A_z}{\partial z}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$\boxed{\mathrm{div}\,\mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A}\cdot \mathrm{d}\mathbf{S}}{\Delta v}.} \Rightarrow \boxed{\mathrm{div}\,\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \nabla\cdot\mathbf{A}.}$$

$$\Delta v = \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

# Divergence theorem

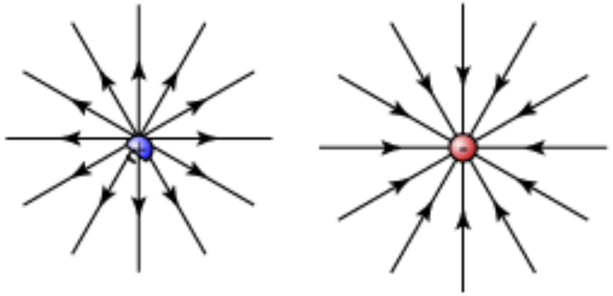
$$\oint_S \mathbf{A} \cdot \mathrm{d}\mathbf{S} = \int_v \nabla \cdot \mathbf{A} \mathrm{d}v$$




$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \sum_i \oint_{S_i} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \sum_i \left(\frac{1}{\Delta V_i} \oint_{S_i} \mathbf{E} \cdot \mathrm{d}\mathbf{S}_i\right) \Delta V_i \to \int \nabla \cdot \mathbf{E} \mathrm{d}V.$$

$$\nabla . E = \frac{\rho}{\varepsilon_0}$$

Gauss' Law



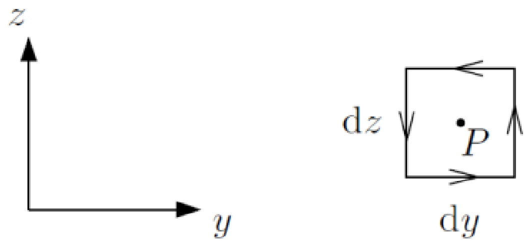
Field pattern of
a pointed electrode

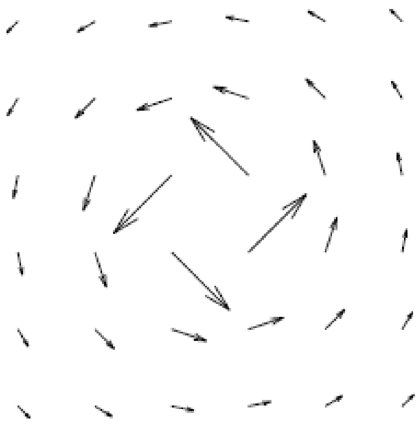
# Curl: how much does a field circulate around a point.

The curl of a field presents the infinitesimal circulation density at each point of the field

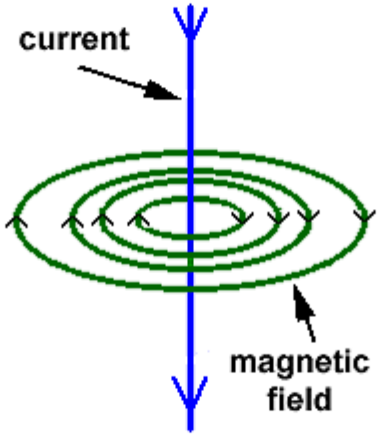
One dimension x

$$(\text{curl}\,\mathbf{A})_x = \lim_{\Delta S \to 0} \frac{\oint_C \mathbf{A} \cdot d\mathbf{l}}{\Delta S}$$



sirkulasjon/curl




Three dimensions x, y and z

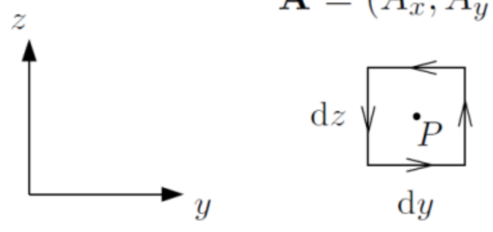
$$\nabla \times A = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}$$

# Curl

**The curl around x-axis, in yz plane**

$$P = (x_0, y_0, z_0)$$

$$\mathbf{A} = (A_x, A_y, A_z)$$




$$\Delta S = \mathrm{d}y\mathrm{d}z$$

$$\oint_C \mathbf{A}\cdot \mathrm{d}\mathbf{l} = \int_{\text{nede}} A_y(\text{nede})\mathrm{d}y - \int_{\text{oppe}} A_y(\text{oppe})\mathrm{d}y - \int_{\text{venstre}} A_z(\text{venstre})\mathrm{d}z + \int_{\text{høyre}} A_z(\text{høyre})\mathrm{d}z$$

$$A_y(\text{nede}) - A_y(\text{oppe}) = A_y(x_0, y_0, z_0 - \mathrm{d}z/2) - A_y(x_0, y_0, z_0 + \mathrm{d}z/2) = -\frac{\partial A_y}{\partial z}\mathrm{d}z,$$

$$A_z(\text{høyre}) - A_z(\text{venstre}) = A_z(x_0, y_0 + \mathrm{d}y/2, z_0) - A_z(x_0, y_0 - \mathrm{d}y/2, z_0) = \frac{\partial A_z}{\partial y}\mathrm{d}y,$$

$$\oint_C \mathbf{A}\cdot \mathrm{d}\mathbf{l} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathrm{d}y\mathrm{d}z,$$

$$(\text{curl}\,\mathbf{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}.$$

Similar to the curl around y and z-axis

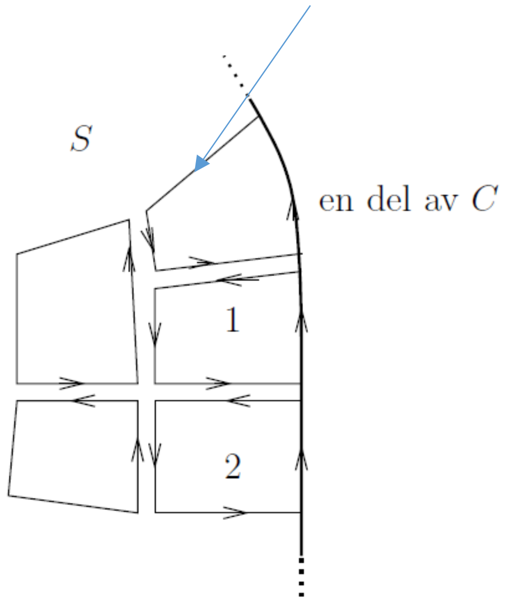
$$\nabla\times A = \left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\hat{z}$$

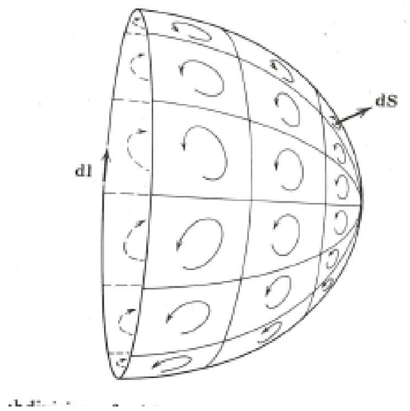
# Stokes' Theorem

$$\oint_C \mathbf{A} \cdot \mathrm{d}\mathbf{l} = \int_S \nabla \times \mathbf{A} \cdot \mathrm{d}\mathbf{S}.$$

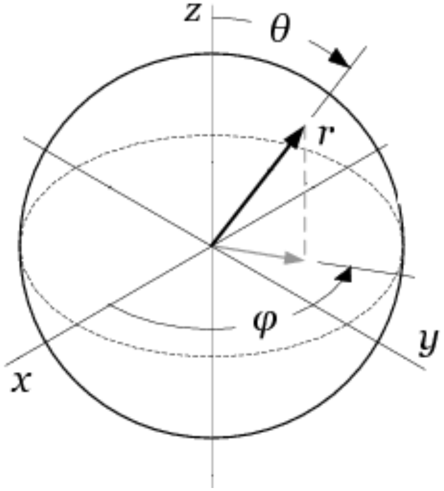
$$\int_S \nabla \times \mathbf{A} \cdot \mathrm{d}\mathbf{S} = \sum_i \oint_{C_i} \mathbf{A} \cdot \mathrm{d}\mathbf{l}.$$

$$\nabla \times \mathbf{A} \cdot \mathrm{d}\mathbf{S} = \oint_{C_1} \mathbf{A} \cdot \mathrm{d}\mathbf{l}.$$

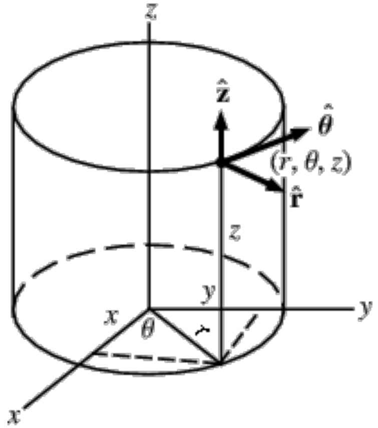


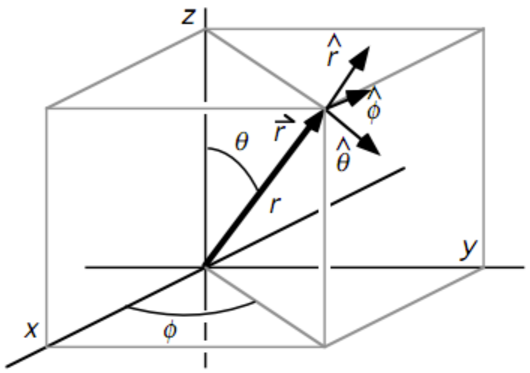
# Different coordinates




Spherial coordinate




Cylindrical coordinate




Cartesia Coordinate

VECTOR DIFFERENTIAL OPERATIONS

*Rectangular Coordinates (x, y, z)*

$$\nabla\Phi = \hat{\mathbf{x}}\frac{\partial\Phi}{\partial x} + \hat{\mathbf{y}}\frac{\partial\Phi}{\partial y} + \hat{\mathbf{z}}\frac{\partial\Phi}{\partial z}$$

$$\nabla\cdot\mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$\nabla\times\mathbf{H} = \hat{\mathbf{x}}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)$$

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}$$

$$\nabla^2\mathbf{A} = \hat{\mathbf{x}}\nabla^2 A_x + \hat{\mathbf{y}}\nabla^2 A_y + \hat{\mathbf{z}}\nabla^2 A_z$$

*Cylindrical Coordinates (r, φ, z)*

$$\nabla\Phi = \hat{\mathbf{r}}\frac{\partial\Phi}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial\Phi}{\partial\phi} + \hat{\mathbf{z}}\frac{\partial\Phi}{\partial z}$$

$$\nabla\cdot\mathbf{D} = \frac{1}{r}\frac{\partial}{\partial r}(rD_r) + \frac{1}{r}\frac{\partial D_\phi}{\partial\phi} + \frac{\partial D_z}{\partial z}$$

$$\nabla\times\mathbf{H} = \hat{\mathbf{r}}\left[\frac{1}{r}\frac{\partial H_z}{\partial\phi} - \frac{\partial H_\phi}{\partial z}\right] + \hat{\boldsymbol{\phi}}\left[\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\right] + \hat{\mathbf{z}}\left[\frac{1}{r}\frac{\partial(rH_\phi)}{\partial r} - \frac{1}{r}\frac{\partial H_r}{\partial\phi}\right]$$

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}$$

$$\nabla^2\mathbf{A} = \hat{\mathbf{r}}\left(\nabla^2 A_r - \frac{2}{r^2}\frac{\partial A_\phi}{\partial\phi} - \frac{A_r}{r^2}\right) + \hat{\boldsymbol{\phi}}\left(\nabla^2 A_\phi + \frac{2}{r^2}\frac{\partial A_r}{\partial\phi} - \frac{A_\phi}{r^2}\right) + \hat{\mathbf{z}}(\nabla^2 A_z)$$

*Spherical Coordinates (r, θ, φ)*

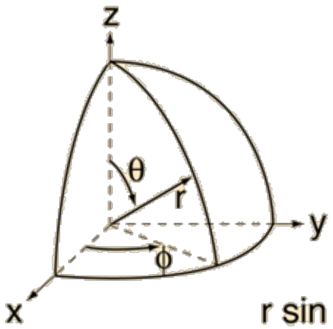
$$\nabla\Phi = \hat{\mathbf{r}}\frac{\partial\Phi}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial\Phi}{\partial\theta} + \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}$$

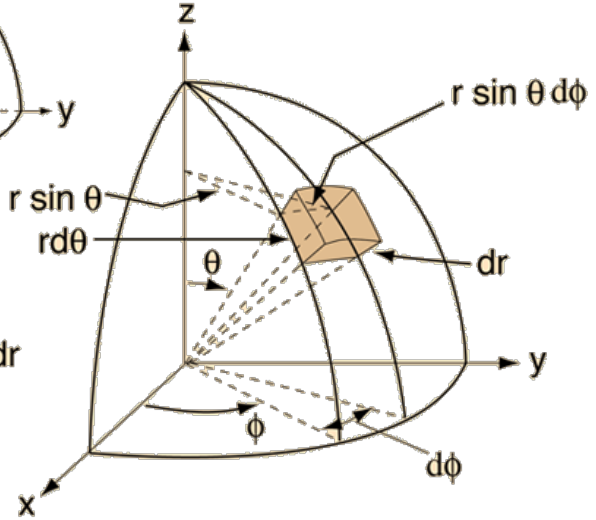
$$\nabla\cdot\mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi}$$

$$\nabla\times\mathbf{H} = \frac{\hat{\mathbf{r}}}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(H_\phi\sin\theta) - \frac{\partial H_\theta}{\partial\phi}\right] + \frac{\hat{\boldsymbol{\theta}}}{r}\left[\frac{1}{\sin\theta}\frac{\partial H_r}{\partial\phi} - \frac{\partial}{\partial r}(rH_\phi)\right] + \frac{\hat{\boldsymbol{\phi}}}{r}\left[\frac{\partial}{\partial r}(rH_\theta) - \frac{\partial H_r}{\partial\theta}\right]$$

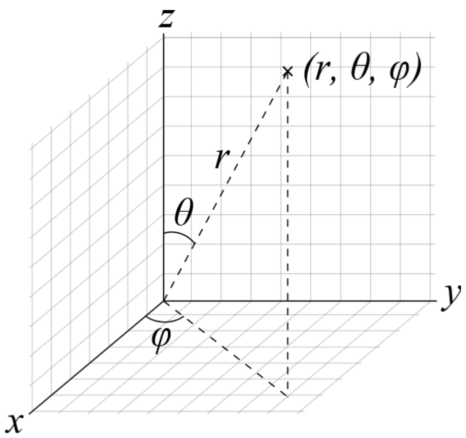
$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

$$\nabla^2\mathbf{A} = \hat{\mathbf{r}}\left[\nabla^2 A_r - \frac{2}{r^2}\left(A_r + \cot\theta A_\theta + \csc\theta\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_\theta}{\partial\theta}\right)\right] + \hat{\boldsymbol{\theta}}\left[\nabla^2 A_\theta - \frac{1}{r^2}\left(\csc^2\theta A_\theta - 2\frac{\partial A_r}{\partial\theta} + 2\cot\theta\csc\theta\frac{\partial A_\phi}{\partial\phi}\right)\right] + \hat{\boldsymbol{\phi}}\left[\nabla^2 A_\phi - \frac{1}{r^2}\left(\csc^2\theta A_\phi - 2\csc\theta\frac{\partial A_r}{\partial\phi} - 2\cot\theta\csc\theta\frac{\partial A_\theta}{\partial\phi}\right)\right]$$




Volume element
$dV = r^2 \sin\theta \, d\theta \, d\phi \, dr$



# Examples: Probelm 3

Calculate the integral

$$I = \int_V (\nabla \cdot \mathbf{F}) \mathrm{d}V \qquad (1)$$

where $\mathbf{F} = r\hat{\mathbf{r}} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$, and the volume $V$ is a sphere with radius $R$ placed in the origin.

$$\mathrm{div}\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \nabla \cdot \mathbf{A}.$$

a) Calculate the integral directly.

b) Calculate the integral using the divergence theorem.

$$\oint_S \mathbf{A} \cdot \mathrm{d}\mathbf{S} = \int_v \nabla \cdot \mathbf{A} \mathrm{d}v$$

# Solutions:

Calculate the integral

$$I = \int_V (\nabla \cdot \mathbf{F}) \mathrm{d}V \qquad (1)$$

where $\mathbf{F} = r\hat{\mathbf{r}} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$, and the volume $V$ is a sphere with radius $R$ placed in the origin.

**a)** Calculate the integral directly.

**b)** Calculate the integral using the divergence theorem.

$$\mathrm{div}\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \nabla \cdot \mathbf{A}.$$

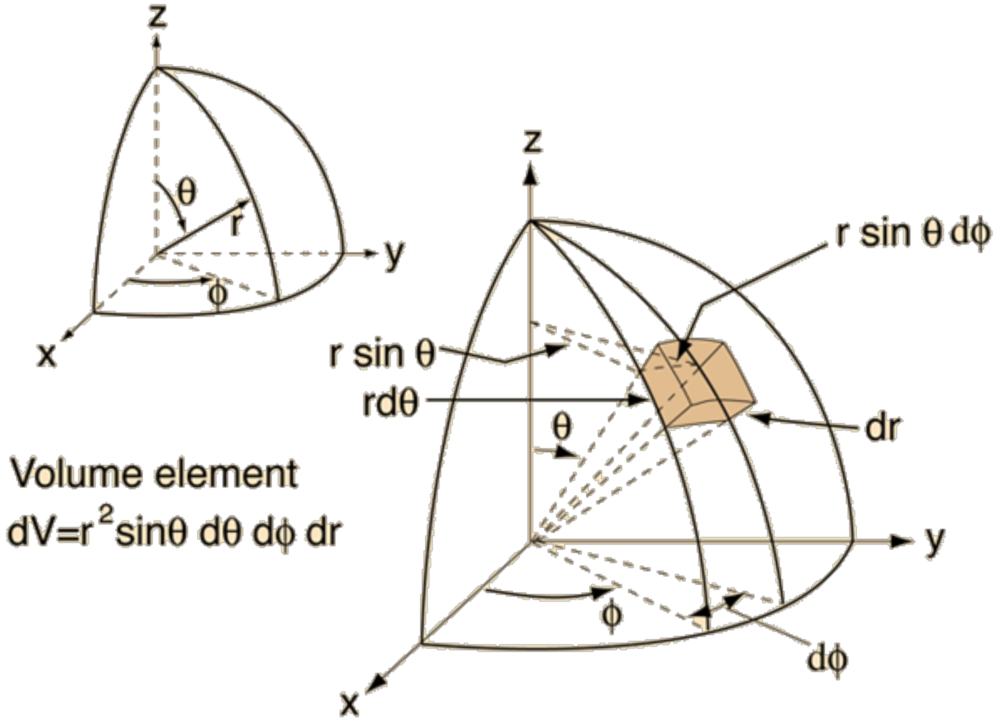
$$\nabla \cdot \mathbf{F} = 3.$$

$$\int_v (\nabla \cdot \mathbf{F}) \mathrm{d}V = \int_0^R \int_0^{2\pi} \int_0^{\pi} 3r^2 \sin\theta \mathrm{d}\varphi \mathrm{d}\theta \mathrm{d}r$$

$$= 3 \int_0^R r^2 \mathrm{d}r \int_0^{\pi} \sin\theta \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi$$

$$= \underline{\underline{4\pi R^3}}.$$

$$\underline{\underline{\int_V (\nabla \cdot \mathbf{F}) \mathrm{d}V = 3 \int_V \mathrm{d}V = 4\pi R^3.}}$$



# Solution for b

$$\int_v (\nabla \cdot \mathbf{F}) \mathrm{d}v = \oint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$
$$= \int_0^{2\pi} \int_0^{\pi} (R\hat{\mathbf{r}}) \cdot (R^2 \sin\theta \mathrm{d}\theta \mathrm{d}\varphi \hat{\mathbf{r}})$$
$$= R^3 \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \sin\theta \mathrm{d}\theta$$
$$= \underline{\underline{4\pi R^3}}.$$

$$\boxed{\oint_S \mathbf{A} \cdot \mathrm{d}\mathbf{S} = \int_v \nabla \cdot \mathbf{A} \mathrm{d}v}$$

# Example:

Calculate the line integral

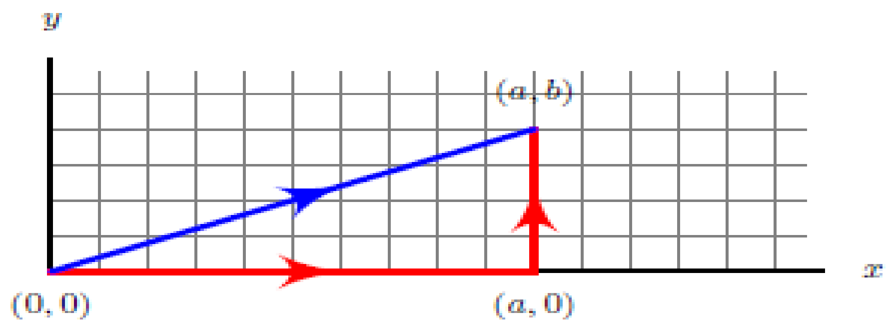
$$I = \int_C \mathbf{F} \cdot d\mathbf{l}, \qquad (2)$$

Where $\mathbf{F} = (xy^2 + 2y)\hat{\mathbf{x}} + (x^2y + 2x)\hat{\mathbf{y}}$,

i) Along the curve $C_1$ which consists of two straight lines connecting the points $(0,0)$, $(a,0)$ and $(a,b)$, see figure below.

ii) Along the curve $C_2$ which consists of one straight line connecting the points $(0,0)$ and $(a,b)$, see figure below.

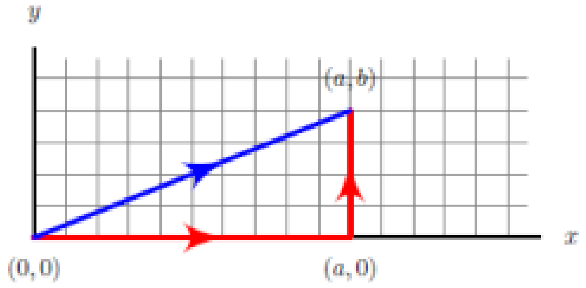
iii) Why do these calculations produce the same answer? Explain using Stoke's theorem.




$$\nabla \times A = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}$$

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{A} \cdot d\mathbf{S}.$$

# Solution for i) and ii)




$$I = \int_V (\nabla \cdot \mathbf{F}) \mathrm{d}V$$

$$\mathbf{F} = (xy^2 + 2y)\hat{\mathbf{x}} + (x^2y + 2x)\hat{\mathbf{y}}$$

i) Along the curve $C_1$ which consists of two straight lines connecting the points $(0, 0)$, $(a, 0)$ and $(a, b)$, see figure below.

$$\begin{aligned} I &= \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{l} \\ &= \int_0^b (a^2y + 2a)\mathrm{d}y \\ &= \underline{\underline{\frac{1}{2}a^2b^2 + 2ab.}} \end{aligned}$$

ii) Along the curve $C_2$ which consists of one straight line connecting the points $(0, 0)$ and $(a, b)$, see figure below.

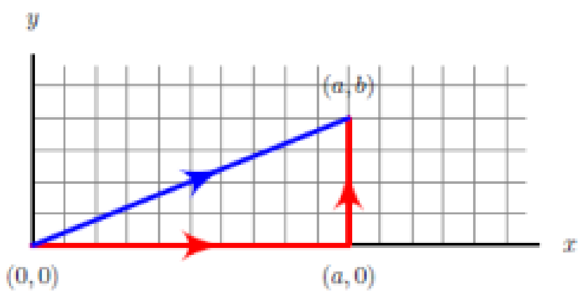
$$\mathrm{d}\mathbf{l} = \mathrm{d}x\hat{\mathbf{x}} + \mathrm{d}y\hat{\mathbf{y}}.$$

$$\begin{aligned} \mathbf{F} \cdot \mathrm{d}\mathbf{l} &= F_x \mathrm{d}x + F_y \mathrm{d}y \\ &= (xy^2 + 2y)\mathrm{d}x + (x^2y + 2x)\mathrm{d}y. \end{aligned}$$

$$y = \frac{bx}{a}$$

$$\begin{aligned} I &= \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{l} \\ &= \int_0^a \left[ x\left(\frac{bx}{a}\right)^2 + 2\left(\frac{bx}{a}\right) \right] \mathrm{d}x + \int_0^b \left[ \left(\frac{ay}{b}\right)^2 y + 2\left(\frac{ay}{b}\right) \right] \mathrm{d}y \\ &= \underline{\underline{\frac{1}{2}a^2b^2 + 2ab.}} \end{aligned}$$

# Conservative vector: solution for iii)




iii) Why do these calculations produce the same answer? Explain using Stoke's theorem.

$$\mathbf{F} = (xy^2 + 2y)\hat{\mathbf{x}} + (x^2y + 2x)\hat{\mathbf{y}}$$

These integrals have equal values since **F** is a conservative field:

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\mathbf{z}}$$
$$= (2xy - 2xy)\,\hat{\mathbf{z}}$$
$$= 0.$$

$$\nabla \times A = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}$$

$$\oint_C \mathbf{A}\cdot d\mathbf{l} = \int_S \nabla \times \mathbf{A}\cdot d\mathbf{S}.$$