TFE4120 Electromagnetics - Preliminary course

Solution proposal, exercise 6

Maxwell's equations are:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \tag{2}$$

$$\nabla \cdot \mathbf{D} = \rho, \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{4}$$

a) First, we use the divergence theorem on (3) to find the corresponding equation on integral form. We integrate across a volume v with its associated closed surface S:

$$\int_v \nabla \cdot \mathbf{D} \mathrm{d}v = \oint_S \mathbf{D} \cdot \mathrm{d}\mathbf{S}, \tag{5}$$

which yields

$$\underline{\underline{\oint_S \mathbf{D} \cdot \mathrm{d}\mathbf{S} = \int_v \rho \mathrm{d}v.}} \tag{6}$$

This is Gauss' law on integral form. It states that the **D**-field flux out of a closed surface is equal to the total charge within the volume surrounded by that surface.

Let's say that the surface S is a sphere with radius r surrounding the point charge. Due to symmetry the **D**-field must be radial, i.e parallel to the surface norm of S, $\mathbf{D} = D(r)\hat{\mathbf{r}}$. For a point charge Q placed in the origin we get:

$$\int_v \rho \mathrm{d}v = Q = \oint_S \mathbf{D} \cdot \mathrm{d}\mathbf{S} = \oint_S D \mathrm{d}S = 4\pi r^2 D, \tag{7}$$

since the area of the spherical surface is $4\pi r^2$. The electric field is then

$$\underline{\underline{\mathbf{E} = \frac{\mathbf{D}}{\epsilon_0} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}.}} \tag{8}$$

From Lorentz' equation of force we find the force acting on charge q:

$$\underline{\underline{\mathbf{F} = \frac{qQ}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}.}} \tag{9}$$

b) Equation (4). Using the analogy from the previous task we can see that equation (4) yields that the **B**-field can not flow out of a closed surface.

c) We can see that the "circulation" of the $\mathbf{E}$-field in the loop C is equal to the rate of change of magnetic flux per unit time through the loop. It is accordingly natural to use (1). To find the corresponding equation on integral form we use Stokes' theorem:

$$\oint_C \mathbf{E} \cdot \mathrm{dl} = \int_S \nabla \times \mathbf{E} \cdot \mathrm{d\mathbf{S}} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d\mathbf{S}} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathbf{B} \cdot \mathrm{d\mathbf{S}}, \tag{10}$$

which gives Faraday's law

$$e = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}. \tag{11}$$

If the resistance R is so great that the current in the loop is negligible (i.e that the induced magnetic field also is negligible, compared to the applied outer field), the total flux density in the loop is given by the applied outer field $\mathbf{B} = (B_0 + B_1 \cos(2\pi f t))\hat{\mathbf{z}}$. The voltage across the resistor is then given by Faraday's law:

$$\underline{\underline{V = -\frac{\mathrm{d}(BS)}{\mathrm{d}t} = 2\pi f S B_1 \sin(2\pi f t)}}. \tag{12}$$

If we reduce R, but maintain the movement, a current will flow in the loop due to the induced voltage. This current will form a magnetic field opposing the applied flux variation (Lenz' law). On the boundary $R \to 0$ the induced current will exactly compensate for the applied field, so that the total flux in the loop is constant. A superconductive loop will therefore always have a constant flux.

d) Here we need to use (2), Stokes' theorem, with a circular integration loop C with a radius $r > a$ around the infinitely long conductor (see figure 1). This yields

$$\oint_C \mathbf{H} \cdot \mathrm{dl} = \int_S (\nabla \times \mathbf{H}) \cdot \mathrm{d\mathbf{S}} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot \mathrm{d\mathbf{S}}. \tag{13}$$

Since all electromagnetic properties are stationary, there is no change in $\mathbf{D}$ with respect to time (i.e. displacement current is zero). We can then find that the right hand side of the equation above is equal to the current I within the integration loop C. Due to symmetry we can conclude that $\oint_C \mathbf{H} \cdot \mathrm{dl} = 2\pi r H$. So the $\mathbf{H}$-field is given by

$$\underline{\underline{\mathbf{H} = \frac{I}{2\pi r} \hat{\boldsymbol{\phi}}}}. \tag{14}$$

The unity vector $\hat{\boldsymbol{\phi}}$ points in the positive rotational direction. The current is assumed to flow in the $\hat{\mathbf{z}}$-direction.

e) Use the divergence of (2) and that the divergence of the curl is exactly equal to zero:

$$\underline{\underline{\nabla \cdot \mathbf{J} = -\nabla \cdot \frac{\partial \mathbf{D}}{\partial t} = -\frac{\partial (\nabla \cdot \mathbf{D})}{\partial t} = -\frac{\partial \rho}{\partial t}}}. \tag{15}$$

In the last part of the equation above we use (3). Then, integrate this equation across a test volume v surrounded by a surface S, and use the divergence theorem:

$$\int_v \nabla \cdot \mathbf{J} \mathrm{d}v = \oint_S \mathbf{J} \cdot \mathrm{d\mathbf{S}}. \tag{16}$$

On the right hand side of (15) we get

$$-\int_v \frac{\partial \rho}{\partial t} \mathrm{d}v = -\frac{\mathrm{d}}{\mathrm{d}t} \int_v \rho \mathrm{d}v = -\frac{\mathrm{d}Q}{\mathrm{d}t}, \tag{17}$$
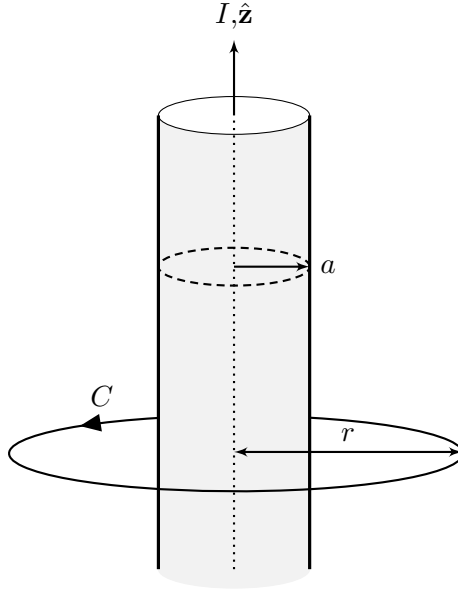
2

Figure 1: Solution figure for task **d)**.

where Q is the total charge in v. I.e

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{dQ}{dt}. \tag{18}$$

In other words is the total current out of the volume v equal to the total charge dissipated from v per unit time, which imply conservation of charge.

f) Here we use Faraday's law (Eq. (1)) on integral form (10):

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S}. \tag{19}$$

We integrate across a small loop as shown in figure 2. For small Δw and Δh we consider the fields constant across the loop. If we let $\Delta h \to 0$ the right hand side is equal to zero, and the left hand side only has the contribution from the two horizontal parts of the loop. Thus, we get that $E_{1t}\Delta w - E_{2t}\Delta w = 0$, or in other words $\underline{\underline{E_{1t} = E_{2t}}}$.

Alternatively (more detailed calculations):

$$\begin{aligned}
\oint_C \mathbf{E} \cdot d\mathbf{l} &= E_{1t}\Delta w - E_{1n}\frac{\Delta h}{2} - E_{2n}\frac{\Delta h}{2} - E_{2t}\Delta w + E_{2n}\frac{\Delta h}{2} + E_{1n}\frac{\Delta h}{2} \\
&= -\frac{\partial}{\partial t}\left(B_{1z}\Delta w\frac{\Delta h}{2} + B_{2z}\Delta w\frac{\Delta h}{2}\right).
\end{aligned} \tag{20}$$

If we let Δh approach 0, we are left with only to parts:

$$\underline{\underline{E_{1t} = E_{2t}}}. \tag{21}$$

g) The potentials $\mathbf{A}$ and V are not unambiguously determined by $\mathbf{E}$ and $\mathbf{B}$. For a random function $f(\mathbf{r}, t)$ we can define new potentials that will give the same $\mathbf{E}$- and $\mathbf{B}$-fields :

$$\mathbf{A}_{\text{new}} = \mathbf{A} + \nabla f \tag{22}$$

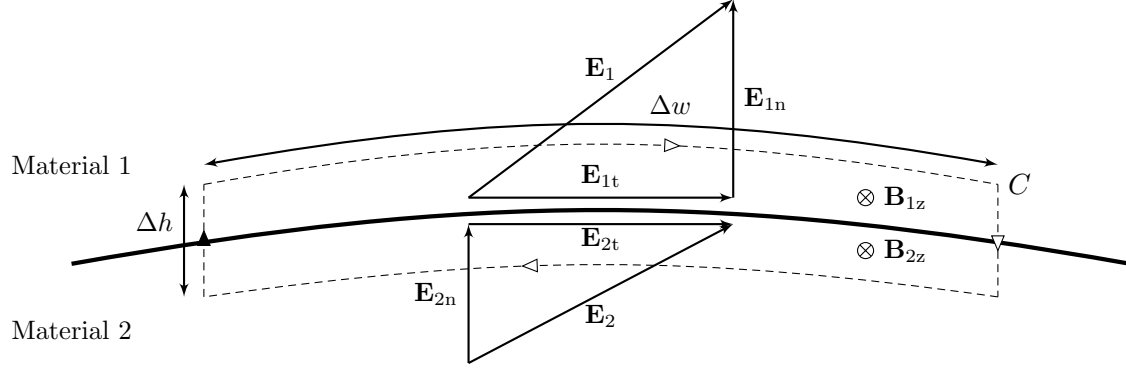$$V_{\text{new}} = V - \frac{\partial f}{\partial t}. \tag{23}$$

3

Figure 2: Solution sketch for task **e)**. The integration curve is given by the curve C that has a width Δw and height Δh.

We can see that

$$\mathbf{B}_{\text{new}} = \nabla \times \mathbf{A}_{\text{new}} = \nabla \times \mathbf{A} + \nabla \times \nabla f = \mathbf{B}, \tag{24}$$

because the curl of a gradient is equal to zero, $\nabla \times \nabla f = 0$. And

$$\mathbf{E}_{\text{new}} = -\nabla V_{\text{new}} - \frac{\partial \mathbf{A}_{\text{new}}}{\partial t} = -\nabla V + \nabla \frac{\partial f}{\partial t} - \frac{\partial \mathbf{A}}{\partial t} - \frac{\partial (\nabla f)}{\partial t} = \mathbf{E}. \tag{25}$$

The equations (3) and (2) is automatically satisfied because the curl of a gradient, and the divergence of a curl, is equal to zero

$$\nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}) = 0, \tag{26}$$

$$\nabla \times \mathbf{E} = \nabla \times \left(-\nabla V - \frac{\partial \mathbf{A}}{\partial t} \right) = -\frac{\partial (\nabla \times \mathbf{A})}{\partial t} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{27}$$

The two other equations gives the wave equation for the potentials.

h) At stationary conditions we have that

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -\nabla V, \tag{28}$$

since $\frac{\partial \mathbf{A}}{\partial t} = 0$. In a linear, isotropic and homogeneous material we can write (3) as

$$\nabla \cdot \mathbf{D} = \epsilon \nabla \cdot \mathbf{E} = \rho. \tag{29}$$

By using (28) we then get

$$\epsilon \nabla \cdot (-\nabla V) = \rho, \tag{30}$$

which yields

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}. \tag{31}$$