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TFE4120 Electromagnetics - Crash course

Solution proposal, Exercise 2

Problem 1

a) The force on each charge is repulsive and has the magnitude

$$F = \frac{Q^2}{4\pi\epsilon_0 a^2}. \tag{1}$$

b) The force between the charge $-Q$ and the two other will in each case be attractive. Let the xy-plane coincinde with the paper plane and we decompose the forces into $\hat{\mathbf{x}}$- and $\hat{\mathbf{y}}$-components (we know that the angles in an equilateral triangle are $\pi/3$)). Thus we find that $\mathbf{F}_y = -\cos(\pi/6)F\hat{\mathbf{y}} = -\sqrt{3}/2F\hat{\mathbf{y}}$ and $\mathbf{F}_x = \mp\sin(\pi/6)F\hat{\mathbf{x}} = \mp 1/2F\hat{\mathbf{x}}$ where F is given by (1). Finally we sum these by superposition and find that the force on the charge is

$$\mathbf{F} = -\frac{Q^2}{4\pi\epsilon_0 a^2}\left[\left(\frac{\sqrt{3}}{2}\hat{\mathbf{y}} + \frac{1}{2}\hat{\mathbf{x}}\right) + \left(\frac{\sqrt{3}}{2}\hat{\mathbf{y}} - \frac{1}{2}\hat{\mathbf{x}}\right)\right] = \frac{-Q^2\sqrt{3}}{4\pi\epsilon_0 a^2}\hat{\mathbf{y}}. \tag{2}$$

Accordingly, the force is pointing down in the $\hat{\mathbf{y}}$-direction.

c) With a positive charge $q = Q$ the force will be equal but in the opposite direction compared to what we found in (b). With a positive q (but not necessarily the same size Q) we find

$$\mathbf{F} = \frac{qQ\sqrt{3}}{4\pi\epsilon_0 a^2}\hat{\mathbf{y}}. \tag{3}$$

From this we can find the electric field

$$\mathbf{E} = \lim_{q\to 0}\frac{\mathbf{F}}{q} = \frac{Q\sqrt{3}}{4\pi\epsilon_0 a^2}\hat{\mathbf{y}}. \tag{4}$$

Because q is assumed to be infinitely small (a so called test charge), the $\mathbf{E}$-field is created solely by the two positive charges Q. If the test charge q is non-negligible, its field would influence the total field.

Problem 2

a) The field from the charge Q is

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} \tag{5}$$

The surface element for spherical coordinates is $d\mathbf{S} = r^2 \sin\phi d\phi d\theta \hat{\mathbf{r}}$, and is pointing parallel to $\mathbf{E}$. Inserted into

$$\int \mathbf{E} \cdot d\mathbf{S} \tag{6}$$

together with the right boundary conditions $\phi \in [0, \pi]$ og $\theta \in [0, 2\pi]$ we get the expression we are looking for:

$$\begin{aligned}\oint \mathbf{E} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^{\pi} \frac{Q}{4\pi\epsilon_0} \sin\phi d\phi d\theta, \\ &= \frac{Q}{4\pi\epsilon_0} \int_0^{2\pi} \Big[-\cos\theta \Big]_0^{\pi} d\phi \\ &= \frac{Q}{\epsilon_0}.\end{aligned} \tag{7}$$

b) Since we know that $\mathbf{E}$ and $d\mathbf{S}$ is parallel across the surface og the sphere, we know that $\mathbf{E} \cdot d\mathbf{S} = EdS$ across the entire integral. On the surface r =constant, so that $E(r)$ is also constant across the entire surface. Thus we can simplify the calculation above like this:

$$\oint \mathbf{E} \cdot d\mathbf{S} = \oint EdS = E(r) \underbrace{\oint dS}_{=4\pi r^2} = \frac{Q}{4\pi\epsilon_0 r^2} 4\pi r^2 = \frac{Q}{\epsilon_0} \tag{8}$$

c) From the divergence theorem we have another way of calculating the flux:

$$\oint \mathbf{E} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{E} dV, \tag{9}$$

i.e by integrating the divergence across the volume of the sphere (right hand side), instead of integrating the field across the surface of the sphere (left hand side). The divergence of the $\mathbf{E}$-field given in spherical coordinates is

$$\nabla \cdot \mathbf{E} = \frac{1}{r^2}\frac{\partial(r^2 E_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta E_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi}, \tag{10}$$

where E_r, E_θ, E_ϕ is the components in spherical coordinates. In our case we have $E_\theta = E_\phi = 0$, and E_r is given by (5). This yields

$$\nabla \cdot \mathbf{E} = \frac{1}{r^2} \cdot 0 = 0, \quad (r \neq 0) \tag{11}$$

The divergence is zero everywhere except for $r = 0$. When $r = 0$ the $\mathbf{E}$-field is not diverging ($E \to \infty$) and we can not calculate $\nabla \cdot \mathbf{E}$ this way (the derivative does not exist when E is not continuous). We write

$$\oint \mathbf{E} \cdot d\mathbf{S} = \underbrace{\int_{r>0} \nabla \cdot \mathbf{E} dV}_{=0} + \int_{r\to 0} \nabla \cdot \mathbf{E} dV, \tag{12}$$

where we divide the area where $r > 0$ inside the sphere volume, and the area where $r \to 0$ (where we need to find the divergence). A trick we can use to calculate the final part without finding the divergence, is to use the divergence theorem over again:

$$\int_{r\to 0} \nabla \cdot \mathbf{E} dV = \oint \mathbf{E} \cdot d\mathbf{S}. \tag{13}$$

We imagine that we integrate the field across a spherical surface with an infinitely small radius, surrounding the charge Q. The solution to this surface integral is the same as in (a) and (b). Thus we find that

$$\int_{r\to 0} \nabla \cdot \mathbf{E} dV = \frac{Q}{\epsilon_0}. \tag{14}$$

Inserted into (12) we get the answer:

$$\oint \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\epsilon_0}. \tag{15}$$

The steps above can be done for a simple closed surface with the same results, which gives Gauss' law.

d) For $r > a$ the answer remains the same. For $r < a$ we use $\rho = \frac{Q}{4\pi a^3/3}$ and find $Q_S = \rho \frac{4}{3}\pi r^3 = Qr^3/a^3$ (the charge within a sphere with radius $r \leq a$). The calculation of the field outside of this can be done in the same way as in (b) above (thanks to spherical symmetry) so that we find

$$\begin{aligned} \mathbf{E}(r) &= \frac{Q_S}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} \\ &= \frac{Q}{4\pi\epsilon_0 a^2}\frac{r}{a}\hat{\mathbf{r}}. \end{aligned} \tag{16}$$

Accordingly we get

$$\underline{\underline{\mathbf{E}(r) = \begin{cases} \frac{Q}{4\pi\epsilon_0 a^2}\frac{r}{a}\hat{\mathbf{r}} & \text{for } r \leq a, \\ \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} & \text{for } r > a. \end{cases}}} \tag{17}$$

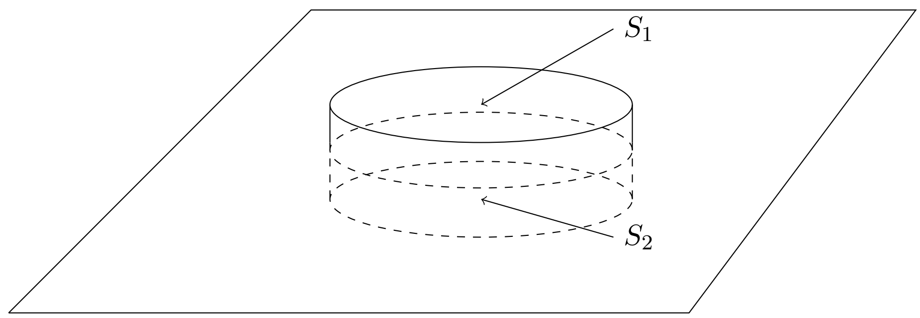
Problem 3

a) We use Gauss' law around the top plane with charge density ρ_s: We draw a cylindric Gaussian surface with a radius r that surrounds a part of the plane (see figure below). We divide the Gaussion surface into three subsurfaces: The top surface S_1, the bottom surface S_2 and the side wall. The charge that is surrounden by this Gaussian surface is consequently given by

$$Q = \int_{S_1} \rho_s dS = \rho_s \int_{S_1} dS = \rho_s \pi r^2. \tag{18}$$

Inserted into Gauss' law, this yields

$$\oint \mathbf{E} \cdot d\mathbf{S} = \frac{\rho_s \pi r^2}{\epsilon_0}. \tag{19}$$



The left hand side can be rewritten

$$\oint \mathbf{E} \cdot d\mathbf{S} = \int_{S_1} \mathbf{E} \cdot d\mathbf{S} + \int_{S_2} \mathbf{E} \cdot d\mathbf{S} + \int_{\text{side wall}} \mathbf{E} \cdot d\mathbf{S}, \tag{20}$$

where we integrate across the top surface S_1, the bottom surface S_2 and the side wall. Considering the symmetries we can show that the **E**-field should point along the surface norm of the plane. In order to show this we first imagine the opposite scenario: Imagine that the **E**-field has a component along the plane (i.e. it is diagonal to the surface). If we rotate the plane 180° around its own surface norm, the **E**-field will be pointing diagonally in the other direction. But because of symmetry, nothing is really changed with respect to the plane, yet the **E**-field is changed. This is then a contradiction. To avoid such a contradiction vi must assume that the **E**-field is pointing normally out of the plane. Note that this reasoning only is valid for all fields on the surface if area of the surface is considered infinite (why?): Generally the field lines along the edges of a surface will not be orthogonal to the surface.

From the above we get that the last integral in (20) is zero since $\mathbf{E} \perp d\mathbf{S}$ along the side walls of the Gaussian surface, while the other integrals is easily solved since $\mathbf{E} \parallel d\mathbf{S}$ on the surfaces S_1 og S_2. We get

$$\oint \mathbf{E} \cdot d\mathbf{S} = E \int_{S_1} dS + E \int_{S_2} dS = E 2\pi r^2 \tag{21}$$

Combined with (19) we get

$$\mathbf{E} = -\frac{\rho_{\text{s}}}{2\epsilon_0} \hat{\mathbf{y}}, \tag{22}$$

where $\hat{\mathbf{y}}$ defines the direction of the surface norm on the top surface. The field from the bottom surface is pointing in the same direction (due to reversed charge), and thus the total field is twice the value

$$\mathbf{E} = -\frac{\rho_{\text{s}}}{\epsilon_0} \hat{\mathbf{y}}. \tag{23}$$

Problem 4

a) First we consider a small surface element dS_i on the disc. On this element we find a charge equal to $Q_i = \rho_s dS_i$, and the electric field from this can be described as:

$$\mathbf{E}_i = \frac{\rho_s dS_i}{4\pi\epsilon_0 R_i^2} \hat{\mathbf{R}}_\mathbf{i}, \tag{24}$$

where R_i is the distance from the surface element dS_i to the observation point. The potential V_i that occurs in the observation point is then given by

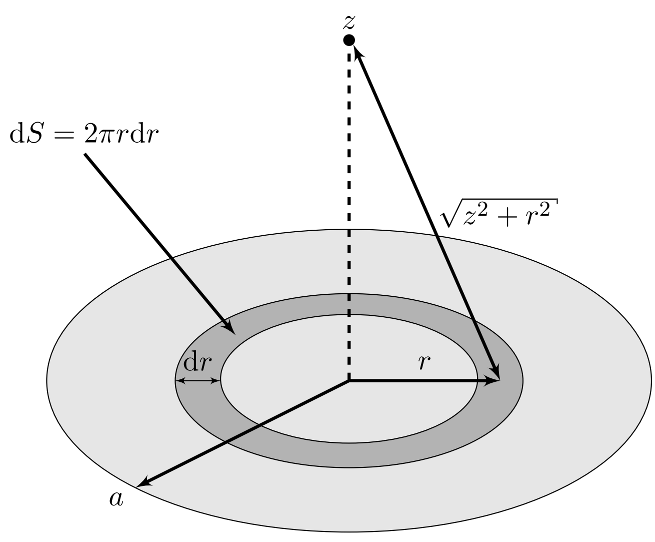
$$\begin{aligned} V_i &= \int_{R_i}^{\infty} E_i dR_i = \int_{R_i}^{\infty} \frac{\rho_s dS_i}{4\pi\epsilon_0 R_i^2} dR_i \\ &= \frac{\rho_s dS_i}{4\pi\epsilon_0 R_i}. \end{aligned} \tag{25}$$

We find the total potential by adding the contribution from all the surface elements dS_i where $i = 1, 2, 3, ...$ (superposition). In practice this is done by integrating across the entire surface of the disc.

$$V = \int_{\text{disc}} \frac{\rho_s dS}{4\pi\epsilon_0 R} = \frac{1}{4\pi\epsilon_0} \int_{\text{disc}} \frac{\rho_s dS}{R}. \tag{26}$$

b) From the figure below we see that $\mathrm{d}S = 2\pi r \mathrm{d}r$ og $R^2 = r^2 + z^2$. Derivation gives $2R\mathrm{d}R = 2r\mathrm{d}r$ so we can express

$$\begin{aligned} V(z) &= \frac{2\pi\rho_s}{4\pi\epsilon_0} \int_0^a \frac{r\mathrm{d}r}{R} \\ &= \frac{\rho_s}{2\epsilon_0} \int_z^{\sqrt{z^2+a^2}} \frac{R\mathrm{d}R}{R} \\ &= \underline{\underline{\frac{\rho_s}{2\epsilon_0} \left(\sqrt{z^2 + a^2} - z \right)}}. \end{aligned} \tag{27}$$



c) The electric field is given by $\mathbf{E} = -\nabla V$. Here $\mathbf{E} = (0, 0, E_z)$ due to the symmetry of the problem. We calculate directly

$$\begin{aligned} E_z &= -\frac{\partial}{\partial z} V(z) \\ &= \frac{\rho_s}{2\epsilon_0} \frac{\partial}{\partial z} \left(z - \sqrt{z^2 + a^2} \right) \\ &= \frac{\rho_s}{2\epsilon_0} \left(1 - \frac{z}{\sqrt{z^2 + a^2}} \right), \quad \text{for } z > 0. \end{aligned} \tag{28}$$

In vector form this is

$$\mathbf{E} = \frac{\rho_\mathrm{s}}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{z^2+a^2}}\right)\hat{\mathbf{z}}. \tag{29}$$

d) When $z \ll a$ will $1 - z/\sqrt{z^2+a^2} \approx 1$. Thus the **E**-field is

$$\mathbf{E}(z \to 0) = \lim_{z\to 0}\frac{\rho_\mathrm{s}}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{z^2+a^2}}\right)\hat{\mathbf{z}} = \frac{\rho_\mathrm{s}}{2\epsilon_0}\hat{\mathbf{z}}. \tag{30}$$

This is eqial to the field from an infinitely large, flat surface with a charge density ρ_s. To investigate the other boundary we use the approximation

$$\frac{z}{\sqrt{z^2+a^2}} = \left(1 + \frac{a^2}{z^2}\right)^{-1/2} \approx 1 - \frac{1}{2}\frac{a^2}{z^2}, \tag{31}$$

when a/z is small. This gives

$$\begin{aligned}\mathbf{E} &= \frac{\rho_\mathrm{s}}{2\epsilon_0}\left[1 - \left(1 - \frac{1}{2}\frac{a^2}{z^2}\right)\right]\hat{\mathbf{z}} \\ &= \frac{\rho_\mathrm{s}}{4\epsilon_0}\frac{a^2}{z^2}\hat{\mathbf{z}} \\ &= \frac{Q}{4\pi\epsilon_0 z^2}\hat{\mathbf{z}},\end{aligned} \tag{32}$$

where $Q = \rho_\mathrm{s}\pi a^2$ is the total charge of the disc. This expression resembles Coloumbs law. This is due to the fact that we are at the limit of being so far away from the disc compared to the radius of the disc, that the disc "looks like a point charge".

Problem 5

By superposition we can find the answer by subtracting the answer from task **4c)**, from the electric field from an ininite plane. Then we find

$$\begin{aligned}\mathbf{E} &= \frac{\rho_\mathrm{s}}{2\epsilon_0}\hat{\mathbf{z}} - \frac{\rho_\mathrm{s}}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{z^2+a^2}}\right)\hat{\mathbf{z}} \\ &= \frac{\rho_\mathrm{s}}{2\epsilon_0}\frac{z}{\sqrt{z^2+a^2}}\hat{\mathbf{z}}.\end{aligned} \tag{33}$$