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TFE4120 Electromagnetics - Crash course

Solution proposal, Exercise 1

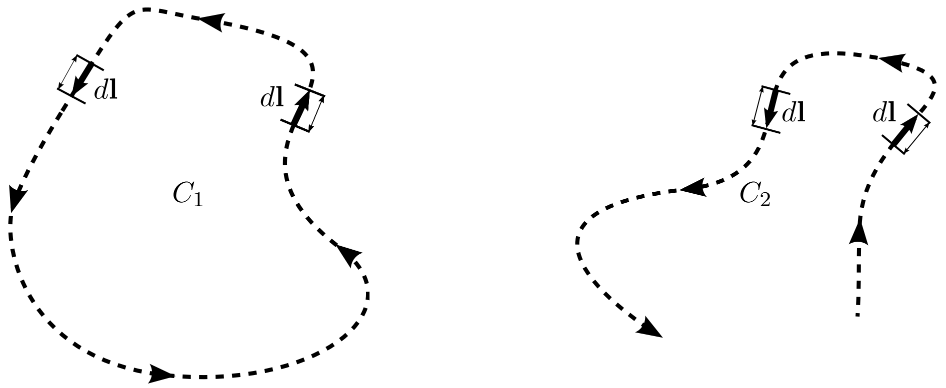
Problem 1

a) i) Here, alternative 1) is correct. If the mass density is F, a line segment $\mathrm{d}l$ will have a mass of $F\mathrm{d}l$. The total mass is then the sum of these segments, thus equal to $\int_C F\mathrm{d}l$.

ii) Here, alternative 4) is correct. Alternative 1) can also be considered correct, if $\mathbf{F}$ is assumed to be parallel with the curve C.

b) If F and $\mathbf{F}$ are constants, they may be put outside the integral sign. The integral of a vectorial line segment around a closed loop is always zero. To understand this you can imagine the integral to be a sum of small movements $\mathrm{d}\mathbf{l}$. The sum of all these movements is the total movement from the start position to the end position. And since the curve ends in the same point where it started the integral is equal to zero. Hence, alternative 2) and 4) will be zero for the closed loop C_1.

For the open loop, curve C_2, all integrals 1)-4) will be different from zero (except the very special case if $\mathbf{F}$ is normal to the vector from the starting to ending point of C_2).

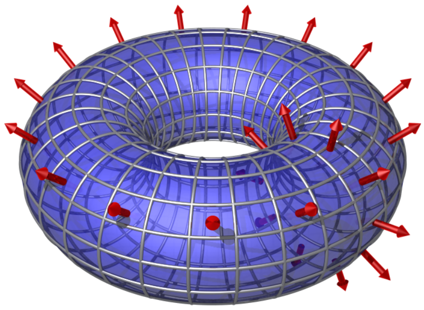


c) Here, the alternatives 2) and 4) will be correct for the closed surface, and both can be proved by the divergence theorem: $\oint_s \mathbf{F}\cdot\mathrm{d}\mathbf{S} = \int_v(\nabla\cdot\mathbf{F})\mathrm{d}v$. Proving that alternative 4) is equal to zero is easy. For this case $\mathbf{F}$ is constant, so the integrand $\nabla \cdot \mathbf{F}$, and by that the integral, is equal to zero.

In order to prove 2) we assume that $\mathbf{F}$ is a constant vector $\mathbf{F} = F\hat{\mathbf{a}}$. F is the absolute value of $\mathbf{F}$ while $\hat{\mathbf{a}}$ is the direction (both are constants). By using the divergence theorem we can write $\hat{\mathbf{a}} \cdot \oint_S F \mathrm{d}\mathbf{S} = \int_v (\nabla \cdot \mathbf{F}) \mathrm{d}v = 0$ since the integrand $\nabla \cdot \mathbf{F} = 0$ ($\mathbf{F}$ is constant). By that, integral 2) $\oint_S F \mathrm{d}\mathbf{S} = 0$ when F is a constant.

The integrals 1) and 3) is FS and $\mathbf{F}S$ respectively.

d) For a closed surface, there are always two normal vector for a given point on the surface, one pointing inwards and one pointing outwards.
Unless specified otherwise, we use the normal vectors pointing outwards.



e) The surface normal for an encircled surface is defined by the positive rotational direction. We use the standard right-hand rule: Let your fingers follow the rotational direction, and the normal vector for the surface is in the direction of your thumb. For the curve C_1 the surface normal vector is out-of-plane.

Problem 2

a) i) The flux of a vector field through a surface S is defined as

$$\text{Flux} = \int_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}, \tag{1}$$

and describes how much $\mathbf{F}$ "flows through" S. For instance, if $\mathbf{F}$ describes the flow of water, the flux describes the total amount of water that is running through S per unit time.

ii) Divergence is defined as

$$\text{div } \mathbf{F} = \lim_{\Delta V \to 0} \frac{\int_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}}{\Delta V} \tag{2}$$

where the volume ΔV is an infinitely small volume, and S is the surface of ΔV. From the definition we see that the divergence is the flux inn/out of the volume divided by the volume. In Cartesian coordinates the expression for divergence is:

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}, \tag{3}$$

where ∇ is defined as $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$.
The divergence states the source properties of $\mathbf{F}$, or in other words, to what extent $\mathbf{F}$ is spreading away from or towards a point. A source has a positive divergence while a drain has a negative divergence.

NOTE: It is only in Cartesian coordinates that the gradient and the divergence can be expressed as ∇F and $\nabla \cdot \mathbf{F}$ respectively, with ∇ defined as above. This is due to that the Cartesian coordinate system is globally orthogonal. For instance, in spherical coordinates, which is locally orthogonal[1] the gradient F is given by

$$\nabla F = \frac{\partial F}{\partial r}\hat{\mathbf{r}} + \frac{1}{r\sin\theta}\frac{\partial F}{\partial \varphi}\hat{\boldsymbol{\varphi}} + \frac{1}{r}\frac{\partial F}{\partial \theta}\hat{\boldsymbol{\theta}}, \tag{4}$$

and thus it is easy to think that the definition $\nabla = \frac{\partial}{\partial r}\hat{\mathbf{r}} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\hat{\boldsymbol{\varphi}} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\boldsymbol{\theta}}$ leads to the following divergence $\nabla \cdot \mathbf{F}$:

$$\frac{\partial F_r}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial \varphi} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta}. \tag{5}$$

This is *not* the correct result. The actual divergence can be derived by using a general coordinate transformation between Cartesian and spherical coordinates (we will not do this), and then we find

$$\text{div } \mathbf{F} = \frac{1}{r^2}\frac{\partial (r^2 F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial \varphi} + \frac{1}{r\sin\theta}\frac{\partial (\sin\theta F_\theta)}{\partial \theta}, \tag{6}$$

which is not the same as the initial expression. This is important to be aware of. So when dealing with cylindric and spherical coordinates in exercises or exams we recommend you to find the expressions for gradient, divergence and curl from a list of formulas.

iii) The x-component of the curl of $\mathbf{F}$ is defined as

$$\underline{\underline{(\text{curl } \mathbf{F})_x = \lim_{\Delta S \to 0} \frac{\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{l}}{\Delta S}}}, \tag{7}$$

where ΔS is a surface element which is normal to the x-axis and C is the closed loop that encircles this surface element. Equivalent definitions is also valid for the y- and z-components, where ΔS is defined as surface elements orthogonal to the y- and z-axis respectively.
The curl of a vector field describes to what extent the field is circulating around a point. This stands in contrast to the divergence which describes to what extent a field is flowing out from a point.
In Cartesian coordinates the curl has a simple expression:

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} \tag{8}$$

Curls in cylindric and spherical coordinates have a more complex form and can be found in lists of formulas.

[1] It is easy to picture this: Two people on earth, one on the north pole and the other on the south pole, are pointing to the sky. In this case they are in fact pointing in the opposite direction from each other!

iv) For *conservative* fields the line integral $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{l}$ is equal to zero for any curve C. By using Stoke's theorem one can prove that $\nabla \times \mathbf{F} = 0$. From the fundamental theorem (or Helmholtz' theorem) of calculus you can also write $\mathbf{F}$ as a simple scalar f so that $\mathbf{F} = \nabla f$.

b) i) The divergence theorem states

$$\underline{\underline{\int_V (\nabla \cdot \mathbf{F}) \mathrm{d}V = \oint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S},}} \tag{9}$$

where S is the closed surface that surrounds the volume V. Thus, the divergence theorem says that the total flux out of the volume is equal to the sum of all sources and drains inside the volume.

ii) Stoke's theorem states

$$\underline{\underline{\int_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{l},}} \tag{10}$$

where S is the surface encircled by the curve C. Stoke's theorem says that the circulation around the curve, $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{l}$, is equal to the sum of all the circulations across the surface S. A mental picture of Stoke's theorem can be imagined if you picture many whirlpools (circulations) in a lake (defined by the surface S). The total circulation along the shore is then the sum of all the circulation from the all the whirlpools!

Problem 3

a) By using the definition of divergence i Cartesian or spherical coordinates, we find that $\nabla \cdot \mathbf{F} = 3$. An infinitesimal volume in spherical coordinates is $\mathrm{d}V = r^2 \sin\theta \mathrm{d}r \mathrm{d}\theta \mathrm{d}\varphi$ so that

$$\begin{aligned} \int_v (\nabla \cdot \mathbf{F}) \mathrm{d}V &= \int_0^R \int_0^{2\pi} \int_0^{\pi} 3r^2 \sin\theta \mathrm{d}\varphi \mathrm{d}\theta \mathrm{d}r \\ &= 3 \int_0^R r^2 \mathrm{d}r \int_0^{\pi} \sin\theta \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi \\ &= \underline{\underline{4\pi R^3}}. \end{aligned} \tag{11}$$

Here, you can picture what the answer will be without solving the integral. Since the divergence is constantly equal to 3 across the entire volume, the answer must be 3 times the volume of the sphere, which gives $4\pi R^3$:

$$\underline{\underline{\int_V (\nabla \cdot \mathbf{F}) \mathrm{d}V = 3 \int_V \mathrm{d}V = 4\pi R^3.}} \tag{12}$$

b) The surface-element of a sphere with radius R is $\mathrm{d}\mathbf{S} = R^2 \sin\theta \mathrm{d}\theta \mathrm{d}\varphi \hat{\mathbf{r}}$. Note that since we integrate over the surface of the sphere, $r = R$ throughout the integral. By that we find

$$\begin{aligned}\int_v (\nabla \cdot \mathbf{F}) \mathrm{d}v &= \oint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} \\ &= \int_0^{2\pi} \int_0^{\pi} (R\hat{\mathbf{r}}) \cdot (R^2 \sin\theta \mathrm{d}\theta \mathrm{d}\varphi \hat{\mathbf{r}}) \\ &= R^3 \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \sin\theta \mathrm{d}\theta \\ &= \underline{\underline{4\pi R^3}}.\end{aligned} \tag{13}$$

The answer is also obvious here, since $\mathbf{F}$ is constant across the surface S equal to $\mathbf{F} = R\hat{\mathbf{r}}$. Since $\mathbf{F}$ and $\mathrm{d}\mathbf{S}$ points in the same direction the answer must be R times the surface of the sphere, and again we find the answer $4\pi R^3$:

$$\underline{\underline{\oint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = R \oint_S \mathrm{d}S = 4\pi R^3.}} \tag{14}$$

Problem 4

a) i) The line integral along the first segment $(0, 0) \to (a, 0)$ be equal to zero because the integrand is zero ($y = 0$, and $\hat{\mathbf{y}}$-component of $\mathbf{F}$ is orthogonal to the line segment). Thus, it is sufficient to integrate from $(a, 0)$ to (a, b) so that

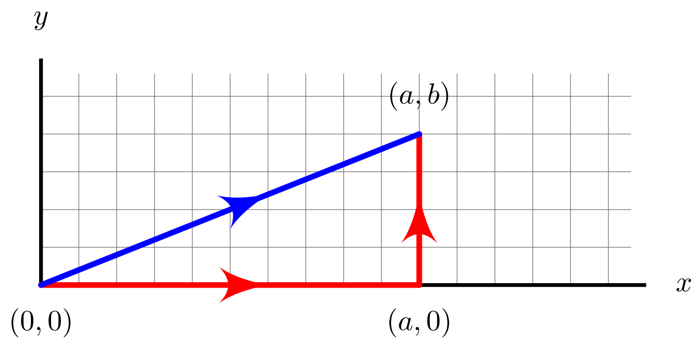
$$\begin{aligned}I &= \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{l} \\ &= \int_0^b (a^2 y + 2a) \mathrm{d}y \\ &= \underline{\underline{\frac{1}{2} a^2 b^2 + 2ab.}}\end{aligned} \tag{15}$$

ii) The straight line segment from (x, y) to $(x + \mathrm{d}x, y + \mathrm{d}y)$ is given by the vector $\mathrm{d}\mathbf{l} = \mathrm{d}x\hat{\mathbf{x}} + \mathrm{d}y\hat{\mathbf{y}}$. From that we find

$$\begin{aligned}\mathbf{F} \cdot \mathrm{d}\mathbf{l} &= F_x \mathrm{d}x + F_y \mathrm{d}y \\ &= (xy^2 + 2y)\mathrm{d}x + (x^2 y + 2x)\mathrm{d}y.\end{aligned} \tag{16}$$

In order to integrate this we need to find the correlation between x and y, which is $y = \frac{bx}{a}$, so that

$$\begin{aligned}I &= \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{l} \\ &= \int_0^a \left[x \left(\frac{bx}{a} \right)^2 + 2 \left(\frac{bx}{a} \right) \right] \mathrm{d}x + \int_0^b \left[\left(\frac{ay}{b} \right)^2 y + 2 \left(\frac{ay}{b} \right) \right] \mathrm{d}y \\ &= \underline{\underline{\frac{1}{2} a^2 b^2 + 2ab.}}\end{aligned} \tag{17}$$



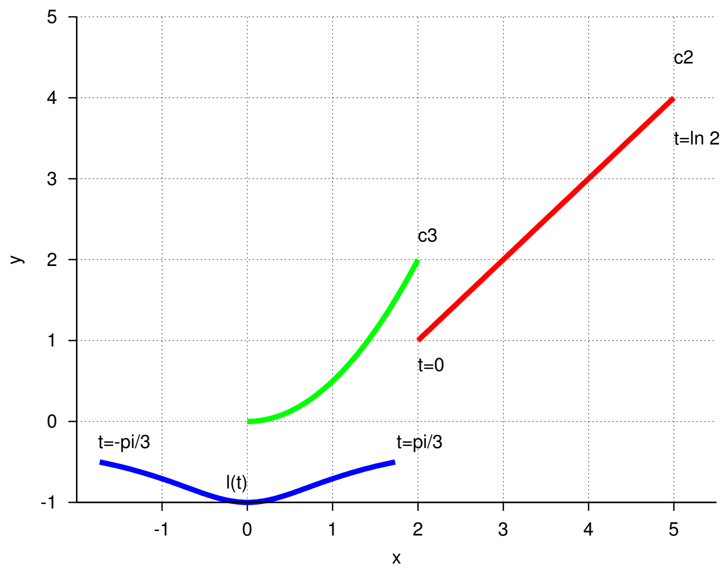
iii) These integrals have equal values since <u>**F** is a conservative field</u>:

$$\begin{aligned} \nabla \times \mathbf{F} &= \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{\mathbf{z}} \\ &= (2xy - 2xy)\,\hat{\mathbf{z}} \\ &= 0. \end{aligned} \tag{18}$$

For a conservative field, you only need to consider the start- and end points to solve the line integral. This is a general answer: We can imagine a closed loop C which consists of two curves C_1 and C_2. When we reverse the direction of one of the curves we find $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{l} = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{l} - \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{l} = 0$ so that

$$\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{l} = \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{l}. \tag{19}$$

b) The curves i)-iii) are shown in the figure below



i) The length of the parametrized line segment dl is

$$
\begin{aligned}
\mathrm{d}l &= \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} \\
&= \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t \\
&= 2\sqrt{2}\,e^{2t}\mathrm{d}t.
\end{aligned} \tag{20}
$$

If we put this together with the function $f[x(t), y(t)] = x(t) + y(t) = 1 + 2e^{2t}$ we find

$$
\begin{aligned}
\int_C f\mathrm{d}l &= \int_0^{\ln 2} (1 + 2e^{2t})2\sqrt{2}\,e^{2t}\mathrm{d}t \\
&= \sqrt{2}\left(e^{4t} + e^{2t}\right)\Big|_0^{\ln 2} \\
&= \underline{\underline{18\sqrt{2}}}.
\end{aligned} \tag{21}
$$

ii) The length of the line segment is $\mathrm{d}l = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \sqrt{1 + (\mathrm{d}y/\mathrm{d}x)^2}\,\mathrm{d}x$. With $y = \frac{1}{2}x^2$ we get $\mathrm{d}l = \sqrt{1 + x^2}\,dx$. If we evaluate $f = \frac{2y}{x}\sqrt{1 + x^2}$ along the line $y = \frac{1}{2}x^2$ we get $f = x\sqrt{1 + x^2}$ so that $f\mathrm{d}l = x(1 + x^2)$. Integration yields

$$
\begin{aligned}
\int_C f(x, y)\mathrm{d}l &= \int_0^2 x(1 + x^2)\mathrm{d}x \\
&= \left(\frac{x^2}{2} + \frac{x^4}{4}\right)\Big|_0^2 \\
&= \underline{\underline{6}}.
\end{aligned} \tag{22}
$$

iii) In this case the line segment is given by

$$
\begin{aligned}
\mathrm{d}\mathbf{l} &= \frac{\mathrm{d}\mathbf{l}}{\mathrm{d}t}\mathrm{d}t \\
&= (\hat{\mathbf{x}}\cos^{-2} t + \hat{\mathbf{y}}\sin t)\mathrm{d}t
\end{aligned} \tag{23}
$$

Thus, we find that $\mathbf{F} \cdot \mathrm{d}\mathbf{l} = (1 + 2\sin^2 t)\mathrm{d}t$ and

$$
\begin{aligned}
I &= \int_C \mathbf{F}(t) \cdot \mathrm{d}\mathbf{l}(t) \\
&= \int_{-\pi/3}^{\pi/3} (1 + 2\sin^2 t)\mathrm{d}t \\
&= (2t - \sin t\cos t)\,\Big|_{-\pi/3}^{\pi/3} \\
&= \underline{\underline{\frac{4\pi}{3} - \frac{\sqrt{3}}{2}}}.
\end{aligned} \tag{24}
$$