Exercise Session 3

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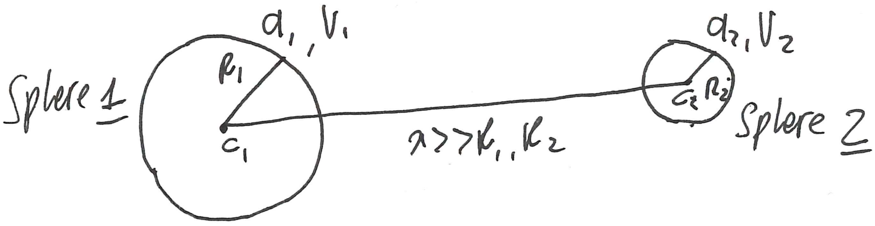
Two conductive spheres having radii R_1 and R_2 respectively are separated by a distance $x \gg R_1, R_2$ (center-to-center). On the sphere with R_1, an electric charge q_1 is deposited, while a charge q_2 is deposited on the sphere with R_2. The potential V_2 on the sphere with R_2 is known. Also, the charge q_1 on the sphere with R_1 is known. Calculate V_1 and q_2 and the interaction force between the spheres.

Suggestion: exploit the condition $x \gg R_1, R_2$



known variables V_2, q_1, R_1, R_2

Solution.

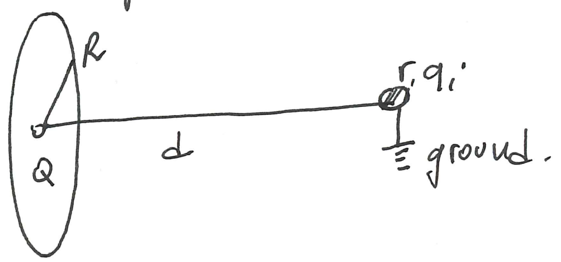
The potential on the surface of Sphere 1 is: $V_1 = \frac{q_1}{4\pi\varepsilon_0 R_1} + \frac{q_2}{4\pi\varepsilon_0 x}$ where $\frac{q_1}{4\pi\varepsilon_0 R_1}$ is the contribution from the charge q_1 on the sphere 1, while $\frac{q_2}{4\pi\varepsilon_0 x}$ is the contribution from the charge q_2, that can be imaged as point-like.
The condition $x \gg R_1$ allows us to assume that all the points on the surface of the sphere 1 are roughly at the same distance from q_2, therefore, the sphere 2 is producing a potential on the sphere 1 that is almost constant.
On sphere 2, the potential is: $V_2 = \frac{q_2}{4\pi\varepsilon_0 R_2} + \frac{q_1}{4\pi\varepsilon_0 x}$ (similar argument of above)
therefore $4\pi\varepsilon_0 V_2 = \frac{q_2}{R_2} + \frac{q_1}{x} \rightarrow q_2 = 4\pi\varepsilon_0 R_2 V_2 - \frac{R_2}{x} q_1$. We replace q_2 in V_1:

$$V_1 = \frac{q_1}{4\pi\varepsilon_0 R_1} + V_2 \frac{R_2}{x} - \frac{R_2 q_1}{4\pi\varepsilon_0 x^2}$$

The interaction force is calculated according to the Coulomb law, and is

$|\vec{F}| = \frac{q_1 q_2}{4\pi\varepsilon_0 x^2}$ where q_1 and q_2 are taken from above.

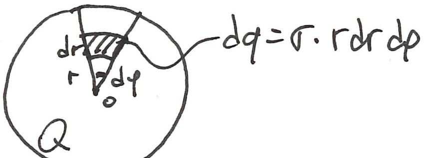
A conductive thin disk having radius R has a charge Q. Along the disk axis, at a distance d from the center, a conductive sphere is placed, having radius $r \ll d$. The conductive sphere is electrically grounded. Express the induced charge q_i on the sphere.

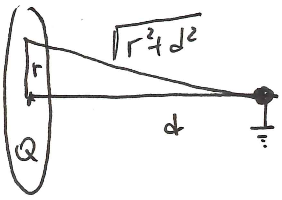


Solution.

The charge Q is homogeneously distributed on the disk surface with density $\sigma = \frac{Q}{\pi R^2}$. The potential produced by the disk at a distance d along the axis is obtained upon integration.

$$dV = \frac{dq}{4\pi\varepsilon\sqrt{d^2+r^2}}$$





$$V_{disk} = \int_0^R \int_0^{2\pi} \frac{\sigma r dr d\varphi}{4\pi\varepsilon\sqrt{d^2+r^2}} = \frac{\sigma}{2\varepsilon}\int_0^R \frac{r dr}{\sqrt{d^2+r^2}} = \frac{\sigma}{2\varepsilon}\left[\sqrt{d^2+r^2}\right]_0^R = \frac{\sigma}{2\varepsilon}\left(\sqrt{R^2+d^2} - d\right).$$

If we assume a charge q_i on the sphere, this charge would produce a potential $\frac{q_i}{4\pi\varepsilon r}$ [attention! r is the sphere radius here and not the integration variable used in the calculation of V_{disk}].

If the sphere is grounded we MUST assume that the sphere has some charge q_i such that the potential due to this charge AND the disk must be summed up to give zero. Therefore.

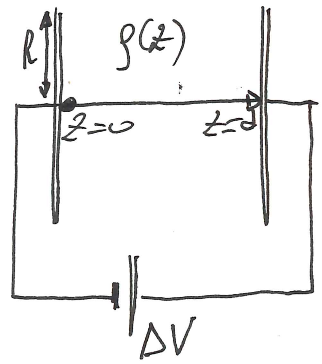
$$V_{disk} + \frac{q_i}{4\pi\varepsilon r} = 0 \quad \text{(since the sphere is grounded)}$$

$$\Rightarrow \quad q_i = -2\pi\sigma r\left(\sqrt{d^2+R^2} - d\right)$$

Note that: - if $Q>0$, then $q_i<0$.
- we associated the potential V_{disk} to all the points of the sphere because $d \gg r$.

A parallel-plate capacitor is constituted by a pair of circular conductive plates having radius R and separated by a distance $d \ll R$. The capacitor is connected to a device (generator) keeping a constant potential difference ΔV between the two plates. In the volume between the plates, there is a volume charge density ρ, which varies in the region $0<z<d$ as $\rho(z)=\rho_0 e^{-z/\lambda}$, ρ_0 and λ constants.

Express:
a) the electrostatic field within the capacitor
b) the electric charges induced on the capacitor plates. Is there complete electrostatic induction between the plates?



Solution.

a) Due to the planar geometry, we can safely assume $\bar{E} = E_z \cdot \bar{u}_z$. In addition, E_z may vary along z, therefore $E_z = E_z(z)$.

From the Gauss law, we have $\nabla \cdot \bar{E} = \frac{\rho}{\varepsilon_0}$, which translates into $\frac{dE_z}{dz} = -\frac{\rho}{\varepsilon_0}$ in our case. Therefore $\frac{dE_z}{dz} = \frac{1}{\varepsilon_0}\rho_0 e^{-z/\lambda}$. Upon integration:

$E_z(z) = \int \frac{\rho_0}{\varepsilon_0} e^{-z/\lambda} dz + A = -\frac{\lambda \rho_0}{\varepsilon_0} e^{-z/\lambda} + A$ where A is the integration constant.

In order to determine A, we consider that the potential difference from $z=d$ and $z=0$ is given by ΔV, as set by the generator.

$$\Delta V = V(d) - V(0) = -\int_0^d E_z(z) \cdot dz, \quad \text{upon integration of } \bar{E} = -\nabla V.$$

By replacing $E_z(z)$ as found above: $\Delta V = \frac{\rho_0 \lambda^2}{\varepsilon_0}\left(1-e^{-\frac{d}{\lambda}}\right) - A \cdot d$, therefore:

$$A = \frac{\rho_0 \lambda^2}{\varepsilon_0 d}\left(1-e^{-\frac{d}{\lambda}}\right) - \frac{\Delta V}{d} \quad \text{and} \quad E_z(z) = -\frac{\Delta V}{d} + \frac{\rho_0 \lambda^2}{\varepsilon_0 d}\left(1-e^{-\frac{d}{\lambda}}\right) - \frac{\rho_0 \lambda}{\varepsilon_0} e^{-\frac{z}{\lambda}}$$

b) By recalling that, for a planar distribution of charge with density σ, the electric field on the plane is $\left|\frac{\sigma}{\varepsilon_0}\right|$ if the plane is a conductive surface, we have

$E_z(0) = \frac{\sigma_1}{\varepsilon_0}$ and $E_z(d) = -\frac{\sigma_2}{\varepsilon_0}$ where σ_1 and σ_2 are the charges on the plate surfaces at $z=0$ and $z=d$ respectively.

$$\sigma_1 = -\frac{\Delta V \varepsilon_0}{d} + \frac{\rho_0 \lambda^2}{d}\left(1-e^{-\frac{d}{\lambda}}\right) - \rho_0 \lambda$$

$$\sigma_2 = \frac{\Delta V \varepsilon_0}{d} - \frac{\rho_0 \lambda^2}{d}\left(1-e^{-\frac{d}{\lambda}}\right) + \rho_0 \lambda e^{-\frac{d}{\lambda}}$$

Since $\sigma_1 \neq \sigma_2$, $Q_1 = \sigma_1 \cdot \pi R^2$ and $Q_2 = \sigma_2 \cdot \pi R^2$ are different → on the two plates there is a different charge (absolute value) → NO complete induction

A parallel-plate capacitor, having plate area $\Sigma = 400\ cm^2$ and plates separated by $d = 0.5\ cm$, is charged such that a $\Delta V = 50V$ is obtained. Then, the capacitor is isolated from the generator. The plates are brought apart to a final separation distance $d' = 2d = 1\ cm$. Calculate:

a) The ~~mean~~ potential difference $\Delta V'$ between the plates
b) the electrostatic fields before (E) and after (E') the separation of the plates
c) the initial and the final electrostatic energy of the systems U_e and U_e'.
d) the work done W to move apart the 2 plates.

Solution.

Once the capacitor is isolated, the same charge is found on the plates. So $q' = q$, where q' is the |charge| on the capacitor plate in the final configuration.

Initial capacitance $C = \frac{\varepsilon \Sigma}{d}$, Final capacitance $C' = \frac{\varepsilon \Sigma}{d'} = \frac{1}{2} C$

a) Therefore $\Delta V = \frac{q}{C}$ (initial) and $\Delta V' = \frac{q'}{C'} = \frac{q}{C'} = 2\Delta V$ (final) $= 100\ V$

b) $|\bar{E}| = \frac{\Delta V}{d}$ because $|\bar{E}|$ is uniform.

$|\bar{E}| = 10^4 \frac{V}{m}$ (initial) and $|\bar{E}'| = \frac{\Delta V'}{d'} = \frac{2\Delta V}{2d} = \frac{\Delta V}{d} = |\bar{E}|$ (final)

c) The electrostatic energy is given by integrating the energy density over the capacitor volume. Since $|\bar{E}|$ is uniform, the density $\frac{1}{2}\varepsilon |\bar{E}|^2$ is also uniform and the total electrostatic energy is

$U_e = \frac{1}{2}\varepsilon |\bar{E}|^2 \cdot T$ where $T = \Sigma \cdot d$ is the capacitor volume (initial)

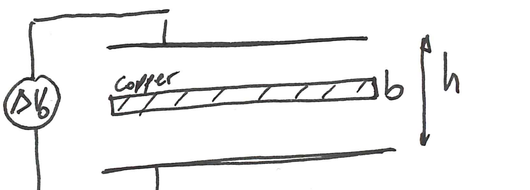
$U_e' = \frac{1}{2}\varepsilon |\bar{E}'|^2 \cdot T'$ where $T' = \Sigma \cdot d' \Rightarrow U_e' = \frac{1}{2}\varepsilon |\bar{E}'|^2 \cdot \Sigma \cdot 2d = 2U_e$

$U_e \cong 8 \cdot 10^{-8}\ J$, $U_e' \cong 1.6 \cdot 10^{-7}\ J$.

d) Since the potential energy of the system is increased, the work done by the electrostatic field is negative; i.e. an external force is needed to separate the plates. $W = |\Delta U_e| \simeq 8 \cdot 10^{-8}\ J$

A copper plate having thickness b= 0.3 cm is introduced within a parallel-plate capacitor having capacitance $C_0 = 100$ nF. The distance between the plates is h=0.5 cm. The capacitor is connected to a generator keeping a $\Delta V_0 = 12$ V between the capacitor plates. Calculate:

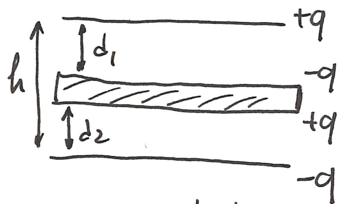
a) the capacitance C after the introduction of the copper plate
b) the electric field in the space between all the plates
c) the charge Δq on the capacitor plates, as provided by the generator.



Solution

In general, for a planar parallel-plate capacitor, $C_0 = \varepsilon_0 \frac{\Sigma}{h}$

When the copper plate is inserted, the electric field inside the copper plate is null, while the upper/lower surfaces of the plate will show inductively-produced charges. The copper plate has an overall zero charge!



a) Therefore this system is a serie of capacitors.

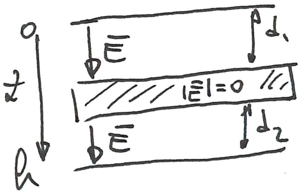
$$C_1 = \varepsilon_0 \frac{\Sigma}{d_1}, \quad C_2 = \varepsilon_0 \frac{\Sigma}{d_2} \qquad \frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} \Rightarrow C_{eq} = \frac{\varepsilon_0 \Sigma}{h-b}, \text{ since } d_2 + d_1 = h-b.$$

b) By expressing ΔV_0 as an integral of the electrostatic field across the gap between the two capacitor plates we have:

$$\Delta V_0 = \int_0^h \bar{E}\cdot d\bar{z} = \int_0^{d_1} \bar{E}\cdot d\bar{z} + \int_{d_1}^{d_1+b} \bar{E}\cdot d\bar{z} \cdot \int_{d_1+b}^{h} \bar{E}\cdot d\bar{z} = E\cdot d_1 + E\cdot d_2 = E(h-b)$$

because $E=0$ for $d_1 < z < d_1+b$

Therefore $|\bar{E}| = \frac{\Delta V_0}{h-b} \; \frac{V}{m}$



c) The potential difference ΔV_0 is kept constant by the generator. The capacitance has changed because the geometry has changed. Therefore, the electric charge on the plates must change as well:

$$q = C\cdot \Delta V_0 \Rightarrow \Delta q = \Delta C \cdot \Delta V_0 = (C_{eq} - C_0)\cdot \Delta V_0$$

Since $C_{eq} = C_0 \cdot \frac{h}{h-b} > C_0 \Rightarrow \Delta q > 0$. The generator must provide such an extra charge in order to keep ΔV_0 = constant.