TOPOLOGY OF SPONTANEOUS SYMMETRY BREAKING

EMERGENT HIGHER-FORM AND HIGHER-GROUP SYMMETRIES

CONSEQUENCES OF SSB

Nambu-Goldstone bosons :

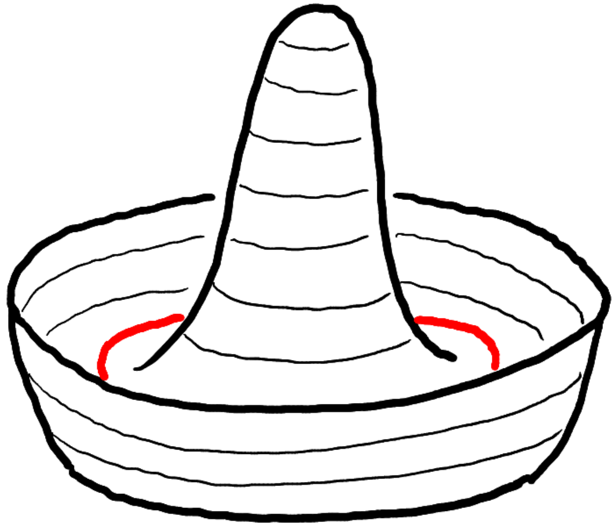
- sound waves in solids & liquids
- spin waves in (anti)ferromagnets
- Bogoliubov mode in superfluids
- pseudoscalar mesons (π, K, η) in nuclear matter
- photons in electromagnetism

... and many other examples!

CONSEQUENCES OF SSB

Emergent symmetries :

associated with the topology
of the vacuum manifold, lead to
topological current / charge

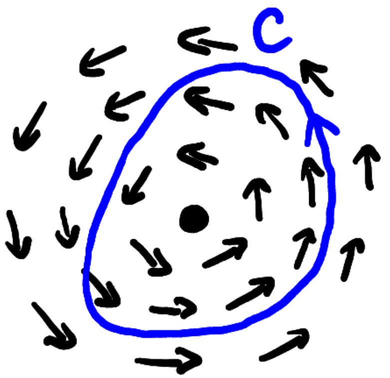


defects classified by
$\pi_1(S^1) = \mathbb{Z}$

EMERGENT SYMMETRY: SUPERFLUIDS

Topological charge : superfluid winding number $w_C[\phi]$

Topological current : $J^{\mu_1 \cdots \mu_{D-1}} \sim \epsilon^{\mu_1 \cdots \mu_D} \partial_{\mu_D} \phi$ or $J \sim * d\phi$



superfluid velocity

$\vec{v} \sim \vec{\nabla} \phi$

$$w_C[\phi] = \frac{1}{2\pi} \oint_C d\phi \sim \oint_C * J$$

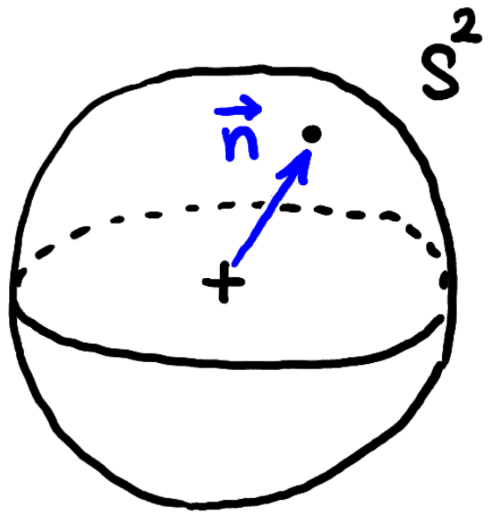
winding number can be nonzero in presence of superfluid vortices

EMERGENT SYMMETRY : SPIN SYSTEMS

Topological charge : winding number $w[\vec{n}]$ on S^2

Topological current : $J^{\mu_1 \cdots \mu_{D-2}} \sim \epsilon^{\mu_1 \cdots \mu_D} \vec{n} \cdot (\partial_{\mu_{D-1}} \vec{n} \times \partial_{\mu_D} \vec{n})$

or $J \sim * \epsilon^{ijk} n_i dn_j \wedge dn_k$



winding number in 2d detects the presence of skyrmions

$$w[\vec{n}] = \frac{1}{4\pi} \int d^2\vec{r} \, \vec{n} \cdot (\partial_x \vec{n} \times \partial_y \vec{n}) \sim \int_{\mathbb{R}^2} *J$$

SUPERFLUIDS: CLOSER LOOK

Low-energy effective theory:

fixed by equation of state

$$S[\phi, A] = \int d^D x \; P\left(\sqrt{D_\mu \phi D^\mu \phi}\right) + \dots$$

$$D_\mu \phi = \partial_\mu \phi - A_\mu$$

gauge field of the U(1) symmetry

Greiter et al. (1989)
Son (2002)

SUPERFLUIDS: CLOSER LOOK

Conserved currents $\longrightarrow$ topological

$\downarrow$

Noether

$$J^\mu = \frac{D^\mu \phi}{\sqrt{D_\nu \phi D^\nu \phi}} P'\left(\sqrt{D_\lambda \phi D^\lambda \phi}\right)$$

- depends on details of action
- conservation equivalent to equation of motion

$$K^{\mu_1 \cdots \mu_{D-1}} \sim \epsilon^{\mu_1 \cdots \mu_D} D_{\mu_D} \phi$$

or as a $(D-1)$-form

$$*K \sim D\phi = d\phi - A$$

- fixed by topology of $U(1)$
- off-shell conservation law

$$d*K \sim -dA = -F$$

- emergent $(D-2)$-form symmetry

SUPERFLUIDS : CLOSER LOOK

Gauging of the dual symmetry :

$$S[\phi, A, B] = \int d^D x\, P\left(\sqrt{D_\mu \phi D^\mu \phi}\right) + \frac{\varkappa}{(D-1)!} \int d^D x\, \underbrace{\epsilon^{\mu_1 \cdots \mu_D} B_{\mu_1 \cdots \mu_{D-1}} \left(\partial_{\mu_D} \phi - A_{\mu_D}\right)}_{\sim B_{\mu_1 \cdots \mu_{D-1}} K^{\mu_1 \cdots \mu_{D-1}}}$$

Background gauge invariance :

$$\phi \to \phi + \alpha$$

$$A_\mu \to A_\mu + \partial_\mu \alpha$$

$$B_{\mu_1 \cdots \mu_{D-1}} \to B_{\mu_1 \cdots \mu_{D-1}} + \left(\partial_{\mu_1} \beta_{\mu_2 \cdots \mu_{D-1}} + \text{cyclic}\right)$$

$$\delta S = \int d^D x\, \frac{(-1)^D}{2} \frac{\varkappa}{(D-2)!} \epsilon^{\mu_1 \cdots \mu_D} F_{\mu_1 \mu_2} \beta_{\mu_3 \cdots \mu_D}$$

mixed 't Hooft anomaly

SUPERFLUIDS : CLOSER LOOK

Gauging of the dual symmetry :

normalization

$$S[\phi, A, B] = \int P\left(\sqrt{D_\mu \phi D^\mu \phi}\right) *1 + \varkappa \int B \wedge (d\phi - A)$$

(D-1)-form source for K

Background gauge invariance :

$$\left.\begin{aligned} \phi &\to \phi + \alpha \\ A &\to A + d\alpha \\ B &\to B + d\beta \end{aligned}\right\} \longrightarrow \delta S = \int (-1)^D \varkappa\, \beta \wedge dA$$

mixed 't Hooft anomaly

HIGHER-FORM SYMMETRY

Continuous p-form symmetry : conserved rank-(p+1) antisymmetric tensor current

$$\partial_\nu J^{\nu\mu_1\dots\mu_p} = 0 \quad \longleftrightarrow \quad d^\dagger J^{(p+1)} = 0$$

$$\updownarrow$$

dual (D-p-1)-form current is closed $\qquad d * J^{(p+1)} = 0$

Zharinov (1992)

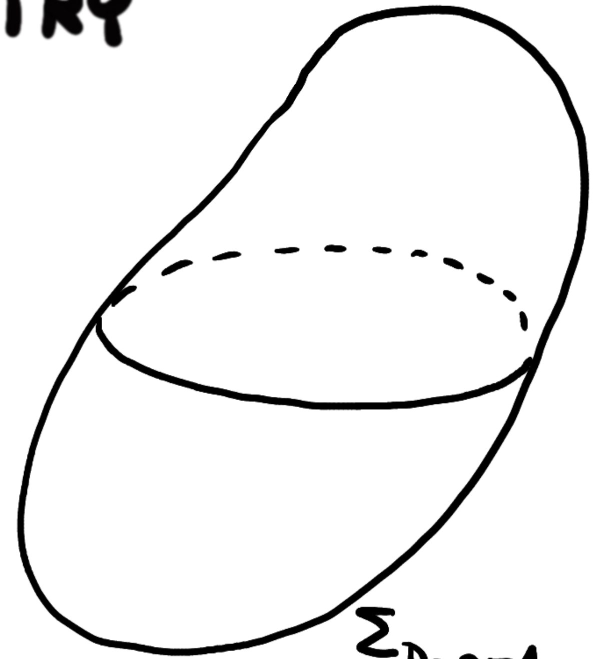
Bryant, Griffiths (1995)

Barnich et al. (1995)

Anderson, Torre (1996)

HIGHER-FORM SYMMETRY

Integral charge : measured on codimension-(p+1) manifolds



$$Q(\Sigma_{D-p-1}) = \int_{\Sigma_{D-p-1}} * J^{(p+1)}$$

$$= \int_{\Sigma_{D-p-1}} \frac{1}{(p+1)!(D-p-1)!} \epsilon_{\mu_1 \cdots \mu_D} J^{\mu_1 \cdots \mu_{p+1}} d\Sigma^{\mu_{p+2} \cdots \mu_D}$$

The charge is invariant under smooth deformations of Σ_{D-p-1} that do not cross a source of the current.

EXAMPLE: U(1) SUPERFLUIDS

0-form U(1) symmetry

$$d^\dagger J^{(1)} = 0 \quad \longleftrightarrow \quad \partial_\mu J^\mu = 0$$

degree

Noether symmetry

(D-2)-form dual symmetry

$$d^\dagger K^{[1]} = 0 \quad \longleftrightarrow \quad \partial_\nu K^{\nu\mu_2\cdots\mu_{D-1}} = 0$$

codegree

where $K^{\mu_1\cdots\mu_{D-1}} \sim \epsilon^{\mu_1\cdots\mu_D}\partial_{\mu_D}\phi$

emergent (topological) symmetry

EXAMPLE : MAXWELL'S THEORY

1-form electric symmetry

$$d^\dagger F^{(2)} = 0 \leftrightarrow \partial_\mu F^{\mu\nu} = 0$$

Noether symmetry

(D-3)-form magnetic symmetry

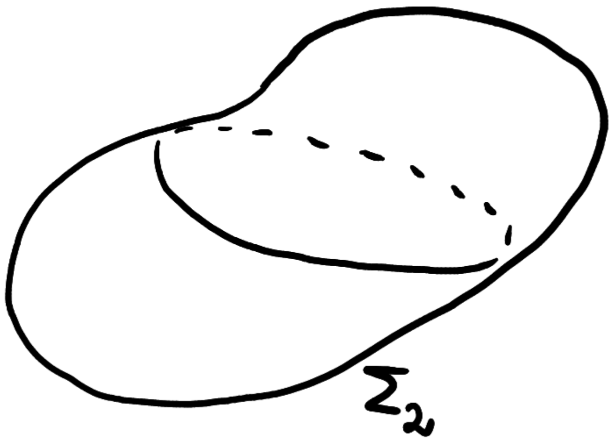
$$d^\dagger * F^{(2)} = 0 \leftrightarrow \epsilon^{\kappa\lambda\mu\nu}\partial_\lambda F_{\mu\nu} = 0$$

emergent (topological) symmetry

In D=4, both are 1-form symmetries!

$\int_{\Sigma_2} *F$ measures

electric flux through Σ_2



$\int_{\Sigma_2} F$ measures

magnetic flux through Σ_2

SSB OF HIGHER-FORM SYMMETRY

Goldstone theorem

SSB of p-form symmetry

$$\downarrow$$

p-form NG mode

$$J^{(p+1)} \sim d\phi^{(p)}$$

Coleman-Hohenberg-Mermin-Wagner theorem

SSB not possible if

$$p \geq D-2$$

(relies on Lorentz invariance)

Gaiotto et al. (2015)

Lake (2018)

Hofman, Iqbal (2019)

COMPOSITE CURRENTS

p-form symmetry

$$d * J^{(p+1)} = 0$$

q-form symmetry

$$d * K^{(q+1)} = 0$$

$\Longrightarrow$ $* J^{(p+1)} \wedge * K^{(q+1)}$ is closed

$\Downarrow$

$* \left[* J^{(p+1)} \wedge * K^{(q+1)} \right]$ is the current of a new $(p+q-D+1)$-form symmetry

requires $p+q \geq D-1$

TB (2021)

COMPOSITE CURRENTS

Simplest example : $p = q = 1$, $D = 3$

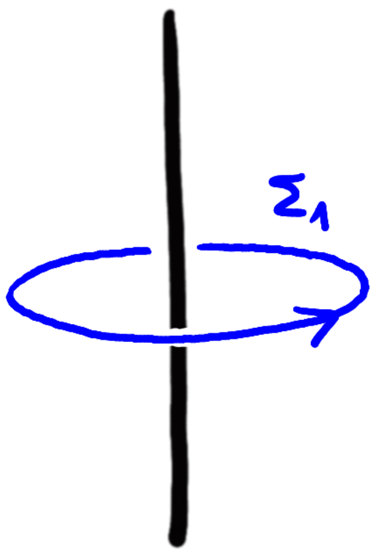
$$\left.\begin{matrix} J^{\mu\nu} \\ K^{\mu\nu} \end{matrix}\right\} \longrightarrow \begin{matrix} \text{composite current} \\ L^{\mu} \sim \epsilon^{\mu\nu\lambda} J_{\nu\alpha} K_{\lambda}{}^{\alpha} \end{matrix}$$

Two-component superfluid mixture :

$$\left.\begin{matrix} J^{\mu\nu} \sim \epsilon^{\mu\nu\lambda} \partial_{\lambda}\phi \\ K^{\mu\nu} \sim \epsilon^{\mu\nu\lambda} \partial_{\lambda}\theta \end{matrix}\right\} \longrightarrow K^{\mu} \sim \epsilon^{\mu\nu\lambda} \partial_{\nu}\phi \, \partial_{\lambda}\theta$$

COMPOSITE CURRENTS

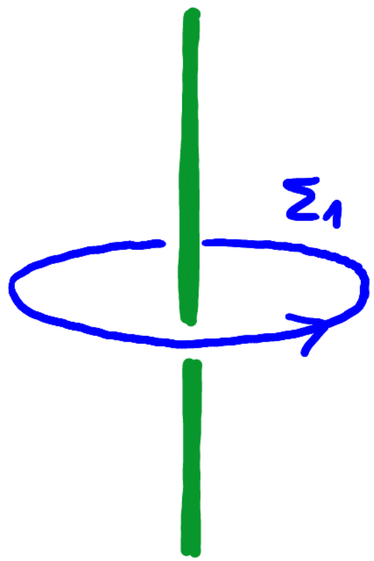
ϕ - vortex



$$Q(\Sigma_1) \sim \int_{\Sigma_1} d\phi$$

winding number of ϕ

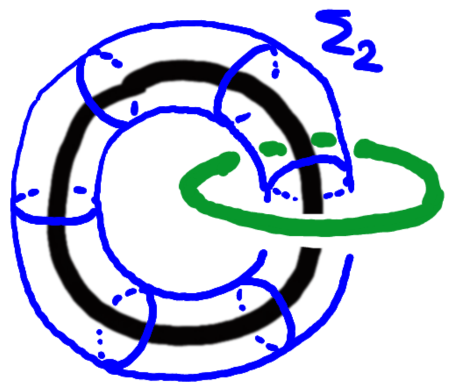
θ - vortex



$$Q(\Sigma_1) \sim \int_{\Sigma_1} d\theta$$

winding number of θ

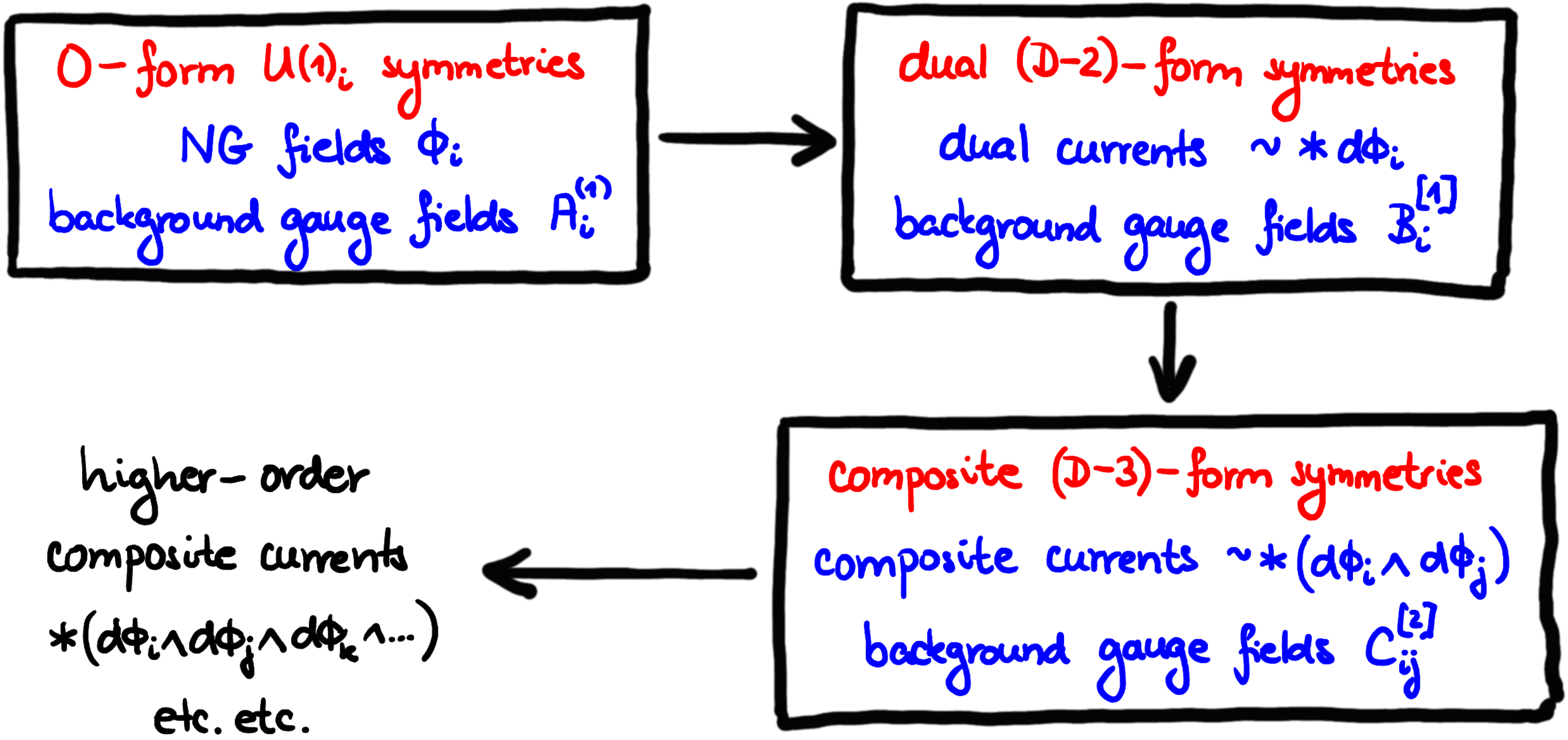
linked vortex rings



$$Q(\Sigma_2) \sim \int_{\Sigma_2} d\phi \wedge d\theta$$

linking of vortices

SUPERFLUID MIXTURES



SUPERFLUID MIXTURES

Low-energy EFT :

$$S[\phi, A, B, C] = S_{inv} + \int \left[\varkappa_1 B_i^{[1]} \wedge (d\phi_i - A_i^{(1)}) + \frac{\varkappa_2}{2} C_{ij}^{[2]} \wedge (d\phi_i - A_i^{(1)}) \wedge (d\phi_j - A_j^{(1)}) \right]$$

built out of $d\phi_i - A_i^{(1)}$

gauged currents to preserve $U(1)_i$ Noether symmetries

Background gauge invariance :

$$\left.\begin{aligned} \phi_i &\longrightarrow \phi_i + \alpha_i \\ A_i^{(1)} &\longrightarrow A_i^{(1)} + d\alpha_i^{(1)} \end{aligned}\right\} \text{ required for invariance of } S_{inv}$$

$$\left.\begin{aligned} B_i^{[1]} &\longrightarrow B_i^{[1]} + d\beta_i^{[2]} \\ C_{ij}^{[2]} &\longrightarrow C_{ij}^{[2]} + d\gamma_{ij}^{[3]} \end{aligned}\right\} \text{ this cannot be right !}$$

SUPERFLUID MIXTURES

Low-energy EFT : TB (2021)

$$S[\phi, A, B, C] = S_{inv} + \int \left[\varkappa_1 B_i^{[1]} \wedge (d\phi_i - A_i^{(1)}) + \frac{\varkappa_2}{2} C_{ij}^{[2]} \wedge (d\phi_i - A_i^{(1)}) \wedge (d\phi_j - A_j^{(1)}) \right]$$

Background gauge invariance :

$$\phi_i \longrightarrow \phi_i + \alpha_i$$

$$A_i^{(1)} \longrightarrow A_i^{(1)} + d\alpha_i$$

$$B_i^{[1]} \longrightarrow B_i^{[1]} + d\beta_i^{[2]} - (-1)^D \frac{\varkappa_2}{\varkappa_1} \gamma_{ij}^{[3]} \wedge dA_j^{(1)}$$

$$C_{ij}^{[2]} \longrightarrow C_{ij}^{[2]} + d\gamma_{ij}^{[3]}$$

Variation of the action : $\delta S = \int (-1)^D \varkappa_1 \beta_i^{[2]} \wedge dA_i^{(1)}$

separate mixed 't Hooft anomaly for each flavor

SUPERFLUID MIXTURES

Background gauge invariance :

$$\phi_i \longrightarrow \phi_i + \alpha_i$$

$$A_i^{(1)} \longrightarrow A_i^{(1)} + d\alpha_i$$

$$B_i^{[1]} \longrightarrow B_i^{[1]} + d\beta_i^{[2]} - (-1)^D \frac{\varkappa_2}{\varkappa_1} \gamma_{ij}^{[3]} \wedge dA_j^{(1)}$$

$$C_{ij}^{[2]} \longrightarrow C_{ij}^{[2]} + d\gamma_{ij}^{[3]}$$

Gauge-invariant field strength tensors :

$$F_i^{(2)} = dA_i^{(1)}$$

$$G_i^{[0]} = dB_i^{[1]} + (-1)^D \frac{\varkappa_2}{\varkappa_1} C_{ij}^{[2]} \wedge dA_j^{(1)}$$

$$H_{ij}^{[1]} = dC_{ij}^{[2]}$$

SUPERFLUID MIXTURES

Ward identities for currents :

$$*J^{(1)}_{Ai} = \frac{\delta S}{\delta A^{(1)}_i} \qquad *J^{[1]}_{Bi} = \frac{\delta S}{\delta B^{[1]}_i} \qquad *J^{[2]}_{Cij} = \frac{\delta S}{\delta C^{[2]}_{ij}}$$

$$d*J^{(1)}_{Ai} = 0 \longrightarrow J^{(1)}_{Ai} \text{ is conserved but not gauge-invariant}$$

$$\left.\begin{aligned} d*J^{[1]}_{Bi} &= -\varkappa_1\, dA^{(1)}_i \\ d*J^{[2]}_{Cij} &= -\frac{\varkappa_2}{\varkappa_1}\left(dA^{(1)}_i \wedge *J^{[1]}_{Bj} - dA^{(1)}_j \wedge *J^{[1]}_{Bi}\right) \end{aligned}\right\} \begin{array}{l} J^{[1]}_{Bi},\ J^{[2]}_{Cij} \text{ are} \\ \text{gauge-invariant but} \\ \text{not conserved} \end{array}$$

Conservation & gauge invariance of currents can be affected by adding local counterterms to the action.

EMERGENT TOPOLOGICAL SYMMETRIES

Nontrivial closed G-invariant p-forms on G/H : de Rham cohomology $H^p(G/H)$

D'Hoker, Weinberg (1994)
D'Hoker (1995)

$\Omega^{(p)}$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$
$D=1$	θ-term	WZ-term	—	—	—
$D=2$	1-form current	θ-term	WZ-term	—	—
$D=3$	2-form current	1-form current	θ-term	WZ-term	—
$D=4$	3-form current	2-form current	1-form current	θ-term	WZ-term

EMERGENT TOPOLOGICAL SYMMETRIES

$\Omega^{(p)}$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$
$D=1$	θ-term	WZ-term	—	—	—
$D=2$	1-form current	θ-term	WZ-term	—	—
$D=3$	2-form current	1-form current	θ-term	WZ-term	—
$D=4$	3-form current	2-form current	1-form current	θ-term	WZ-term

$(D-2)$-form dual symmetry in $U(1)$ superfluids

skyrmion current in (anti)ferromagnets

skyrmion current in QCD

EMERGENT TOPOLOGICAL SYMMETRIES

Decomposable cohomology generators: higher-degree generators as products of lower-degree ones

$$\Omega^{(p)} \sim \Omega^{(p_1)} \wedge \Omega^{(p_2)} \wedge \Omega^{(p_3)} \wedge \ldots \quad , \quad p = p_1 + p_2 + p_3 + \ldots$$

$\Omega^{(p)}$	$p=1$	$p=2$	$p=3$
primitive	$\Omega^{(1)}$	$\Omega^{(2)}$	$\Omega^{(3)}$
decomposable	—	$\Omega^{(1)} \wedge \Omega^{(1)}$	$\Omega^{(1)} \wedge \Omega^{(2)}$ $\Omega^{(1)} \wedge \Omega^{(1)} \wedge \Omega^{(1)}$

GENERALIZATION

Non-Abelian primitive symmetries :

- topological currents $*\Omega_a^{(p_a)}$
- composite currents $*(\Omega_a^{(p_a)} \wedge \Omega_b^{(p_b)})$
- background fields $B_a^{[p_a]}$, $C_{ab}^{[p_a+p_b]}$

What exactly are $\Omega_a^{(p_a)}$?

- generators of de Rham cohomology of G/H
- generally nonlinear functions of NG modes of G/H
- fully gauged under G
- may not be closed :

$$\begin{cases} d\Omega^{(1)} \sim F^{(2)} \\ d\Omega^{(2)} = 0 \\ d\Omega^{(3)} \sim F^{(2)} \wedge F^{(2)} \end{cases}$$

lead to mixed 't Hooft anomaly

GENERALIZATION

Non-Abelian primitive symmetries :

$$S = S_{inv} + \int \left(\varkappa_1 \sum_a B_a^{[p_a]} \wedge \Omega_a^{(p_a)} + \frac{\varkappa_2}{2} \sum_{a,b} C_{ab}^{[p_a+p_b]} \wedge \Omega_a^{(p_a)} \wedge \Omega_b^{(p_b)} \right)$$

Background gauge invariance :

$$A^{(1)} \longrightarrow g A^{(1)} g^{-1} + g\, d g^{-1} \,, \quad g \in G$$

$$B_a^{[p_a]} \longrightarrow B_a^{[p_a]} + d\beta_a^{[p_a+1]} + \frac{\varkappa_2}{\varkappa_1} \sum_b (-1)^{D+(p_a+1)(p_b+1)} \gamma_{ab}^{[p_a+p_b+1]} \wedge d\Omega_b^{(p_b)}$$

$$C_{ab}^{[p_a+p_b]} \longrightarrow C_{ab}^{[p_a+p_b]} + d\gamma_{ab}^{[p_a+p_b+1]}$$

$$\delta S = \int \sum_a (-1)^{D+p_a} \varkappa_1 \beta_a^{[p_a+1]} \wedge d\Omega_a^{(p_a)}$$

No modification of gauge transformation without mixed 't Hooft anomalies !

SUMMARY & OPEN QUESTIONS

Summary :

SSB of 0-form symmetry → Emergent topological symmetry → Hierarchy of composite currents → "Higher group" symmetry

Open questions :

- precise mathematical structure ?
- specific physical predictions ?