Polarization Sums and Unitarity in QCD

Magnus N. Malmquist

May 7, 2021

Outline

- Introduction: Polarization sums in QED.

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- What about QCD?

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- Unitarity in QCD.
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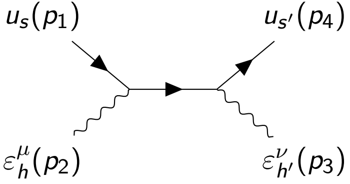
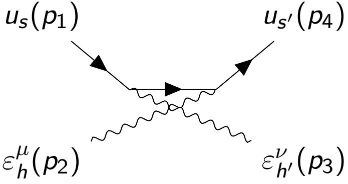
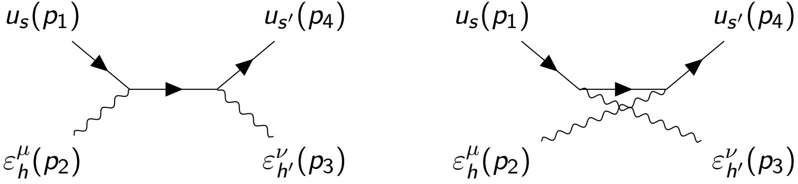
Polarization sums in QED





Polarization sums in QED



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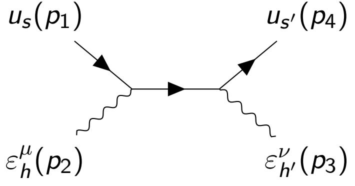

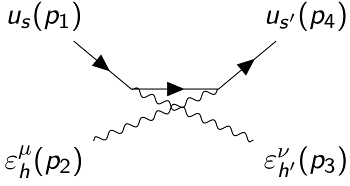

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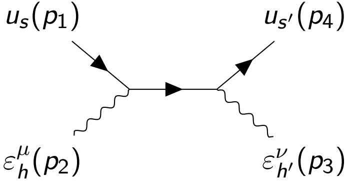
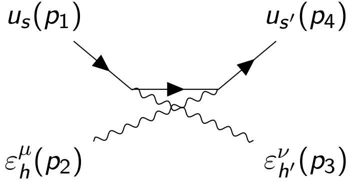
Polarization sums in QED





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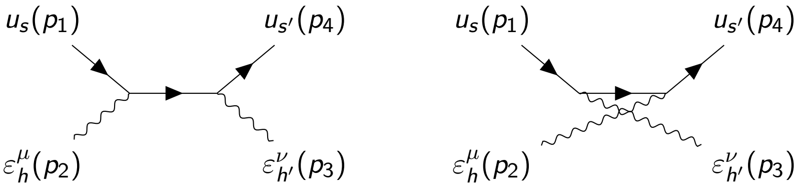
Polarization sums in QED



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$$\varepsilon_+^{\mu}(p) = (\omega_p, \mathbf{p})/(\sqrt{2}\omega_p) \qquad \varepsilon_-^{\mu}(p) = (\omega_p, -\mathbf{p})/(\sqrt{2}\omega_p)$$

$$\mathcal{P}^{\mu\mu'} - \left(\varepsilon_+^{\mu*}\varepsilon_-^{\mu'} + \varepsilon_-^{\mu*}\varepsilon_+^{\mu'}\right) = -\eta^{\mu\mu'}$$

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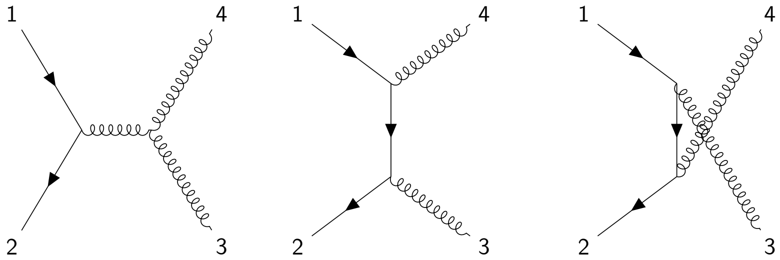
...and the Ward identity

$$\varepsilon_+^{\mu}(p)\mathcal{A}_{\mu} = 0.$$

What about QCD?

Our main example:



The simplest example (maybe too simple?).

What about QCD?

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What about QCD?

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$$\varepsilon(p_3) \to \varepsilon_+(p_3) \qquad \Longrightarrow \qquad \mathcal{A} \to \mathcal{A}' \propto p_4 \cdot \varepsilon(p_4)$$

No.

What about QCD?

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No.

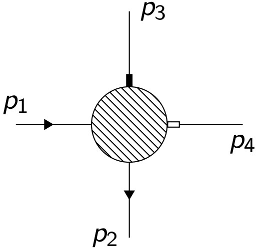
Using $\mathcal{P}^{\mu\mu'} \to -\eta^{\mu\mu'}$ on both gluons adds

$$\varepsilon(p_3) \to \varepsilon_+(p_3), \quad \varepsilon(p_4) \to \varepsilon_-(p_4)$$

$$\mathcal{A} \to g^2(T^c T^d - T^d T^c)\frac{\bar{v}(p_2)\not{p}_3 u(p_1)}{s} \neq 0$$

What about QCD?

Some notation:



$$= g^2(T^c T^d - T^d T^c)\frac{\bar{v}(p_2)\not{p}_3 u(p_1)}{s}$$

Unitarity

The optical theorem

$$2\,\mathrm{Im}\{\mathcal{A}(i \to i)\} = \sum_{\xi_1\xi_2} \int |\mathcal{A}(i \to \xi_1\xi_2)|^2 \, \mathrm{d}\Phi^{(2)} + \dots$$

Unitarity

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$$\mathrm{Disc}\{\mathcal{A}(s)\} = \mathcal{A}(s + \mathrm{i}\epsilon) - \mathcal{A}(s - \mathrm{i}\epsilon) = 2\mathrm{i}\,\mathrm{Im}\{\mathcal{A}(s + \mathrm{i}\epsilon)\}$$

Cutkosky's rule

Cutkosky (1960) gives us the rule for finding Disc$\{\mathcal{A}(s)\}$.

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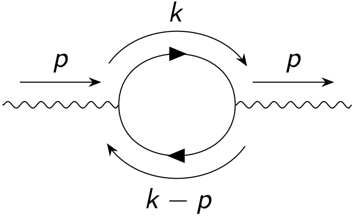

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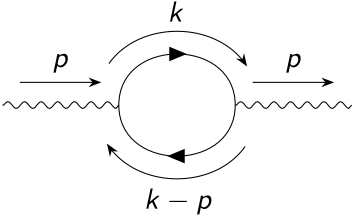

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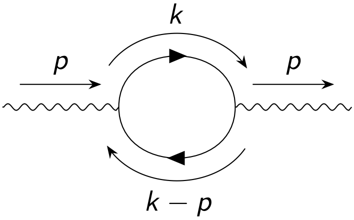

$$s = p^2 = (q_1 + q_2)^2 = (k - (k - p))^2.$$

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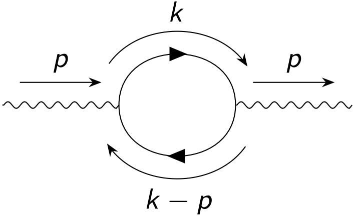

$$s = p^2 = (q_1 + q_2)^2 = (k - (k-p))^2.$$

$$\frac{1}{q_i^2 - m^2} \to 2\pi \mathrm{i} \delta^{(+)}(q_i^2 - m^2).$$

Cutkosky's rule

Propagator numerator:

$$\not{q}_1 + m = \sum_s u_s(q_1)\bar{u}_s(q_1) \qquad \not{q}_2 - m = \sum_s v_s(q_2)\bar{v}_s(q_2).$$

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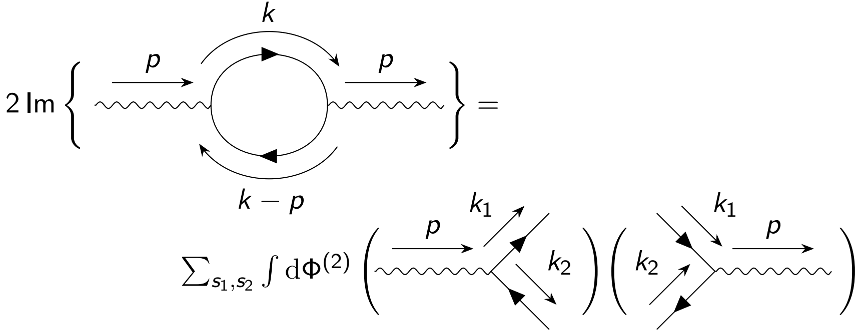

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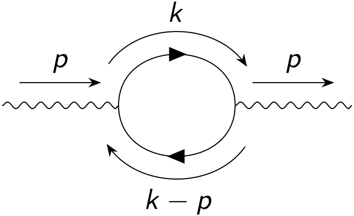
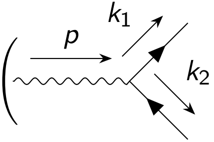
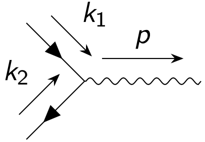
$2\,\mathrm{Im}\Big\{$  $\Big\} =$

$\sum_{s_1,s_2} \int \mathrm{d}\Phi^{(2)} \Big($  $\Big)\Big($  $\Big)$

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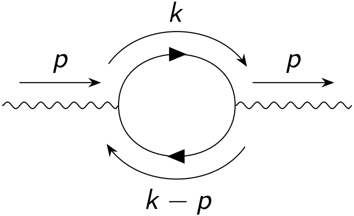
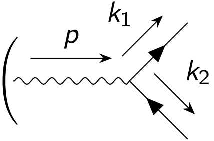
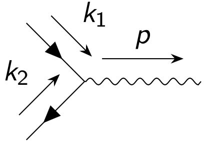
$$(\bar{u}\gamma^{\mu_1}\dots\gamma^{\mu_n}v)^* = \bar{v}\gamma^{\mu_n}\dots\gamma^{\mu_1}u,$$

$2\,\mathrm{Im}\Big\{$  $\Big\} =$

$\sum_{s_1,s_2}\int \mathrm{d}\Phi^{(2)}\Big($  $\Big)\Big($  $\Big)$

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Incoming $\leftrightarrow$ Outgoing

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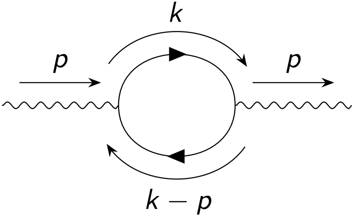

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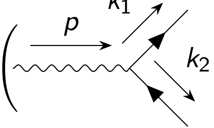

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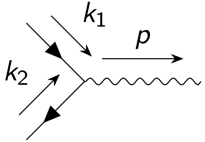

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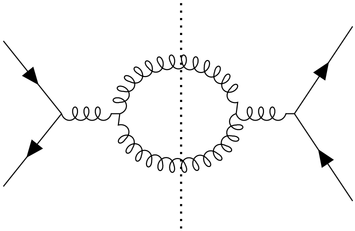
Unitarity in QCD



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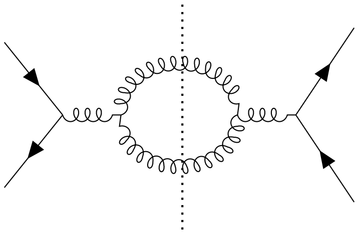

Propagator numerator: (Feynman-t' Hooft gauge)

$$-\eta^{\mu\mu'}.$$

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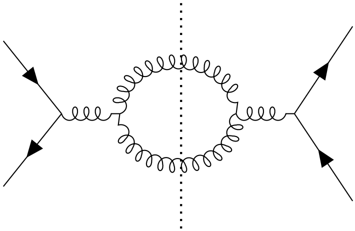

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Cutkosky's rule gives

$$\int -\eta_{\mu\mu'}\mathcal{A}_s^{\mu}(q\bar{q}\to gg)\mathcal{A}_s^{\mu'}(q\bar{q}\to gg)^*\,\mathrm{d}\Phi^{(2)}$$

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What is missing?

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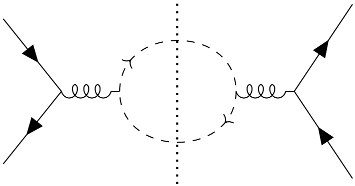

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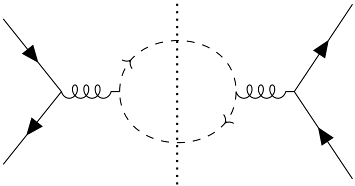
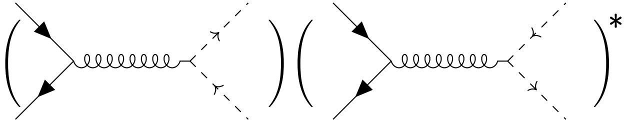
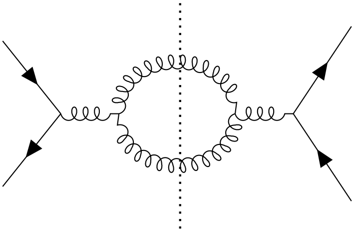


$=$

$-\int \mathrm{d}\Phi^{(2)}$



Unitarity in QCD



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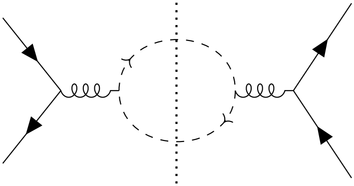

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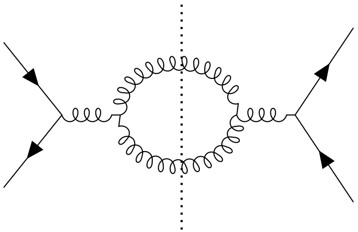
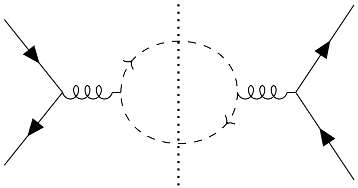


-2 

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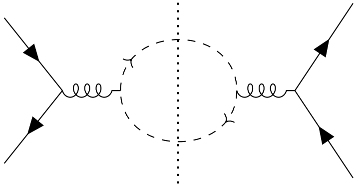
-2 

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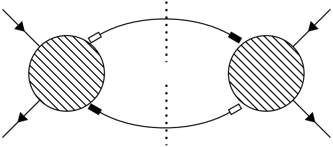
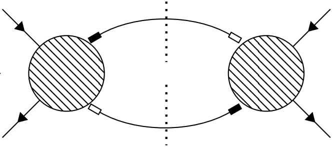
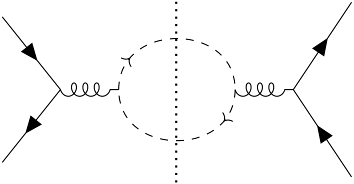
$+$  $+$ 

-2 

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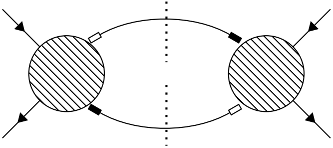
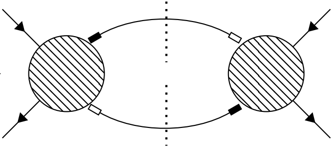
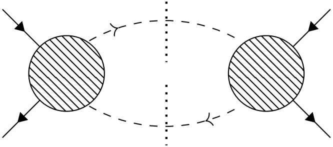
$+$  $+$ 

-2 

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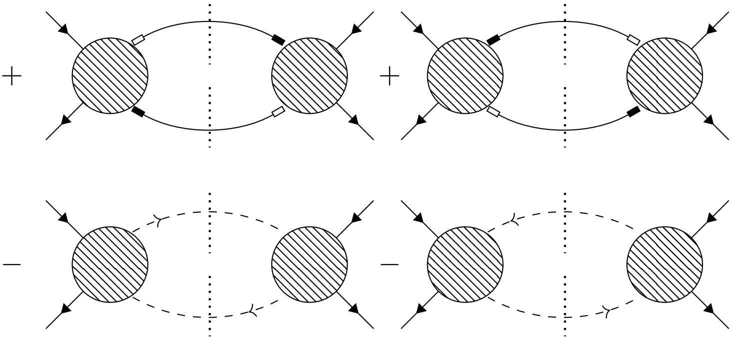
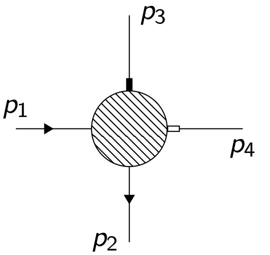


Unitarity in QCD

We found earlier:



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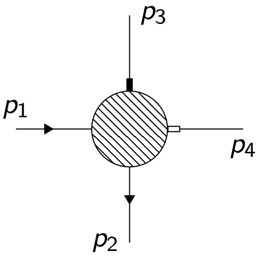
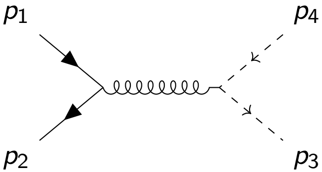
Unitarity in QCD

We found earlier:



$$= g^2(T^c T^d - T^d T^c)\frac{\bar{v}(p_2)\not{p}_3 u(p_1)}{s}$$

...which is minus



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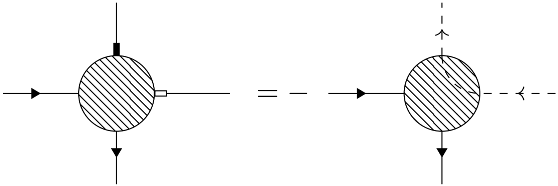

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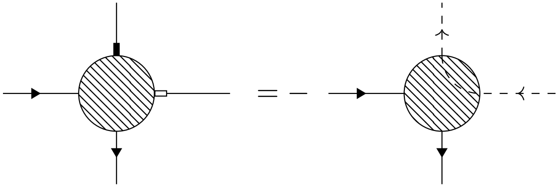
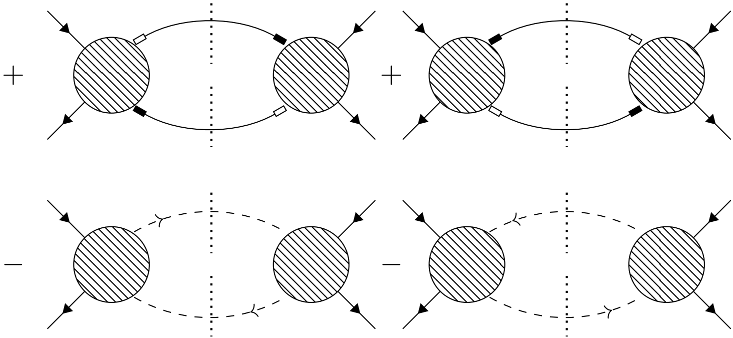


$\int \sum_{h,h'} |\mathcal{A}_s(q\bar{q} \to gg)|^2 \, \mathrm{d}\Phi^{(2)}$



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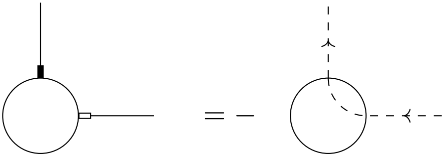
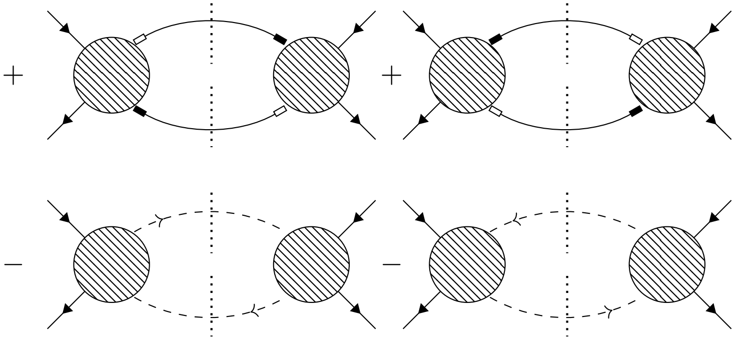


$\int \sum_{h,h'} |\mathcal{A}_s(q\bar{q} \to gg)|^2 \, \mathrm{d}\Phi^{(2)}$



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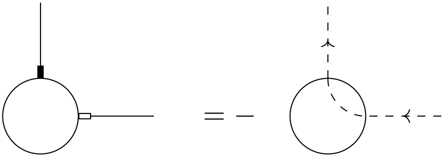
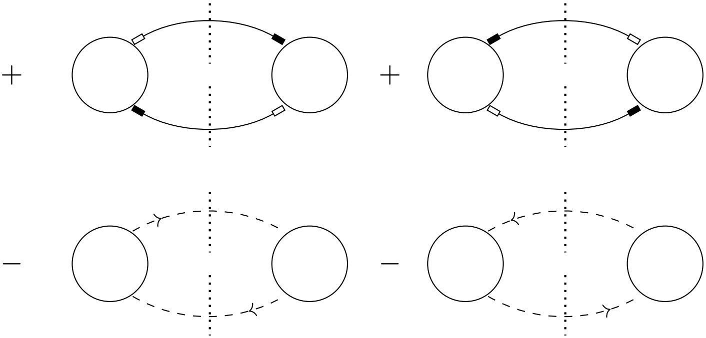


$\int \sum_{h,h'} |\mathcal{A}|^2 \, \mathrm{d}\Phi^{(2)}$



$\mathcal{P}^{\mu\mu'} \to -\eta^{\mu\mu'}$ in QCD

$$\eta_{\mu\mu'}\eta_{\nu\nu'}\mathcal{A}^{\mu\nu}(q\bar{q} \to gg)\mathcal{A}^{\mu'\nu'}(q\bar{q} \to gg)^*$$

$\mathcal{P}^{\mu\mu'} \to -\eta^{\mu\mu'}$ in QCD

$$\eta_{\mu\mu'}\eta_{\nu\nu'}\mathcal{A}^{\mu\nu}(q\bar{q} \to gg)\mathcal{A}^{\mu'\nu'}(q\bar{q} \to gg)^*$$

$$+ \ \ \mathcal{A}(q\bar{q} \to c\bar{c})\mathcal{A}(q\bar{q} \to \bar{c}c)^*$$

$\mathcal{P}^{\mu\mu'} \to -\eta^{\mu\mu'}$ in QCD

$$\eta_{\mu\mu'}\eta_{\nu\nu'}\mathcal{A}^{\mu\nu}(q\bar{q} \to gg)\mathcal{A}^{\mu'\nu'}(q\bar{q} \to gg)^*$$

$$+\,2\mathcal{A}(q\bar{q} \to c\bar{c})\mathcal{A}(q\bar{q} \to \bar{c}c)^*$$

$\mathcal{P}^{\mu\mu'} \to -\eta^{\mu\mu'}$ in QCD

$$\sum_{hh'} |\mathcal{A}(q\bar{q} \to gg)|^2 = \eta_{\mu\mu'}\eta_{\nu\nu'}\mathcal{A}^{\mu\nu}(q\bar{q} \to gg)\mathcal{A}^{\mu'\nu'}(q\bar{q} \to gg)^* \\ + 2\mathcal{A}(q\bar{q} \to c\bar{c})\mathcal{A}(q\bar{q} \to \bar{c}c)^*$$

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