

Polarization Sums and Unitarity in QCD

Magnus N. Malmquist

May 7, 2021

- ▶ Introduction: Polarization sums in QED.

Outline

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- ▶ What about QCD?

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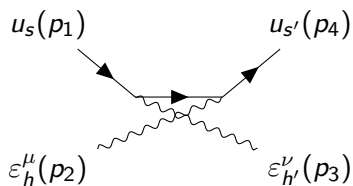
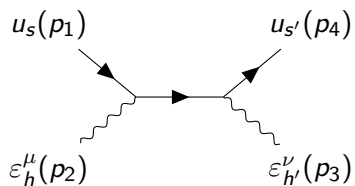
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- ▶ Introduction: Polarization sums in QED.
- ▶ What about QCD?
- ▶ Unitarity.
- ▶ Unitarity in QCD.

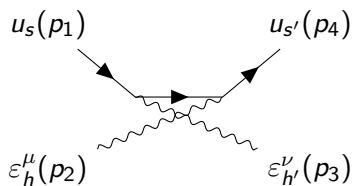
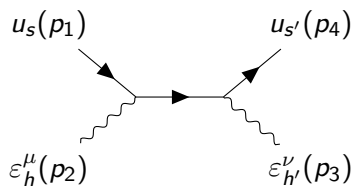
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- ▶ Unitarity.
- ▶ Unitarity in QCD.
- ▶ Polarization sums in QCD.

Polarization sums in QED

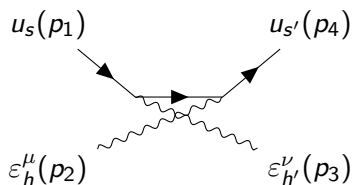
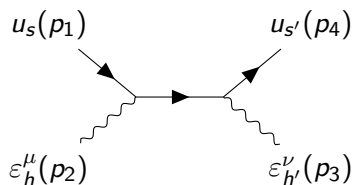


Polarization sums in QED



$$\mathcal{A} = \varepsilon_h^\mu(p_2) \mathcal{A}_\mu$$

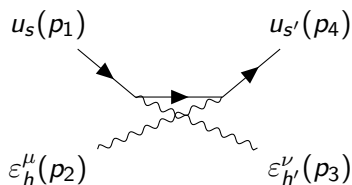
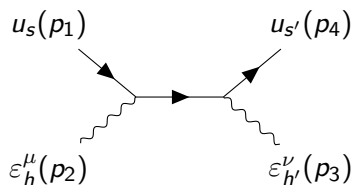
Polarization sums in QED



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$$\frac{1}{2^2} \sum_{s,s'} \sum_{h,h'} |\mathcal{A}|^2$$

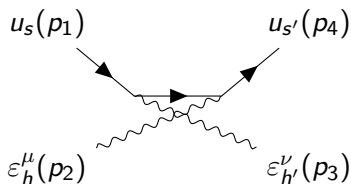
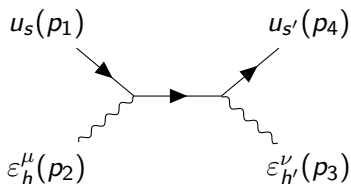
Polarization sums in QED



$$\mathcal{A} = \varepsilon_h^\mu(p_2) \mathcal{A}_\mu$$

$$\frac{1}{2^2} \sum_{s,s'} \sum_{h,h'} |\mathcal{A}|^2 = \frac{1}{2^2} \sum_{s,s'} \sum_{h,h'} \varepsilon_h^\mu(p_2) \varepsilon_{h'}^{\mu'}(p_2)^* \mathcal{A}_\mu \mathcal{A}_{\mu'}^*$$

Polarization sums in QED



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Polarization sums in QED

$$\mathcal{P}^{\mu\mu'} = \sum_{h,h'} \varepsilon_h^\mu(p_2) \varepsilon_h^{\mu'}(p_2)^* \rightarrow -\eta^{\mu\mu'}$$

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Result of:

$$\varepsilon_+^\mu(\mathbf{p}) = (\omega_p, \mathbf{p}) / (\sqrt{2}\omega_p) \quad \varepsilon_-^\mu(\mathbf{p}) = (\omega_p, -\mathbf{p}) / (\sqrt{2}\omega_p)$$

$$\mathcal{P}^{\mu\mu'} - \left(\varepsilon_+^{\mu*} \varepsilon_-^{\mu'} + \varepsilon_-^{\mu*} \varepsilon_+^{\mu'} \right) = -\eta^{\mu\mu'}$$

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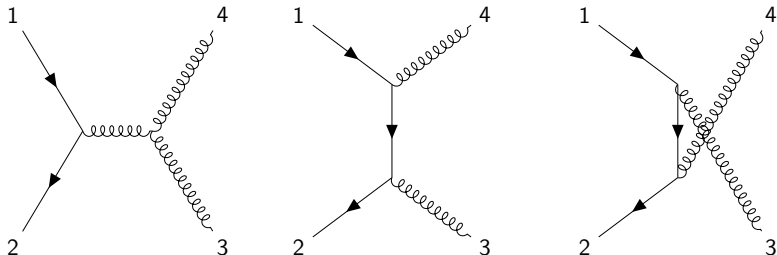
$$\mathcal{P}^{\mu\mu'} - \left(\varepsilon_+^{\mu*} \varepsilon_-^{\mu'} + \varepsilon_-^{\mu*} \varepsilon_+^{\mu'} \right) = -\eta^{\mu\mu'}$$

...and the Ward identity

$$\varepsilon_+^\mu(\mathbf{p}) \mathcal{A}_\mu = 0.$$

What about QCD?

Our main example:



The simplest example (maybe too simple?).

What about QCD?

Does the Ward identity hold?

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Does the Ward identity hold?

$$\varepsilon(p_3) \rightarrow \varepsilon_+(p_3) \quad \Longrightarrow \quad \mathcal{A} \rightarrow \mathcal{A}' \propto p_4 \cdot \varepsilon(p_4)$$

No.

What about QCD?

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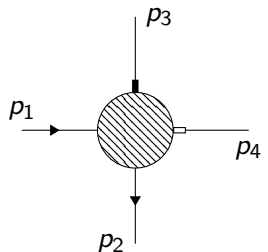
Using $\mathcal{P}^{\mu\mu'} \rightarrow -\eta^{\mu\mu'}$ on both gluons adds

$$\varepsilon(p_3) \rightarrow \varepsilon_+(p_3), \quad \varepsilon(p_4) \rightarrow \varepsilon_-(p_4)$$

$$\mathcal{A} \rightarrow g^2 (T^c T^d - T^d T^c) \frac{\bar{v}(p_2) \not{p}_3 u(p_1)}{s} \neq 0$$

What about QCD?

Some notation:



A Feynman diagram showing a central shaded circular vertex with four external lines. An incoming line from the left is labeled p_1 with an arrow pointing right. An outgoing line to the right is labeled p_4 . An incoming line from the top is labeled p_3 with an arrow pointing down. An outgoing line to the bottom is labeled p_2 with an arrow pointing down.

$$= g^2(T^c T^d - T^d T^c) \frac{\bar{v}(p_2) \not{p}_3 u(p_1)}{s}$$

The optical theorem

$$2 \operatorname{Im}\{\mathcal{A}(i \rightarrow i)\} = \sum_{\xi_1 \xi_2} \int |\mathcal{A}(i \rightarrow \xi_1 \xi_2)|^2 d\Phi^{(2)} + \dots$$

Unitarity

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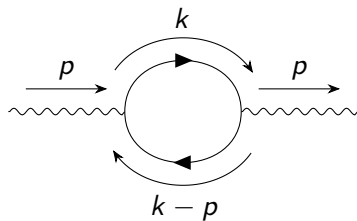
$$\operatorname{Disc}\{\mathcal{A}(s)\} = \mathcal{A}(s + i\epsilon) - \mathcal{A}(s - i\epsilon) = 2i \operatorname{Im}\{\mathcal{A}(s + i\epsilon)\}$$

Cutkosky's rule

Cutkosky (1960) gives us the rule for finding $\text{Disc}\{\mathcal{A}(s)\}$.

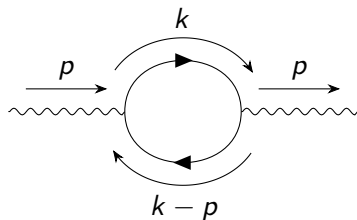
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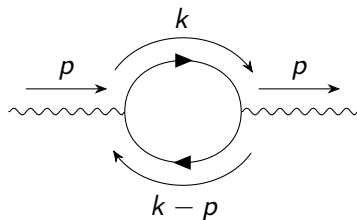
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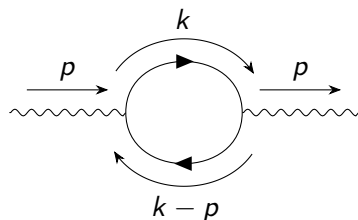
Cutkosky (1960) gives us the rule for finding $\text{Disc}\{\mathcal{A}(s)\}$.



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$$s = p^2 = (q_1 + q_2)^2 = (k - (k - p))^2.$$

$$\frac{1}{q_i^2 - m^2} \rightarrow 2\pi i \delta^{(+)}(q_i^2 - m^2).$$

Cutkosky's rule

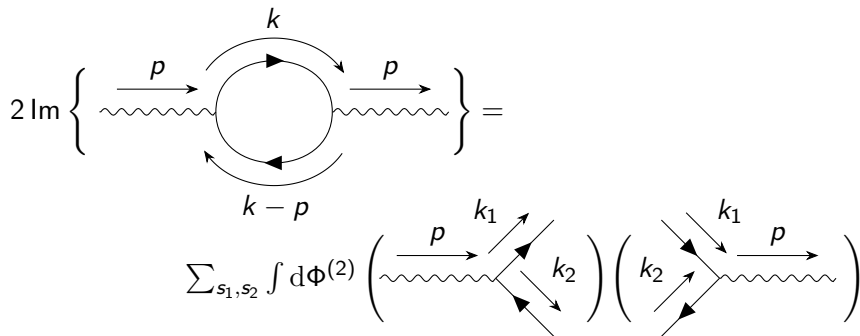
Propagator numerator:

$$q_1 + m = \sum_s u_s(q_1) \bar{u}_s(q_1) \qquad q_2 - m = \sum_s v_s(q_2) \bar{v}_s(q_2).$$

Cutkosky's rule

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Cutkosky's rule

$$2\text{Im} \left\{ \text{Diagram} \right\} = \sum_{s_1, s_2} \int d\Phi^{(2)} \left(\text{Diagram}_1 \right) \left(\text{Diagram}_2 \right)$$

The diagram on the left shows a wavy line with momentum p entering from the left and exiting to the right. It is connected to a circular loop. The top arc of the loop has momentum k and an arrow pointing right. The bottom arc has momentum $k-p$ and an arrow pointing left. The right side of the loop is connected to another wavy line with momentum p exiting to the right.

The diagram on the right is a sum over two terms. The first term shows a wavy line with momentum p entering from the left and splitting into two particles with momenta k_1 and k_2 . The second term shows two particles with momenta k_1 and k_2 entering from the left and merging into a wavy line with momentum p exiting to the right.

Cutkosky's rule

$$(\bar{u}\gamma^{\mu_1} \dots \gamma^{\mu_n} v)^* = \bar{v}\gamma^{\mu_n} \dots \gamma^{\mu_1} u,$$

$$2\text{Im} \left\{ \begin{array}{c} \text{---} \xrightarrow{p} \text{---} \\ \text{---} \xrightarrow{p} \text{---} \end{array} \right\} = \sum_{s_1, s_2} \int d\Phi^{(2)} \left(\begin{array}{c} \text{---} \xrightarrow{p} \text{---} \\ \text{---} \xrightarrow{k_1} \text{---} \\ \text{---} \xrightarrow{k_2} \text{---} \end{array} \right) \left(\begin{array}{c} \text{---} \xrightarrow{p} \text{---} \\ \text{---} \xrightarrow{k_1} \text{---} \\ \text{---} \xrightarrow{k_2} \text{---} \end{array} \right)$$

The diagram illustrates Cutkosky's rule for the imaginary part of a loop diagram. On the left, a self-energy loop diagram is shown with two external wavy lines of momentum p and two internal fermion lines of momentum k and $k-p$. The right side shows a sum over two-particle phase space $\sum_{s_1, s_2} \int d\Phi^{(2)}$ of two diagrams: one with a cut in the k propagator and one with a cut in the $k-p$ propagator.

Cutkosky's rule

$$(\bar{u}\gamma^{\mu_1} \dots \gamma^{\mu_n} v)^* = \bar{v}\gamma^{\mu_n} \dots \gamma^{\mu_1} u,$$

Incoming \leftrightarrow Outgoing

$$2\text{Im} \left\{ \begin{array}{c} \text{---} \xrightarrow{p} \text{---} \\ \text{---} \xrightarrow{p} \text{---} \end{array} \right\} = \sum_{s_1, s_2} \int d\Phi^{(2)} \left(\begin{array}{c} \text{---} \xrightarrow{p} \text{---} \\ \text{---} \xrightarrow{k_1} \text{---} \\ \text{---} \xrightarrow{k_2} \text{---} \end{array} \right) \left(\begin{array}{c} \text{---} \xrightarrow{p} \text{---} \\ \text{---} \xrightarrow{k_1} \text{---} \\ \text{---} \xrightarrow{k_2} \text{---} \end{array} \right)$$

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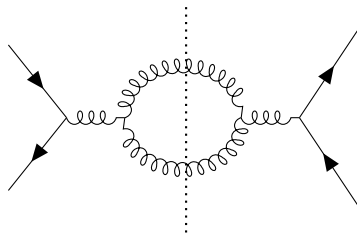
- ▶ $\mathcal{A}(\dots \rightarrow \dots \phi(p))^* = \mathcal{A}(\phi(p) \dots \rightarrow \dots)$
- ▶ The propagator numerator contains a sum over external line factors.

Cutkosky's rule

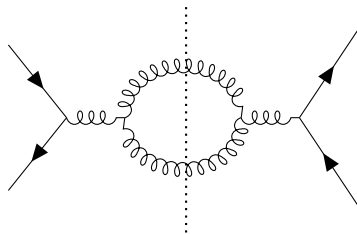
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Unitarity in QCD



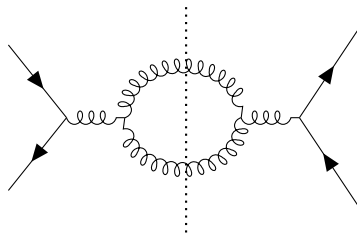
Unitarity in QCD



Propagator numerator: (Feynman-t' Hooft gauge)

$$-\eta^{\mu\mu'}$$

Unitarity in QCD



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$$-\eta^{\mu\mu'}$$

Cutkosky's rule gives

$$\int -\eta_{\mu\mu'} \mathcal{A}_s^\mu(q\bar{q} \rightarrow gg) \mathcal{A}_s^{\mu'}(q\bar{q} \rightarrow gg)^* d\Phi^{(2)}$$

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$$\int \sum_{h,h'} |\mathcal{A}_s(q\bar{q} \rightarrow gg)|^2 d\Phi^{(2)}$$

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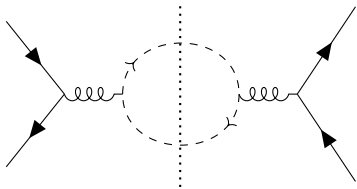
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...but unitarity requires

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What is missing?

Unitarity in QCD

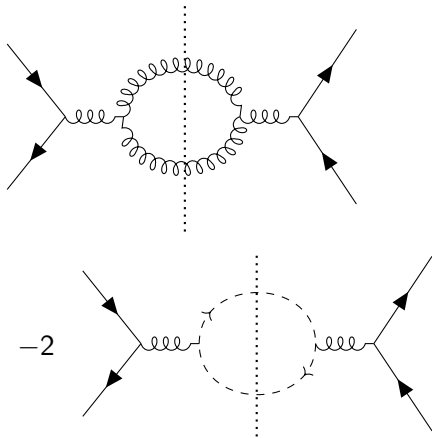


Unitarity in QCD

The image illustrates the unitarity condition in QCD. The top part shows a tree-level process (left) with a loop correction (middle) and an equals sign. The bottom part shows the unitarity cut represented as an integral over phase space of two tree-level diagrams.

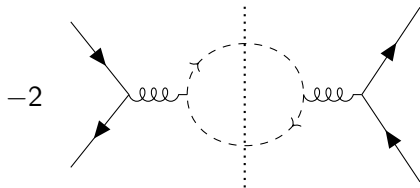
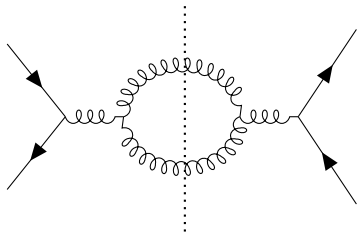
$$- \int d\Phi^{(2)} \left(\text{Tree Diagram} \right) \left(\text{Tree Diagram} \right)^*$$

Unitarity in QCD



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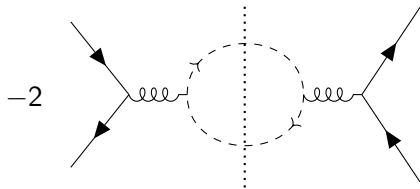
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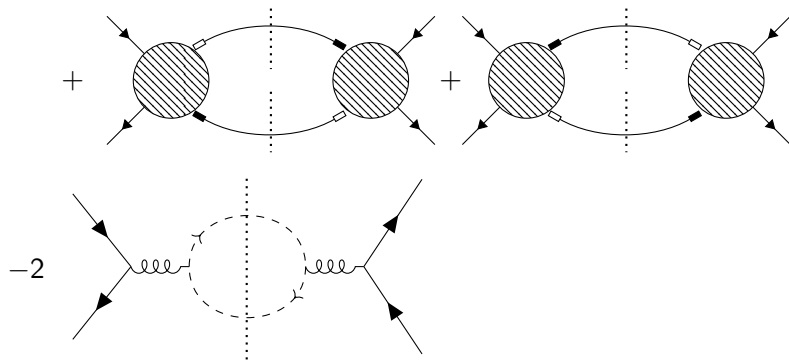
$$\int \sum_{h,h'} |\mathcal{A}_s(q\bar{q} \rightarrow gg)|^2 d\Phi^{(2)}$$



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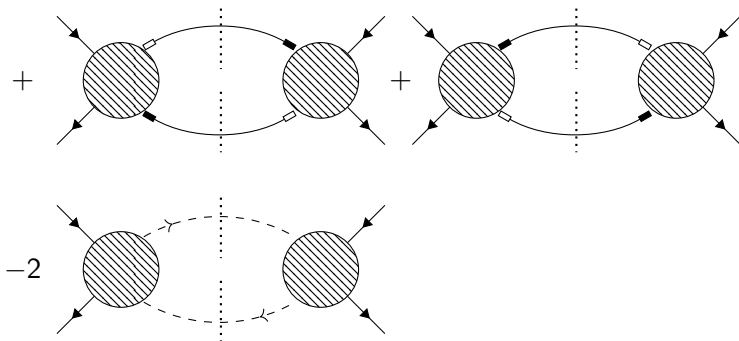
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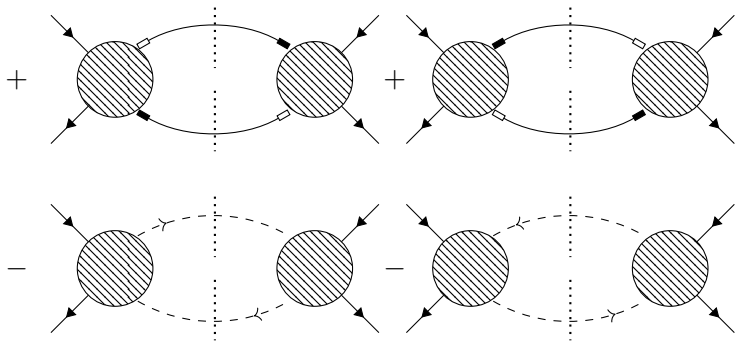
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Unitarity in QCD

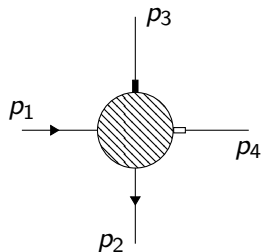
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Unitarity in QCD

We found earlier:

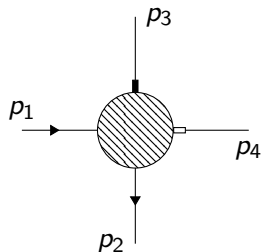


A Feynman diagram showing a quark loop. The loop is represented by a circle with diagonal hatching. Four external lines are attached to the loop: an incoming line from the left labeled p_1 , an outgoing line to the right labeled p_4 , an incoming line from the top labeled p_3 , and an outgoing line to the bottom labeled p_2 . The diagram is followed by an equals sign and a mathematical expression.

$$= g^2 (T^c T^d - T^d T^c) \frac{\bar{v}(p_2) \not{p}_3 u(p_1)}{s}$$

Unitarity in QCD

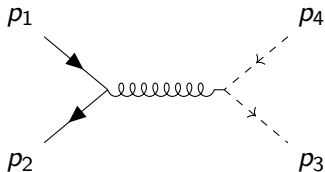
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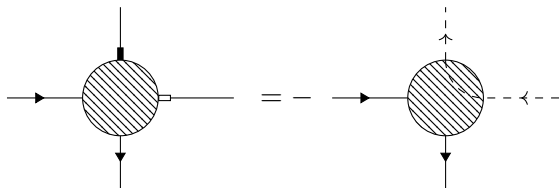
A Feynman diagram showing a four-point contact interaction. A central shaded circle has four external lines: an incoming line from the left labeled p_1 , an outgoing line to the right labeled p_4 , an incoming line from the top labeled p_3 , and an outgoing line to the bottom labeled p_2 .

$$= g^2(T^c T^d - T^d T^c) \frac{\bar{v}(p_2) \not{p}_3 u(p_1)}{s}$$

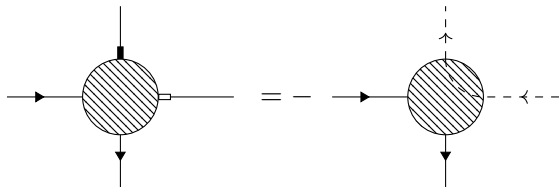
...which is minus



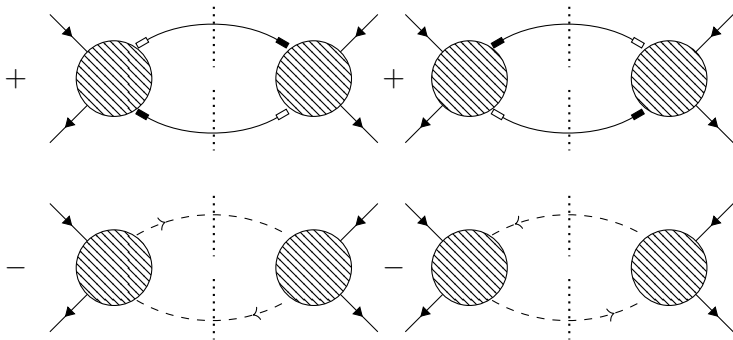
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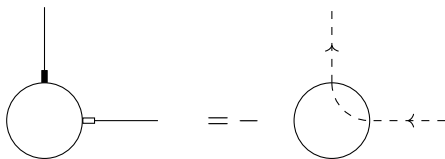
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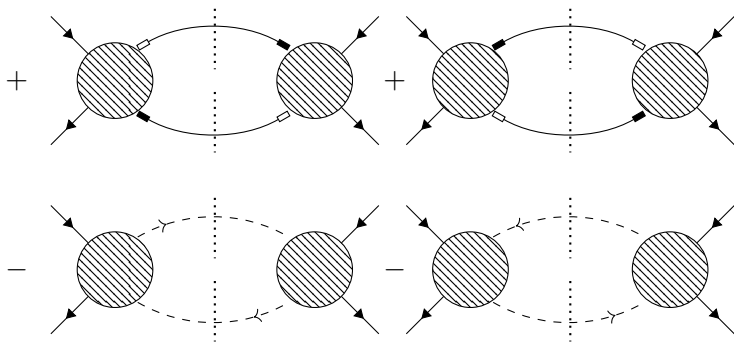
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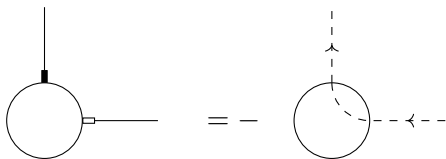
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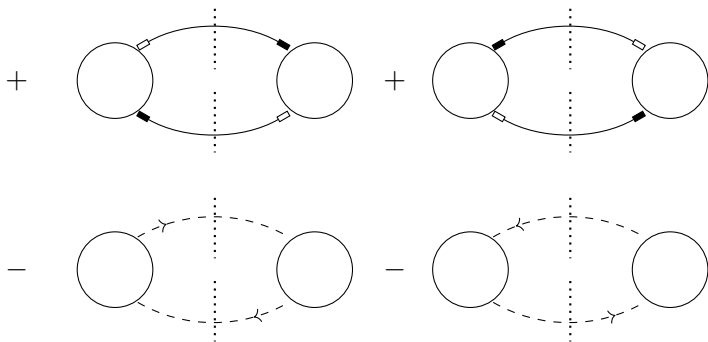
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Unitarity in QCD



$$\int \sum_{h,h'} |\mathcal{A}|^2 d\Phi^{(2)}$$



$\mathcal{P}^{\mu\mu'} \rightarrow -\eta^{\mu\mu'}$ in QCD

$$\eta_{\mu\mu'}\eta_{\nu\nu'}\mathcal{A}^{\mu\nu}(q\bar{q} \rightarrow gg)\mathcal{A}^{\mu'\nu'}(q\bar{q} \rightarrow gg)^*$$

$\mathcal{P}^{\mu\mu'} \rightarrow -\eta^{\mu\mu'}$ in QCD

$$\eta_{\mu\mu'}\eta_{\nu\nu'}\mathcal{A}^{\mu\nu}(q\bar{q} \rightarrow gg)\mathcal{A}^{\mu'\nu'}(q\bar{q} \rightarrow gg)^* \\ + \mathcal{A}(q\bar{q} \rightarrow c\bar{c})\mathcal{A}(q\bar{q} \rightarrow \bar{c}c)^*$$

$\mathcal{P}^{\mu\mu'} \rightarrow -\eta^{\mu\mu'}$ in QCD

$$\eta_{\mu\mu'}\eta_{\nu\nu'}\mathcal{A}^{\mu\nu}(q\bar{q} \rightarrow gg)\mathcal{A}^{\mu'\nu'}(q\bar{q} \rightarrow gg)^* \\ + 2\mathcal{A}(q\bar{q} \rightarrow c\bar{c})\mathcal{A}(q\bar{q} \rightarrow \bar{c}c)^*$$

$\mathcal{P}^{\mu\mu'} \rightarrow -\eta^{\mu\mu'}$ in QCD

$$\sum_{hh'} |\mathcal{A}(q\bar{q} \rightarrow gg)|^2 = \eta_{\mu\mu'} \eta_{\nu\nu'} \mathcal{A}^{\mu\nu}(q\bar{q} \rightarrow gg) \mathcal{A}^{\mu'\nu'}(q\bar{q} \rightarrow gg)^* \\ + 2\mathcal{A}(q\bar{q} \rightarrow c\bar{c}) \mathcal{A}(q\bar{q} \rightarrow \bar{c}c)^*$$

Magnus N. Malmquist
magnusnym97@gmail.com