



TFE4120 Electromagnetics - Crash course

Exercise 6

During this course we have become familiar with 4 equations, Maxwell's equations, which is electromagnetics in a nutshell. We've also considered Coulombs law and the corresponding law for the magnetic force between two conducting objects. The objective for this exercise is to get even more familiar with Maxwell's equations.

Classic electromagnetics can be summarized by Maxwell's equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (1)$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \quad (2)$$

$$\nabla \cdot \mathbf{D} = \rho, \quad (3)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (4)$$

Where ρ and \mathbf{J} is the free charge density and the free current density respectively.

Additionally we have the general correlation between the flux densities and the fields:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \quad (5)$$

$$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}). \quad (6)$$

Where \mathbf{P} is the polarization density and \mathbf{M} is the magnetization density in the material. We also need Lorentz' force equation,

$$\mathbf{F} = Q (\mathbf{E} + \mathbf{v} \times \mathbf{B}), \quad (7)$$

which describes the force acting on a charge Q that is moving freely with a speed \mathbf{v} in an electric and magnetic field.

For a linear and isotropic material the polarization density is proportional to the electric field,

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad (8)$$

and that the magnetizing density is proportional to the \mathbf{H} -field,

$$\mathbf{M} = \chi_m \mathbf{H}. \quad (9)$$

When (8) and (9) are put into (5) and (6) respectively, we get

$$\mathbf{D} = \epsilon \mathbf{E}, \quad (10)$$

$$\mathbf{B} = \mu \mathbf{H}, \quad (11)$$

where $\epsilon = \epsilon_0 \epsilon_r = \epsilon_0 (1 + \chi_e)$ and $\mu = \mu_0 \mu_r = \mu_0 (1 + \chi_m)$.

By using the equations above you should now be able to solve the tasks below.

- a) Use (3) to show Gauss' law on integral form, i.e.,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_{\text{free in } S}. \quad (12)$$

Additionally find the field from a point charge Q located in the origin. Show that the force acting on a small charge q with a distance r , is given by Coulomb's law:

$$\mathbf{F} = \frac{Qq}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}}, \quad (13)$$

where $\hat{\mathbf{r}}$ is a unit vector in the \mathbf{r} -direction.

- b) Which of Maxwell's equations states that the \mathbf{B} -field lines "bites their own tail"?
- c) Given a circular conducting loop C with one turn, where C is constant. Show Faraday's law

$$e = -\frac{d\Phi}{dt}, \quad (14)$$

where e is the induced electromotive force in the loop, and Φ is the total magnetic flux density through the loop, given by

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S}. \quad (15)$$

e is defined as

$$e \equiv \oint_C \mathbf{E} \cdot d\mathbf{l}, \quad (16)$$

and S is the surface encircled by the loop C . At first we assume that the resistance R in the loop is so great that we can neglect the current (see figure 1 on the next page). Then we move a permanent bar magnet back and forth towards the loop so that the magnetic flux in the loop is changing correspondent to the movement. Assume that the magnetic flux density from the magnet is approximately homogeneous in the loop and can be described as

$$\mathbf{B} = [B_0 + B_1 \cos(2\pi ft)] \hat{\mathbf{z}}, \quad (17)$$

where $\hat{\mathbf{z}}$ is the normal vector for the encircled surface. B_0 and B_1 is constants, and the area of the loop is S . Find the voltage across the resistor R as a function of time t . If we reduce R but continue the movement, what will happen with the total magnetic flux in the loop (cf. Lenz' law)?

- d) Find the magnetic flux density outside a infinitely long straight conductor which is carrying a constant current I . Assume the conductor to be round with radius a .
- e) Show that

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0, \quad (18)$$

is valid and that this involves current conservation in a test volume v . You can assume that v does not change with time.

- f) Show that the tangential component of the \mathbf{E} -field is continuous on a boundary interface between two arbitrary surfaces.
- g) Now we want to describe electromagnetics by using the potentials V and \mathbf{A} . As usual we define the potentials so that

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad (19)$$

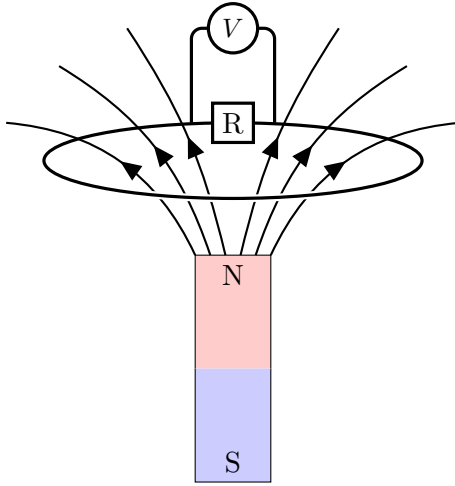


Figure 1: Illustration for task c).

and

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}. \quad (20)$$

Are these potentials unambiguously defined by \mathbf{E} and \mathbf{B} ? Explain your answer. Reason why the Maxwell's equations $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ now automatically are satisfied. What will the the final two Maxwell-equations, $\nabla \cdot \mathbf{D} = \rho$ and $\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$ mean for the potential (write your answer with words)? Assume a linear, isotropic and homogeneous material.

- h)** Assume stationary conditions. Show that the scalar potential V in a linear, isotropic and homogeneous material satisfies Poisson's equation

$$\nabla^2 V = -\frac{\rho}{\epsilon}. \quad (21)$$