

TFE4120 Electromagnetics - Crash course

Solution proposal, exercise 4

Problem 1

- a) The current is given by $I = \int_S \mathbf{J} \cdot d\mathbf{S} = JS$, if J is evenly distributed across S . Thus, the current density is given by

$$J = \frac{I}{\pi a^2}. \quad (1)$$

- b) We integrate \mathbf{J} across the cross section:

$$I = \int_S \mathbf{J} \cdot d\mathbf{S} = 2J_0 \int_0^a \frac{r^2}{a^2} 2\pi r dr = J_0 \pi a^2. \quad (2)$$

Problem 2

The \mathbf{B} -field in the center of the loop can be considered as the sum of two contributions (superposition). It is easiest to use two different cylindric coordinate systems for the two contributions. One with the z -axis along the straight conductor, and the other with its z -axis pointing out of the plane with its origin in the center of the loop (we use small subscripts s and l in order to differentiate the two)

\mathbf{B}_s , represents the contribution from the circular loop with current I , while \mathbf{B}_l represents the contribution from the infinitely long, straight, thin conductor carrying the current I . We have then

$$\mathbf{B} = \mathbf{B}_s + \mathbf{B}_l. \quad (3)$$

For \mathbf{B}_s we can use Biot-Savart's law

$$\mathbf{B}_s = \frac{\mu_0}{4\pi} \oint_C \frac{I d\mathbf{l} \times \hat{\mathbf{r}}_s}{r_s^2}, \quad (4)$$

where C is the circular loop in the task. When our observation point is in the center of the loop we can say that $d\mathbf{l} \times \hat{\mathbf{r}}_s = dl \hat{\mathbf{z}}_s$ and $r_s = a$ for the whole loop so that we get

$$\mathbf{B}_s = \frac{\mu_0 I}{4\pi a^2} \oint_C dl \hat{\mathbf{z}}_s = \frac{\mu_0 I}{4\pi a^2} 2\pi a \hat{\mathbf{z}}_s = \frac{\mu_0 I}{2a} \hat{\mathbf{z}}_s. \quad (5)$$

For an infinitely long, straight, thin conductor we can use Ampere's law

$$\oint_C \mathbf{B}_1 \cdot d\mathbf{l} = 2\pi r_1 B_1 = \mu_0 \oint \mathbf{J} \cdot d\mathbf{S} = \mu_0 I, \quad (6)$$

and thus

$$\mathbf{B}_1 = \frac{\mu_0 I}{2\pi r_1} \hat{\phi}_1. \quad (7)$$

For $r_1 = a$ we find

$$\mathbf{B}_1(a) = \frac{\mu_0 I}{2\pi a} \hat{\phi}_1. \quad (8)$$

In the center of the loop we have $\hat{\phi}_1 = \hat{\mathbf{z}}_s$. Thus, the flux density there is

$$\underline{\underline{\mathbf{B} = \frac{\mu_0 I}{2a} \left(1 + \frac{1}{\pi}\right) \hat{\mathbf{z}}.}} \quad (9)$$

Problem 3

- a) Due to cylindric symmetry we use the cylindric coordinates r and ϕ . We can also say that the field only has a ϕ -component as a function of r , so that

$$\mathbf{B} = B(r) \hat{\phi}. \quad (10)$$

is valid everywhere. In this task we use Ampère's law

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{S}, \quad (11)$$

where $\int_S \mathbf{J} \cdot d\mathbf{S}$ is the total current flowing through an area encircled by the integration loop. In this case we choose to have an integration loop C to be a circle with its center in the middle of the coaxial cable, and with a radius r .

For $r < a$:

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = 2\pi r B(r) = \mu_0 \frac{\pi r^2}{\pi a^2} I, \quad (12)$$

which yields

$$\mathbf{B} = \frac{\mu_0 r I}{2\pi a^2} \hat{\phi} \quad \text{for } r < a. \quad (13)$$

For $a < r < b$ we find

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = 2\pi r B(r) = \mu_0 I, \quad (14)$$

which yields

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r} \hat{\phi} \quad \text{for } a < r < b. \quad (15)$$

For $b < r < b + t$ we find

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = 2\pi r B(r) = \mu_0 \left(I - \frac{\pi r^2 - \pi b^2}{\pi(b+t)^2 - \pi b^2} I \right) = \mu_0 I \frac{(b+t)^2 - r^2}{2bt + t^2}, \quad (16)$$

which yields

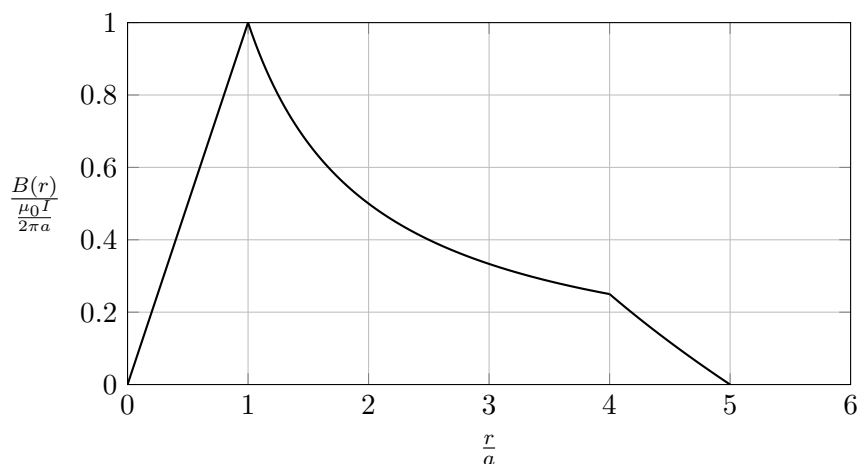
$$\mathbf{B} = \frac{\mu_0 I}{2\pi r} \frac{(b+t)^2 - r^2}{2bt + t^2} \hat{\phi} \quad \text{for } b < r < b+t. \quad (17)$$

For $r > b+t$ we find

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = 2\pi r B(r) = \mu_0(I - I) = 0. \quad (18)$$

To sum up:

$$\mathbf{B} = \begin{cases} \frac{\mu_0 r I}{2\pi a^2} \hat{\phi} & \text{for } r < a, \\ \frac{\mu_0 I}{2\pi r} \hat{\phi} & \text{for } a < r < b, \\ \frac{\mu_0 I}{2\pi r} \frac{(b+t)^2 - r^2}{2bt + t^2} \hat{\phi} & \text{for } b < r < b+t, \\ \mathbf{0} & \text{for } r > b+t. \end{cases} \quad (19)$$



Plott av $B(r)$ for $b = 4a$ og $t = a$.

- b) Since the current is assumed to be evenly distributed across the conductors, the current in the outer conductor will not contribute to the field inside of the outer conductor. This is easily verified by imagining the field contribution on the inside of the outer conductor when there is no inner conductor. Due to symmetry any field contribution will have the form $\mathbf{B} = B(r)\hat{\phi}$. Since there is no current flowing through a hollow inner conductor Ampere's law gives us $\oint_C \mathbf{B} \cdot d\mathbf{l} = 0$ for a closed loop inside the cylinder. By choosing a circular loop around the cylindrical axis we get $B = 0$ inside the hollow outer conductor.

The field inside of the outer conductor should therefore be concentric circles around the inner conductor's axis. Sketch I, V and VI can therefore be eliminated.

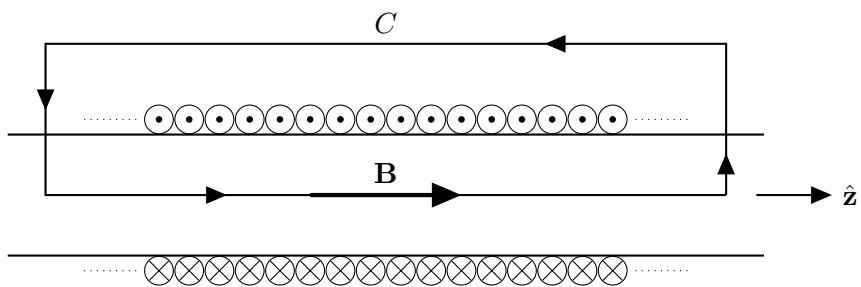
Sketch III is not valid since the field lines outside of the outer conductor does not contain any contribution from the current in the outer conductor. (The sketch only shows the field contribution from the inner conductor)

Sketch IV is not valid since the field has a divergence by the outer conductor (all magnetic field lines should "bite their own tail").

Sketch I, IV and VI are not valid since the field outside of the outer conductor should be different from zero (The field from the two conductor can not counteract each other completely).

Thus, sketch II is correct

Problem 4



We use Ampère's law

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{S}, \quad (20)$$

where the integration loop C is as shown in the figure above. We assume that the current is I . By using Ampère's law and the equation $\nabla \cdot \mathbf{B} = 0$ it can be claimed that the magnetic flux density \mathbf{B} only have a $\hat{\mathbf{z}}$ -component. Since the \mathbf{B} -field is zero outside the solenoid, we get from Ampère's law that $\mathbf{B} = B\hat{\mathbf{z}}$ inside the solenoid, where

$$Bl = \mu_0 NI, \quad (21)$$

which yields

$$B = \frac{\mu_0 NI}{l} \hat{\mathbf{z}}. \quad (22)$$