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TFE4120 Electromagnetics - Crash course

Lecture 4: Magnetic fields, Biot-Savart's law, Ampere's law

Thursday August 11th, 9-12 am.

Exercise 4: Problem 1. 10 min + 5 min solution.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = 0, \tag{1}$$

and

$$\nabla \cdot \mathbf{E} = \frac{\rho_{\text{tot}}}{\epsilon_0} \tag{2}$$

are the two laws concerning $\mathbf{E}$ in electrostatics. The corresponding laws for the magnetic field $\mathbf{B}$ are:

$$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}_{\text{tot}}, \tag{3}$$

and

$$\nabla \cdot \mathbf{B} = 0. \tag{4}$$

In "magnetostatics", i.e. if all currents are constant in time (so that the electromagnetic fields are time-independent) we have

$$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \mathbf{J}_{\text{tot}}. \tag{5}$$

Biot-Savart's law (the magnetic equivalent to Coulomb's law)

If we have to moving charges, Q_1 moving with velocity $\mathbf{v}_1$, and Q_2 moving with $\mathbf{v}_2$, the magnetic force on Q_2 from Q_1 is

$$\mathbf{F} = Q_2\mathbf{v}_2 \times (\frac{\mu_0}{4\pi}\frac{Q_1\mathbf{v}_1 \times \hat{\mathbf{r}}}{r^2}), \tag{6}$$

where $\mu_0 = 4\pi \cdot 10^7 \text{Ns}^2\text{C}^2$. This value is exact, because this equation is used as a standard for 1 C (the unit for charge). If $\mathbf{v}_1$ is parallel to $\mathbf{v}_2$, and $\mathbf{v}_1$ is normal to $\hat{\mathbf{r}}$ (the unit vector from Q_1 to Q_2) we get

$$\mathbf{F} = -\frac{\mu_0}{4\pi} Q_2 v_2 \frac{Q_1 v_1}{r^2}\hat{\mathbf{r}}. \tag{7}$$

The magnetic force between two positive charges moving parallel to eachother is thus attractive.

We define the magnetic field B (the magnetic flux density) produced by a charge Q_1 moving at $\mathbf{v}_1$:

$$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{Q_1 \mathbf{v}_1 \times \hat{\mathbf{r}}}{r^2}. \tag{8}$$

This is called "Biot-Savart's law". The magnetic field $\mathbf{B}$ describes the force on a charge Q moving with a velocity $\mathbf{v}$:

$$\mathbf{F} = Q\mathbf{v} \times \mathbf{B}. \tag{9}$$

Example: Charge moving in a constant $\mathbf{B}$-field: The charge will follow a circular path.

B-fields from different current distributions

a) Current along a line:

$$\mathrm{d}\mathbf{B} = \frac{\mu_0}{4\pi} \frac{Q' \mathrm{d}l \mathbf{v} \times \hat{\mathbf{r}}}{r^2}, \tag{10}$$

is the contribution to the total $\mathbf{B}$-field from a line segment $\mathrm{d}\mathbf{l}$ along the thin conductor. We have

$$Q' \mathrm{d}l \mathbf{v} = \mathrm{d}q \frac{\mathrm{d}\mathbf{l}}{\mathrm{d}t} = \frac{\mathrm{d}q}{\mathrm{d}t} \mathrm{d}\mathbf{l} = I \mathrm{d}\mathbf{l}. \tag{11}$$

This gives

$$\mathbf{B} = \int_C \mathrm{d}\mathbf{B} = \frac{\mu_0}{4\pi} \int_C \frac{I \mathrm{d}\mathbf{l} \times \hat{\mathbf{r}}}{r^2}. \tag{12}$$

This is the magnetic field $\mathbf{B}$ from a line current ("Biot-Savart's law for a line current").

b) Current along a surface: If the current instead is running along a thin surface we have

$$\mathbf{J}_S = \frac{nq}{\Delta S} \mathbf{v} = N_S q \mathbf{v} = \rho_s \mathbf{v}. \tag{13}$$

This gives

$$\mathbf{B} = \int_S \mathrm{d}\mathbf{B} = \frac{\mu_0}{4\pi} \int_S \frac{\mathbf{J}_S \mathrm{d}S \times \hat{\mathbf{r}}}{r^2}. \tag{14}$$

This is the magnetic field $\mathbf{B}$ from a surface current ("Biot-Savart's law for a surface current").

c) Current through a volume:

$$\mathbf{J} = Nq\mathbf{v}, \tag{15}$$

which gives

$$\mathbf{B} = \int_V \mathrm{d}\mathbf{B} = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J} \times \hat{\mathbf{r}}}{r^3} \mathrm{d}V. \tag{16}$$

This is the magnetic field $\mathbf{B}$ from a volume current ("Biot-Savart's law for a volume current").

Example: Magnetic field outside a cable with current I in the $\hat{\mathbf{y}}$-direction. We find $\mathbf{B}$ at a given point $(x, y) = (a, b)$:

$$\begin{aligned}
\mathbf{B} &= \frac{\mu_0}{4\pi}\int_{-l/2}^{l/2} \frac{I\mathrm{d}\mathbf{y}\times\hat{\mathbf{r}}}{r^2}, \\
&= \frac{\mu_0}{4\pi}\int_{-l/2}^{l/2} \frac{\sin\theta I\mathrm{d}y\hat{\boldsymbol{\phi}}}{r^2}, \\
&= \frac{\mu_0}{4\pi}\int_{-l/2}^{l/2} \frac{aI\mathrm{d}y}{r^3}\hat{\boldsymbol{\phi}}, \\
&= \frac{\mu_0 aI}{4\pi}\int_{-l/2}^{l/2} \frac{1}{(a^2+(b-y)^2)^{3/2}}\mathrm{d}y\hat{\boldsymbol{\phi}}, \\
&= \frac{\mu_0 aI}{4\pi}\left[\frac{\frac{l}{2}-b}{a^2\sqrt{a^2+(b-\frac{l}{2})^2}} + \frac{\frac{l}{2}+b}{a^2\sqrt{a^2+(b+\frac{l}{2})^2}}\right]\hat{\boldsymbol{\phi}}.
\end{aligned} \tag{17}$$

If we assume that the cable is **long**, i.e. $l \gg a, b$[1] we get

$$\mathbf{B} = \frac{\mu_0 I}{2\pi a}\hat{\boldsymbol{\phi}}. \tag{18}$$

Exercise 4: Problem 2. 15 min + 5 min solution.

Note: Using Biot-Savart's law often involves complicated calculations. If the problem has some degree of symmetry we rather use "Ampere's law".

More about the B-field:

The magnetic force does no work!

$$\mathrm{d}W = \mathbf{F}_{\mathrm{mag}}\cdot\mathrm{d}\mathbf{l} = (q\mathbf{v}\times\mathbf{B})\cdot\frac{\mathrm{d}\mathbf{l}}{\mathrm{d}t}\mathrm{d}t = (q\mathbf{v}\times\mathbf{B})\cdot\mathbf{v}\mathrm{d}t = 0. \tag{19}$$

Magnetic forces may change the direction, but not the velocity of a moving charge.

There are no magnetic monopoles. It may be shown from Biot-Savart's law that

$$\nabla\cdot\mathbf{B} = 0 \tag{20}$$

everywhere! This is one of Maxwell's equations. It means that magnetic fields cannot flow out from a point. That is, there are not "magnetic charges". All magetic fields must "bite their own tale". In integral form:

$$\int_V \nabla\cdot\mathbf{B}\mathrm{d}V = \oint_S \mathbf{B}\cdot\mathrm{d}\mathbf{S} = 0, \tag{21}$$

i.e. the total magnetic flux through any closed surface S is always zero!

[1] in fact we MUST do this, since we have ignored the return-current!

Amperes law for constant currents

Applyling Stoke's theorem to (5) gives:

$$\int_S \nabla \times \mathbf{B} \cdot \mathrm{d}\mathbf{S} = \int_C \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S}. \tag{22}$$

This law tells us that the circulation of magnetic field around a closed path is proportional to the total current through the path.

Example: Magnetic field outside a cable (again).

$$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 \int \mathbf{J} \cdot \mathrm{d}\mathbf{S}. \tag{23}$$

Let C be a circular curve centered at the cable. We then have

$$\mu_0 \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} = \mu_0 \int_S J \mathrm{d}S = \mu_0 I, \tag{24}$$

and

$$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \oint_C B\hat{\boldsymbol{\phi}} \cdot \mathrm{d}l\hat{\boldsymbol{\phi}} = \oint_C B \mathrm{d}l = B \oint_C \mathrm{d}l = B(r)2\pi r. \tag{25}$$

We here used that $\mathbf{B}$ is directed along the $\hat{\boldsymbol{\phi}}$-direction, i.e. along the circular path around the cable. Further, we used that $\mathbf{B}$ is independent on ϕ and z, and thus only depends on r. Since r is constant along C we could thus move B outside the integral. Equations (5), (24) and (25) gives

$$\mathbf{B}(r) = \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\phi}}, \tag{26}$$

which is the same as we got last time.

We now argue why $\mathbf{B}(r, \phi, z) = B(r)\hat{\boldsymbol{\phi}}$. We assume that the cable is infinitely long. We thus don't have to bother about the returing current, or what happens at the end points.

First, $\mathbf{B}$ must be independent on z, since the cable is infinitely long. Considering $\mathbf{B}$ at different values for z does not alter the physical situation, and thus our solution $\mathbf{B}(r, \phi, z)$ should be independent on z. Similarly, (B) must be independent on ϕ, since if we rotate the cable by an angle ϕ wrt. the $\hat{\mathbf{z}}$-axis, the physical situation is unaltered, so $\mathbf{B}(r, \phi)$ must be independent on ϕ.

If $\mathbf{B}$ has a $\hat{\mathbf{r}}$-component, this will thus be equal for all ϕ and z. This means we have a net flow of magnetic field out from a closed cylinder surface around the cable, which contradics (4).

Finally, $\mathbf{B}$ cannot have a $\hat{\mathbf{z}}$-component. This is seen from Biot-Savart's law:

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int_C \frac{I \mathrm{d}\mathbf{l} \times \hat{\mathbf{r}}}{r^2}. \tag{27}$$

Since the current is along the $\hat{\mathbf{z}}$-axis, we have $I\mathrm{d}\mathbf{l} = I\mathrm{d}l\hat{\mathbf{z}}$, which gives no $\hat{\mathbf{z}}$-component in $\mathbf{B}$.

Example 2: Cable with final thickness. Assume the current I runs through a cable with radius a, and that the current is evenly distributed across the cross section: $\mathbf{J} = \frac{I}{\pi a^2}\hat{\mathbf{z}}$. Ampere's law gives

$$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S}. \tag{28}$$

As in the previous example, one may argue that $\mathbf{B} = B(r)\hat{\boldsymbol{\phi}}$. This gives

$$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{l} = 2\pi r B(r). \tag{29}$$

The value of the right hand side depends on r. For $r > a$ we have $\int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} = I$, while for $r < a$ we have $\int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} = \frac{I\pi r^2}{\pi a^2}$. This gives for $r < a$:

$$\mathbf{B} = \frac{\mu_0 I r}{2\pi a^2}\hat{\boldsymbol{\phi}}, \tag{30}$$

and for $r > a$:

$$\mathbf{B} = \frac{\mu_0 I}{2\pi a}\hat{\boldsymbol{\phi}}. \tag{31}$$

Exercise 4: Problem 3 and 4. 45 min + 15 min solution.