NTNU

Dpt. of Electronics and Telecommunications,
IME faculty,
NTNU

TFE4120 Electromagnetics - Crash course

Lecture 1: Maxwell's equtaions, Coulombs law, vector calculus

Monday August 8th 10-12 am.

Welcome to NTNU and to this crash course in Electromagnetism. It will be assumed that you have a BSc-degree in electronics/electrical engineering/power engineering or similar from a university/collage. The course will be taught in English. The aim of this course is to give you a minimum of prerequisities to follow a 2-year master program in electronics or electrical power engineering here at NTNU.

- Webpage: https://www.ntnu.no/wiki/x/bBh2BQ. All information is posted there.
- Teacher: Hans Olaf Haagenvik, PhD-student. hans.hagenvik@iet.ntnu.no. Office: B417 Electro-building.
- Syllabus: Lecture notes, posted on webpage before each lecture. Will try to keep them updated. You will also find suggestions to supplementary litterature, but don't go and buy expensive books unless you're going to use them in later courses.
- The exercises will be included in the lectures, where you will be given breaks (≈ 15 min) after each new topic has been covered to solve some the relevant problems from the exercise.

Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{3}$$

$$\frac{1}{\mu_0} \nabla \times \mathbf{B} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}. \tag{4}$$

These equations describe every electromagnetic, including optical, devices and phenomena. From these one may derive the wave equation for **E** and **B**, and also the antenna equations, which will be the final results in this course. These equations should be familiar, but maybe not in this notation?

Coulombs law ($\nabla \cdot \mathbf{E} = \rho/\epsilon_0$)

The electric force $\mathbf{F}$ from a charge Q acting on a charge q is given by

$$\mathbf{F} = \frac{qQ}{4\pi\epsilon_0 R^2}\hat{\mathbf{R}}, \tag{5}$$

where R is the distance between the charges, and $\hat{\mathbf{R}}$ is the unit vector along the line from Q to q.

The total electric force acting on q from several charges Q_i is given by vector addition of the forces from each charge seperately. If there are N charges Q_i:

$$\mathbf{F}_{\text{tot}} = \sum_{i=1}^{N} \frac{qQ_i}{4\pi\epsilon_0 R_i^2}\hat{\mathbf{R}}_{\mathbf{i}}, \tag{6}$$

where each R_i is the distance between q and Q_i, and $\hat{\mathbf{R}}_i$ is the unit vector along the line from Q_i to q.

Notation:

- Scalar: $R = |\mathbf{R}|$.
- Vector: $\mathbf{R}$.
- Unit vector: $\hat{\mathbf{R}} = \mathbf{R}/|\mathbf{R}|$, so $|\hat{\mathbf{R}}| = 1$.

Electric fields

Define electric field: $\mathbf{E} = \lim_{q\to 0} \frac{\mathbf{F}}{q}$. The electric field describes the direction and magnitude of the force a test charge will "feel". The field from a point charge is given by:

$$\mathbf{E} = \lim_{q\to 0} \frac{\frac{qQ}{4\pi\epsilon_0 R^2}\hat{\mathbf{R}}}{q} = \frac{Q}{4\pi R^2}\hat{\mathbf{R}}. \tag{7}$$

If the electric field $\mathbf{E}$ is known, the force acting on a test charge q is given by $\mathbf{F} = q\mathbf{E}$.

Vector calculus: line-, surface- and volume integrals

Electromagnetism takes place in space (and time), so
$\mathbf{E} = E_x(x,y,z)\hat{\mathbf{x}} + E_y(x,y,z)\hat{\mathbf{y}} + E_z(x,y,z)\hat{\mathbf{z}}$. Scalar functions: $V = V(x,y,z)$.

Line integrals: e.g. work required to move a charge through an electric field:
$W = \int_A^B q\mathbf{E} \cdot \mathrm{dl}$.

Surface integrals: e.g. flux of $\mathbf{E}$-field through the surface of a sphere: $\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S}$.

Volume integrals: e.g. charge inside a bounded volume: $Q = \int_V \rho(x,y,z)\mathrm{d}V$, where ρ is the charge density.

Exercise 1: Problem 1. 15 min + 5 min solution.
Hint: always look for simplifications:

- If $\mathbf{E}||\mathrm{d}\mathbf{S}$, we have $\oint \mathbf{E}\cdot\mathrm{d}\mathbf{S} = \oint E\mathrm{d}S$ (got rid of vectors).
- If $\mathbf{E}(R)$ is constant over a sphere, $\oint \mathbf{E}\cdot\mathrm{d}\mathbf{S} = \oint E\mathrm{d}S = E\oint \mathrm{d}S = E\cdot 4\pi R^2$.

Application: Find E-field from different charge distributions

a) Charge distributed over volume

$$\mathbf{E} = \sum_{i=1}^{N} \frac{Q_i}{4\pi\epsilon_0 R_i^2}\hat{\mathbf{R}}_i. \tag{8}$$

Define $Q_i = \rho_i \Delta V_i$, where ρ_i is the charge density. Then

$$\mathbf{E} = \sum_{i=1}^{N} \frac{\rho_i}{4\pi\epsilon_0 R_i^2}\hat{\mathbf{R}}_i \Delta V_i \to \int_V \frac{\rho}{4\pi\epsilon_0 R^2}\hat{\mathbf{R}}\mathrm{d}V. \tag{9}$$

b) Charge distributed over a surface (plane)

$$\mathbf{E} = \int_V \frac{\rho_S}{4\pi\epsilon_0 R^2}\hat{\mathbf{R}}\mathrm{d}S, \tag{10}$$

where in this case $Q_i = \rho_{S,i}\Delta S_i$, and ρ_S is charge per unit surface area.

c) Charge distributed along a line

$$\mathbf{E} = \int_V \frac{Q'}{4\pi\epsilon_0 R^2}\hat{\mathbf{R}}\mathrm{d}l, \tag{11}$$

where in this case $Q_i = Q_i'\Delta l$, and Q' is charge per unit length.

Example: Electric field along the z-axis from charged ring

At the ring: $Q' = \frac{Q}{2\pi a}$. Line charge: $\mathbf{E} = \int_C \frac{Q'}{4\pi\epsilon_0 R^2}\hat{\mathbf{R}}\mathrm{d}l = \int_C \frac{Q'}{4\pi\epsilon_0 R^3}\mathbf{R}\mathrm{d}l = \frac{Q'}{4\pi\epsilon_0 R^3}\int_C \mathbf{R}\mathrm{d}l$.
Here R is the distance from the ring to our observation point along the **z**-axis. From Pythagoras we have $R^2 = a^2 + z^2$. Note that R is constant along C, so it may be taken outside the ingeral along with the other constants.

The integral $\int \mathbf{R}\mathrm{d}l$ is a sum of all the **R**-vectors from all the points along the ring to the observation point. This may be simplified by noting that all the **x**- and **y**-components will cancel out. This gives

$$\int \mathbf{R}\mathrm{d}l = \int \mathbf{z}\mathrm{d}l = 2\pi\mathbf{z}, \tag{12}$$

which finally gives

$$\begin{aligned}\mathbf{E} &= \frac{Q'}{4\pi\epsilon_0 R^3}\cdot 2\pi a z\hat{\mathbf{z}} \\ &= \frac{\frac{Q}{2\pi a}}{4\pi\epsilon_0 R^3}\cdot 2\pi a z\hat{\mathbf{z}} \\ &= \frac{Qz}{4\pi\epsilon_0 (a^2+z^2)^{3/2}}\hat{\mathbf{z}}, \end{aligned} \tag{13}$$

where we used that $R^2 = a^2 + z^2$.

This is an ugly expression. One test to see of it is correct is to examine the limit $a \to 0$. This should give the field from a point charge (a very tiny ring looks like a point).

Consider

$$\lim_{a\to 0}\mathbf{E} = \frac{Qz}{4\pi\epsilon_0 z^3}\hat{\mathbf{z}} = \frac{Q}{4\pi\epsilon_0 z^2}\hat{\mathbf{z}}, \tag{14}$$

where $R \approx z$ as $a \to 0$. This expression is thus the same as the Couloumb-field. OK!

We may also verify that $\mathbf{E}(z=0) = 0$, as it should be since the charge is spread evenly out along the ring, so the contribution from charges opposite one another exactly cancel out. OK!

Properties of the fields: Divergence and curl

Gradient: $\nabla = < \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} >$. "3-dimensional derivative of a scalar function".

The gradient points in the direction of the greatest rate of increase of the function, and its magnitude is the slope of the graph in that direction.

Divergence:

How much does the field flow out of a point?

$\mathrm{div}\mathbf{E} = \nabla\cdot\mathbf{E} = \lim_{\Delta V\to 0}\frac{\oint_S \mathbf{E}\cdot \mathrm{d}\mathbf{S}}{\Delta V}$.

Curl:

How much does the field circulate around a point?

$\mathrm{curl}\mathbf{E} = \nabla\times\mathbf{E} = \lim_{\Delta S\to 0}\frac{\oint_C \mathbf{E}\cdot \mathrm{d}\mathbf{l}}{\Delta S}$.

Useful theorems

Divergence theorem:

$$\oint_S \mathbf{E}\cdot \mathrm{d}\mathbf{S} = \int_V \nabla\cdot\mathbf{E}\mathrm{d}V. \tag{15}$$

"Proof": Two ways to find the flux out of a sphere.

a) Flux out of **one** surface.

b) Flux out of many small surfaces.

$$\oint_S \mathbf{E}\cdot \mathrm{d}\mathbf{S} = \sum_i \oint_{S_i}\mathbf{E}\cdot \mathrm{d}\mathbf{S} = \sum_i (\frac{1}{\Delta V_i}\oint_{S_i}\mathbf{E}\cdot \mathrm{d}\mathbf{S}_i)\Delta V_i \to \int \nabla\cdot\mathbf{E}\mathrm{d}V. \tag{16}$$

Stokes theorem:

$$\oint_C \mathbf{E} \cdot \mathrm{d}\mathbf{l} = \int_V \nabla \times \mathbf{E} \cdot \mathrm{d}\mathbf{S}. \tag{17}$$

"Proof": Two ways to calculate the integral of $\mathbf{E}$ pointing along a closed loop.

a) Integrate along the edge of the surface.

b) ... or along the edges of many small surfaces (no gaps between them).

Since the integral along any "inner edges" cancel out, these two calculations must be equal.

$$\oint_C \mathbf{E} \cdot \mathrm{d}\mathbf{l} = \sum_i \oint_{C_i} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = \sum_i (\frac{\oint_{S_i} \mathbf{E} \cdot \mathrm{d}\mathbf{S}_i}{\Delta S_i}) \Delta S_i \to \int \nabla \times \mathbf{E} \cdot \mathrm{d}\mathbf{S}. \tag{18}$$

This "proof" was for a flat surface, but the theorem is valid in general.

Exercise 1: Problem 2-4. 15 min + 5 min solution.