TDT4127 Programming and Numerics Week 44

Solving ordinary differential equations

- More dimensions, stability and accuracy

Kunnskap for en bedre verden

Some facts about the exam

- **November 30, 09:00-13:00.**
  - Check location at **studentweb** (may not be available yet)
- The exam will be **digital**
  - Same as the auditorium exercises
  - https://innsida.ntnu.no/wiki/-/wiki/norsk/digital+eksamen
- Theory questions will have **multiple choice** answers
  - Mostly numerics, possibly some programming related
- Theory questions and numerical exercise(s) will be similar to those in **Auditorium exercise 2**
- You do **not** need to remember formulas for the numerics
  - But it will of course *help* if you are familiar with them!

Learning goals

- Goals
  - Solving **ordinary differential equations**
  - Analysis of algorithms:
    - ***Explicit Euler method***
    - ***Implicit Euler method***
    - ***Heun's method***
  - Stability and accuracy

- Curriculum
  - Exercise set 9
    - But only in the interpretation of results

Numerical methods for ODEs

- Last week: **explicit Euler**, **implicit Euler**, **Heun's method**
- These schemes numerically solve the ODE

$$\dot{x}(t) = f(x(t), t)$$

- The **explicit Euler** method:

$$x^{j+1} = x^j + hf\left(x^j, t_j\right)$$

- The **implicit Euler** method:

$$x^{j+1} = x^j + hf(x^{j+1}, t_{j+1})$$

- **Heun's method**:

$$s^{j+1} = x^j + hf\left(x^j, t_j\right)$$

$$x^{j+1} = x^j + \frac{h}{2}\left(f\left(x^j, t_j\right) + f\left(s^{j+1}, t_{j+1}\right)\right)$$

Treating ODEs in many dimensions

- In more than one dimensions, equations are **vectorized**

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), t)$$

- The methods are exactly the same, ***but with vectors***
- The **explicit Euler** method:

$$\boldsymbol{x}^{j+1} = \boldsymbol{x}^{j} + h\boldsymbol{f}\left(\boldsymbol{x}^{j}, t_j\right)$$

- The **implicit Euler** method:

$$\boldsymbol{x}^{j+1} = \boldsymbol{x}^{j} + h\boldsymbol{f}(\boldsymbol{x}^{j+1}, t_{j+1})$$

- **Heun's method**:

$$\boldsymbol{s}^{j+1} = \boldsymbol{x}^{j} + h\boldsymbol{f}\left(\boldsymbol{x}^{j}, t_j\right)$$

$$\boldsymbol{x}^{j+1} = \boldsymbol{x}^{j} + \frac{h}{2}\left(\boldsymbol{f}\left(\boldsymbol{x}^{j}, t_j\right) + \boldsymbol{f}\left(\boldsymbol{s}^{j+1}, t_{j+1}\right)\right)$$

- Implementation difference: **vector addition. Numerical solver in several dimensions** for the **implicit Euler** method

Stability

- When solving an ODE numerically, we need to choose an appropriate time **step size** h (i.e. **number of steps** N).
  - Too small h → takes long time to compute solution
  - Too large h → inaccurate (bad) or unstable (worse) solutions

- What is meant by ***stability*** and ***instability***?
  - Instability is when the solution «blows up» when it's not supposed to
  - Stability is when it doesn't

- Test example: Apply the numerical method to the ODE

$$\dot{x}(t) = -\lambda x(t), \qquad \lambda > 0, \qquad x(0) = x_0$$

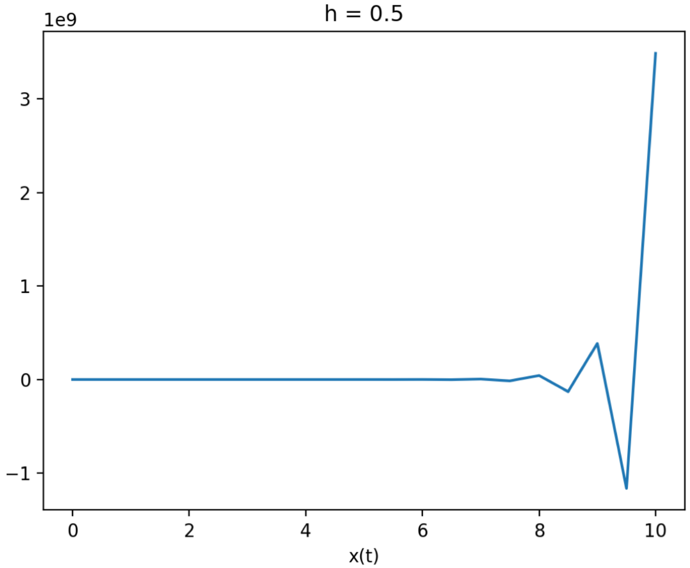
- This equation has the **strictly decreasing** solution

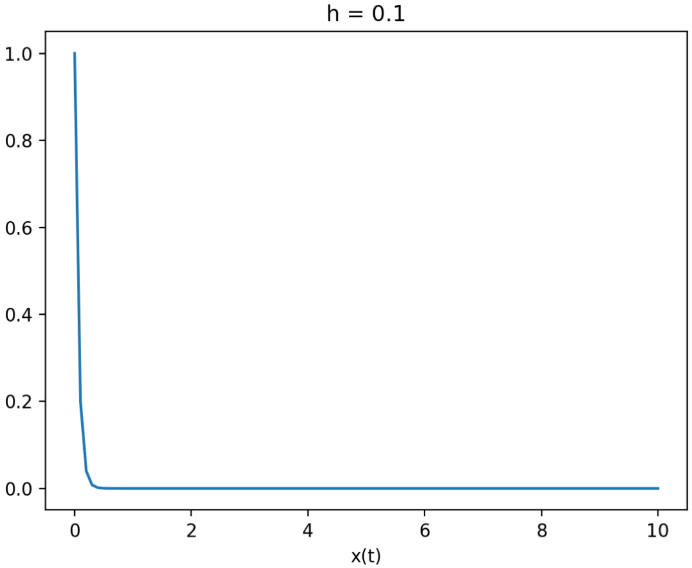
$$x(t) = x_0 e^{-\lambda t}$$

Numerical instability example

- Below: explicit Euler applied with various h to the ODE

$$\dot{x}(t) = -8x(t), \qquad \lambda = 8, \qquad x(0) = 1$$





Left: instability.

Right: stability

Numerical stability example

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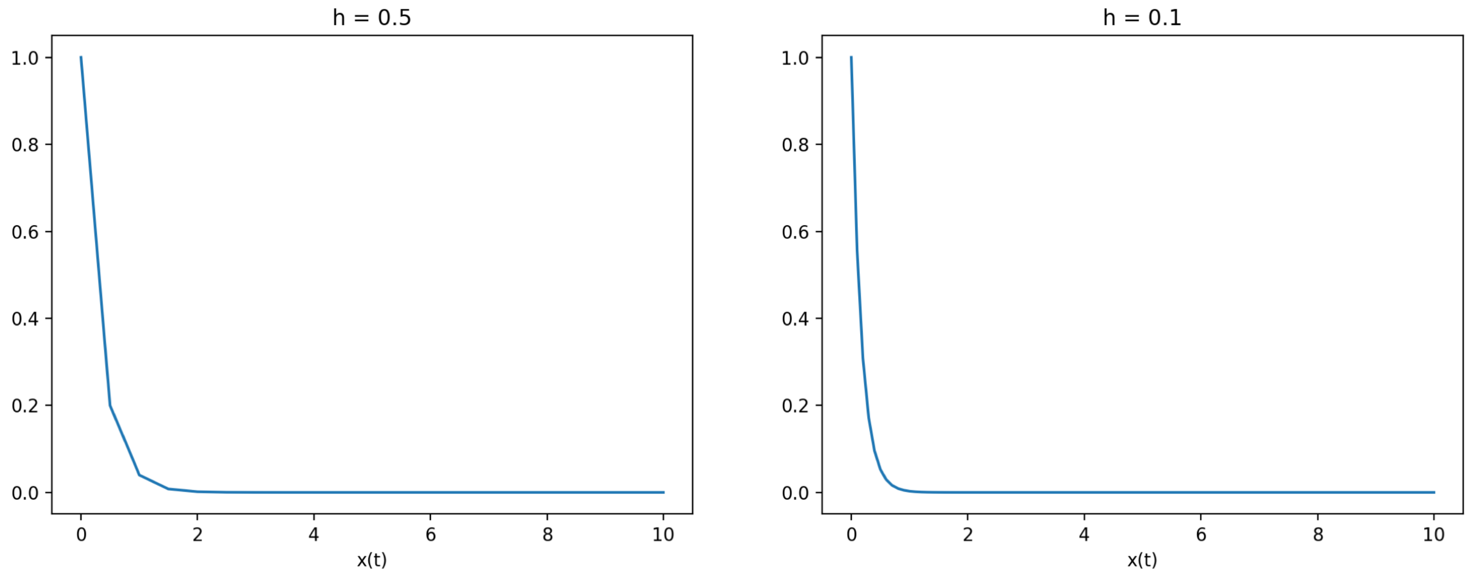


Left: stability.

Right: stability

Numerical stability explained

- So what happens? Compare solutions for this particular ODE:

$$\dot{x}(t) = -\lambda x, \qquad \lambda > 0, \qquad x(0) = x_0$$

- Explicit Euler:

$$x_{n+1} = (1 - \lambda h)x_n = (1 - \lambda h)^2 x_{n-1} = \cdots = (1 - \lambda h)^n x_0$$

- Blows up if $|1 - \lambda h| > 1$, i.e. if $\lambda h > 2$.
  - Must take $h < 2/\lambda$! This can be restrictive.

- Implicit Euler:

$$x_{n+1} = x_n - \lambda h x_{n+1}$$

$$x_{n+1} = \frac{1}{1 + \lambda h} x_n = \left(\frac{1}{1 + \lambda h}\right)^2 x_{n-1} = \cdots = \left(\frac{1}{1 + \lambda h}\right)^n x_0$$

- Always decreasing, no matter the value of λ or h

Stability summarized

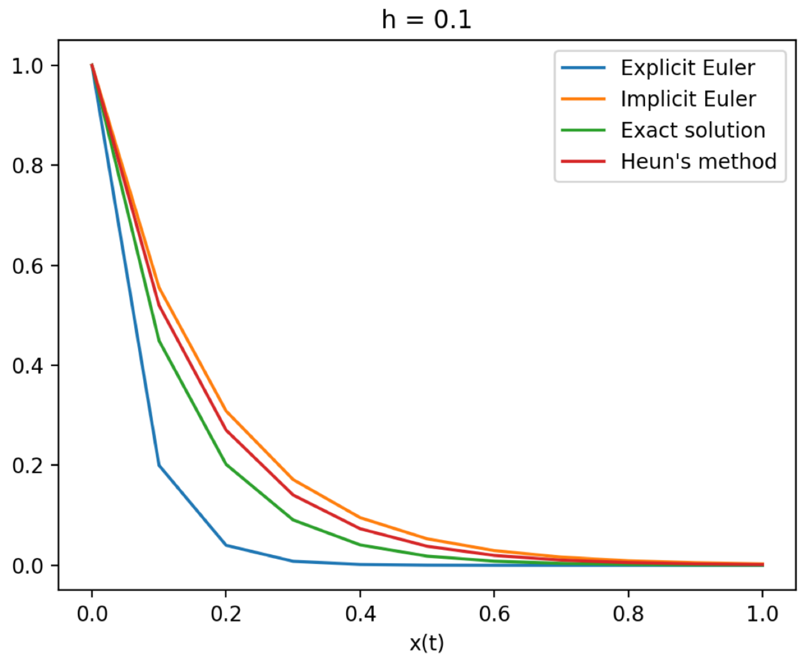
- Intuition: *different methods have different stability properties*.
  - Some methods are stable for larger or even **all** h
  - There is a large literature on this for *Runge-Kutta* methods
  - **Implicit** methods are typically **more stable** than explicit ones
  - No **explicit** method is stable for **all** h
  - All methods work with **small enough** h
    - But it may mean a restrictively very long computation time

- What about Heun's method?
  - **More stable** than explicit Euler, but it is **explicit** and has restrictions

- **Practical tips:** If you see unwanted blow-up or oscillations, try a smaller h first, before switching to another solver.

Accuracy

- Assuming our solutions don’t blow up, the next question is: how **accurate** are they?
- Accuracy is a function of h



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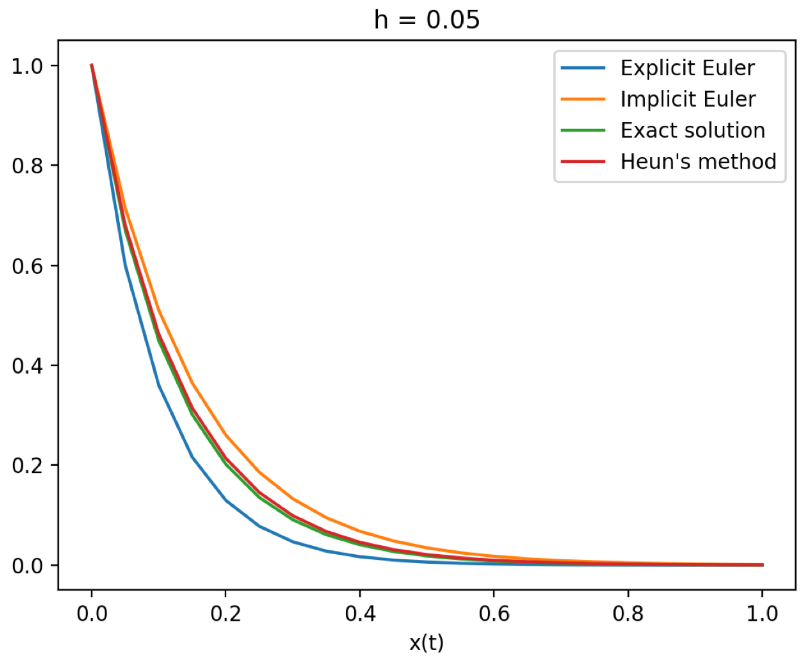

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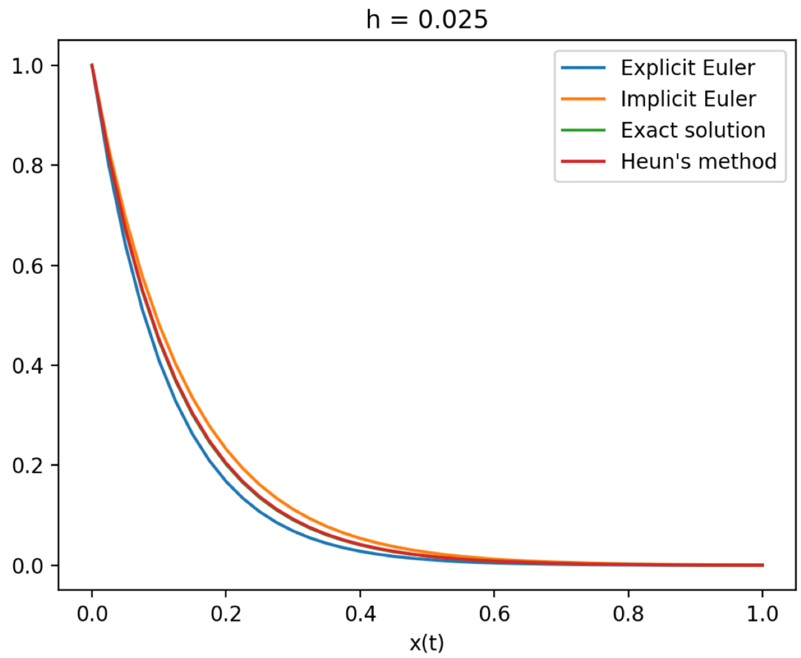

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Accuracy

- Accuracy is measured in terms of the **global error**

$$e_j = \left|x(t_j) - x_j\right|, \qquad t_j = jh$$

- A method is said to be of **order** p if for some constant C_j,

$$e_j \le h^p C_j$$

- Explicit and implicit Euler are both of **order 1**:

$$e_j^{\text{Euler}} \le hC_j, \qquad C_j = \frac{\mathrm{e}^{L(t_j - t_0)} - 1}{L}$$

- Heun's method is of order 2:

$$e_j^{\text{Heun}} \le h^2 C_j, \qquad C_j = \frac{\mathrm{e}^{L(t_j - t_0)} - 1}{L}$$

- Note: These C_j estimates are **very** conservative and should not be used in practice. ***The order is what's important***.

Accuracy

- **Practical consequence**: *order p methods improve errors by a factor 2^p when halving the step size*.
- The table below demonstrates the orders by considering the **global error** at $T = 1$ for the test problem from before.
- explicit/implicit Euler: $p = 1$, Heun's method: $p = 2$

h	Explicit Euler	Implicit Euler	Heun's method
0.01	$9.62*10^{-5}$	$11.91*10^{-5}$	$30.54*10^{-7}$
0.005	$5.09*10^{-5}$	$5.66*10^{-5}$	$7.38*10^{-7}$
0.0025	$2.61*10^{-5}$	$2.76*10^{-5}$	$1.82*10^{-7}$

- **High-order methods** gain a lot from smaller time steps!
  - Methods with **order 4** are often used!

Summary of ODE solvers

- **Explicit/implicit** methods
  - **Explicit**: Can calculate directly. Explicit Euler and Heun's method
  - **Implicit**: Need to solve an equation per step. Implicit Euler
- Multi-stage
  - Methods can have more than one **stage**. Heun's method
- Stability
  - Does the method blow up when applied to $\dot{x}(t) = -\lambda x$?
  - **Explicit** methods are **unstable** for too **large step sizes** h
  - **Implicit** methods are generally **more stable**
- Accuracy
  - A method is **order** p accurate if $e_j \leq h^p C_j$.
  - implicit/explicit Euler methods are **order 1**
  - Heun's method is **order 2**
  - **Can construct** integration methods of even **higher order**

Next weeks

- Three lectures left
  - **Adaptive Simpson** next week (November 9)
    - Last regular lecture
  - **Repetition** and **exam prep** on November 16 and November 23!
  - I will go through the numerics from **auditorium exercise 2** in detail
  - **Suggest other topics** you want me to cover
    - Otherwise, I’ll pick them myself

Questions?