TDT4127 Programming and Numerics Week 38

Equation solving

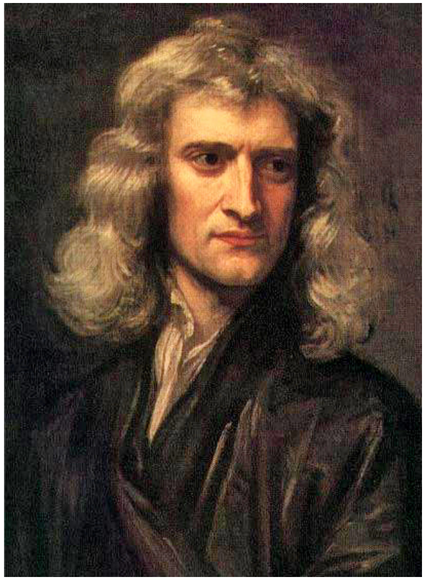
Algorithms: Newton's method, secant method

News

- **Exercises** are now scheduled for hand-in on Mondays
  - Due to better capacity of teaching assistants on Mondays
- **QueueMe** is replaced by blackboards/pen and paper
- **Lectures** will stay at this time and location
- Remember that you can ask questions on **Piazza**
  - Our teaching assistants will answer as quickly as they can
  - Allows your questions to help others
  - Reduces load on teaching assistants vs. email

Learning goals

- Goals
  - Solving equations numerically
    - Algorithm statements
      - ***Newton's method, secant method***
    - Implementation tips
    - Convergence analysis – *next week*



- Curriculum
  - Exercises 3 and 5

Solving equations numerically

- The prototypical equation to solve:

$$f(x) = 0$$

  - *Any other equation* can be reduced to this; if we wish to solve

$$g(x) = h(x)$$

    for x, simply reduce it to $f(x) = 0$ by setting $f(x) = g(x) - h(x)$.
  - For now, we consider one-dimensional equations.
    - More dimensions to come in week 42!
- Many different algorithms exist
  - Bisection method (week 36)
  - Newton's method (today)
  - Secant method (today)

Newton's method

- Last week: Numerical integration
  - We approximated f by constant, linear and quadratic functions
  - Can we do something similar to solve equations here?
  - A linear approximation to $f(x)$, based on Taylor expansion:

$$g(x) = f(x_0) + f'(x_0)(x - x_0)$$

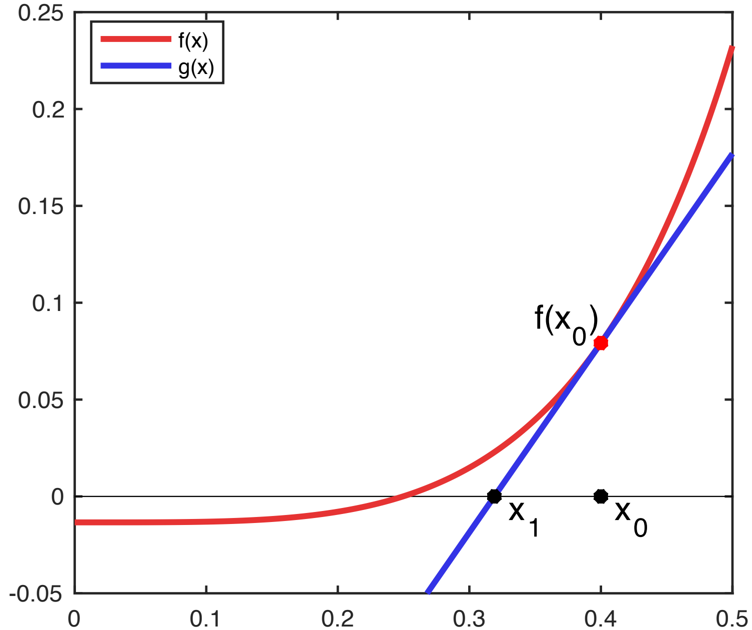
$$g(x) = 0 \quad solved\ by\ x = x_0 - f(x_0)/f'(x_0)$$

- The linear approximation $g(x)$ is exactly the tangent line
  - Interpretation: follow the tangents toward a zero!

Example

$$f(x) = \tan\left(e^{x^4}\right), \qquad f'(x) = \frac{4x^3 e^{x^4}}{\tan\left(e^{x^4}\right)}$$

$$x_1 = x_0 - f(x_0)/f'(x_0)$$



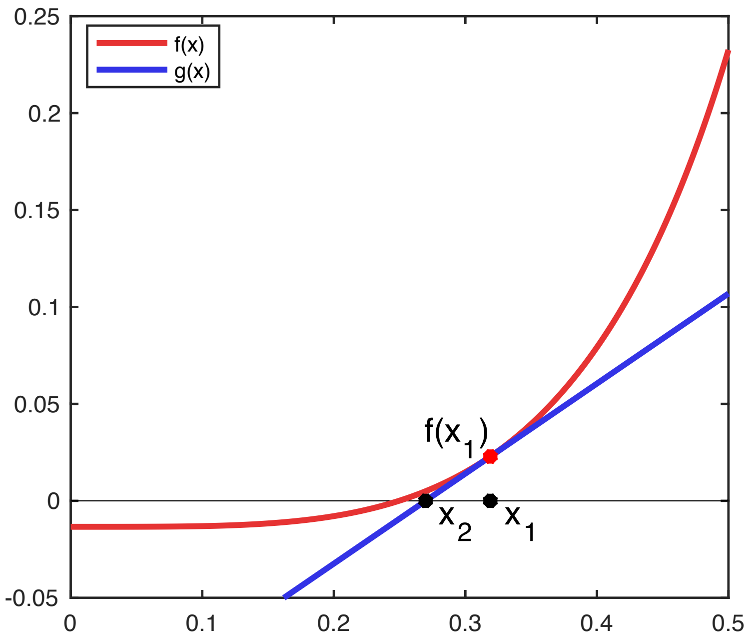
$x_0 = 0.4000, \qquad x_1 = 0.3190$

$f(x_0) = 0.0792, \qquad f(x_1) = 0.0229$

Example

$$f(x) = \tan\left(e^{x^4}\right), \qquad f'(x) = \frac{4x^3 e^{x^4}}{\tan(e^{x^4})}$$

$$x_2 = x_1 - f(x_1)/f'(x_1)$$



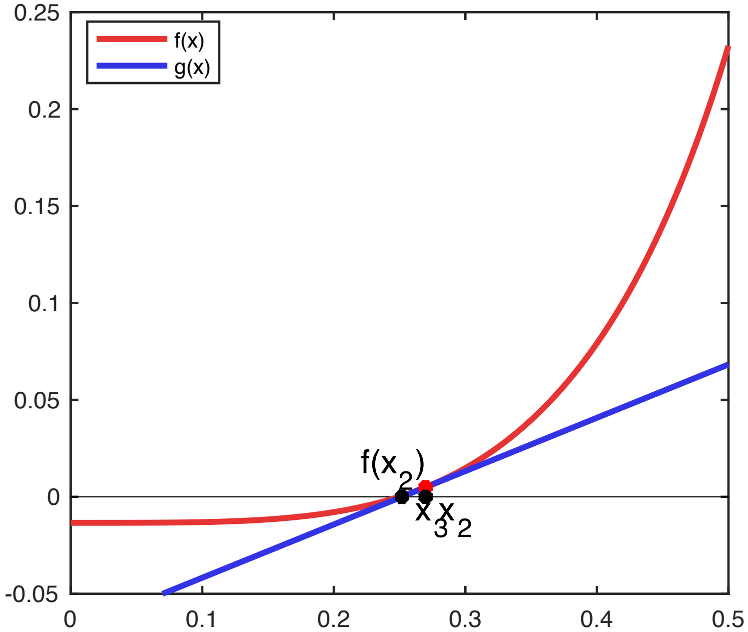
$$x_1 = 0.3190, \qquad x_2 = 0.2698$$

$$f(x_1) = 0.0229, \qquad f(x_2) = 0.0050$$

Example

$$f(x) = \tan\left(e^{x^4}\right), \qquad f'(x) = \frac{4x^3 e^{x^4}}{\tan(e^{x^4})}$$

$$x_3 = x_2 - f(x_2)/f'(x_2)$$



$x_2 = 0.2698,$ $\qquad x_3 = 0.2518$

$f(x_2) = 0.0050,$ $\qquad f(x_3) = 0.0005$

Stopping criteria

- Updates are given by

$$x_{k+1} = x_k - f(x_k)/f'(x_k)$$

  - Unlike numerical integration, number of iterations is not set.
- When do we stop?
  - When iterations give sufficiently small improvements

$$|x_{k+1} - x_k| < \delta \text{, possibly } |x_{k+1} - x_k| < \delta|x_1 - x_0|$$

  - When the function value is small enough

$$|f(x_k)| < \epsilon \text{, possibly } |f(x_k)| < \epsilon|f(x_0)|$$

  - Or a combination of the two

$$|x_{k+1} - x_k| < \delta \; or \; |f(x_k)| < \epsilon$$

- Typically, a combination is used
  - But at least one check must be there, otherwise we'll never stop!

Implementation

- Open-ended iterations with a criterion to check every iteration is perfect for a `while` loop
- Pseudocode:

```
x = …
epsilon = …
stop_criterion = False
while stop_criterion is False:
     x = x – f(x)/f′(x)
     stop_criterion = bool(… < epsilon)
```

The secant method

- Newton's method requires knowledge of $f'(x)$
  - This is not always available.
- Use a finite difference approximation instead
  - Introduced in Exercise 1, not explicitly curriculum

$$f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

  - Using this, we get:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \approx x_k - f(x_k)\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}$$

- Example of a quasi-Newton method
  - A class of methods often used in practice, not curriculum

Comparisons between three methods

- We have seen three root finding methods so far:
  - Bisection method, Newton's method and secant method
  - When do you use one over the others?
- Algorithmic restrictions
  - Newton's method requires $f'(x)$ in addition to $f(x)$
- Safety of use
  - Bisection is safest
    - As long as f is continuous, it converges
  - Both Newton's method and the secant method can fail
    - Depending on the starting point, more about this next week
- Speed of convergence
  - Given that they work, Newton's method is fastest, followed by the secant method and, far behind, the bisection method.

Newton's method in optimization

- Optimization is the task of finding the minimum or maximum of a function $f(x)$
- Basic knowledge: At an extreme point, $f'(x) = 0$.
- Use Newton's method on $f'(x)$ to solve this:

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}$$

- Very fast algorithm, when applicable

Summary

- Newton's method is an algorithm for root finding
  - Starts from an initial guess and produces better and better guesses from there
- Simpson's method is a derivative-free version of Newton's method
- Implementing root finders is best done using `while` loops.
- Choose the algorithm that is appropriate to your problem
- Next week: convergence analysis

Questions?