TDT4127 Programming and Numerics Week 36

Floating point numbers

Basic concepts in numerical mathematics

Kunnskap for en bedre verden

NTNU

Learning goals

- Goals
  - Floating point numbers
  - Refresh mathematical concepts
  - General knowledge of numerics

- Curriculum
  - Exercise 1
  - Exercise 2

Floating point numbers

- Continued from last week's lecture
- Decimal numbers can be both infinitely *large* and *long*
  - For example, π is infinitely long
    - π = 3.14159265359…
  - We can still use it mathematically:
    - $A = \pi r^2$
  - When computing, we use a *truncated* value with an uncertainty:
    - π = 3.14 (± 0.005)
  - We do this for other infinitely long numbers as well:
    - 1/3 = 0.3333 (± 0.0005)
- Our representation of decimal numbers must balance *magnitude* and decimal point *precision.*

Floating point numbers

- Floats are a tradeoff between *size range* and *accuracy*
- Based on scientific notation for numbers
  - Avogadro's number: $10^{23} \times 6.022140857$
  - Electron rest mass: $10^{-31} \times 9.109383561$
  - Large range of numbers, here using only 12 digits (base 10 numbers).
  - *Uncertainty* lies in the last digit
- Floating point numbers use the same idea, but in base 2
  - $a = (-1)^{sg} \times 2^{e-b} \times s$
    - Sign: *sg* is 1 bit representing 0 or 1, allows negative/postive numbers
    - Exponent: *e* is a positive integer, adjusts size
    - Bias: *b* is a *predetermined* integer allowing for negative exponents
    - Significand: s is a number between 1 and 2 of the form

$$s = 1.s_1 s_2 s_3 s_4 s_5 s_6 \ldots$$
$$= 1 + s_1 \times 2^{-1} + s_2 \times 2^{-2} + s_3 \times 2^{-3} + s_4 \times 2^{-4} + s_5 \times 2^{-5} + s_6 \times 2^{-6} + \ldots$$

- This is like scientific notation in base 2, with uncertainty in the last digit.
- More in Exercise 1, after which we will mostly not have to worry about them.

Operations with floating point numbers

- Addition/subtraction **requires care** due to roundoff error
  - When adding, the smaller number loses significance
  - Example in base 10: 12345.67 + 1.224567 with 7 digit precision:

$$\begin{array}{r} 12345.67 \\ +\quad 1.224567 \\ \hline = 12346.894567 \end{array} \approx 12346.89$$

  - Same effect as adding 1.22 since *the last four digits are lost*.
  - When adding many small numbers to a larger number, we lose precision unless it is done carefully.
    - Workarounds such as Kahan's algorithm is an algorithm for doing so. Not curriculum.

Operations with floating point numbers

- Multiplication/division are **safe**
  - We add/subtract exponents and multiply/divide the significands.
- Checking for equality is **very unsafe**
  - If *a* and *b* are floats, $a = b$ if all their bits are the same.
  - Due to imprecision, numbers that *should* be equal after some computation, may not be equal.
  - Example: Are $d = (a + b) + c$ and $e = a + (b + c)$ equal?

    $a = 123456.7$, $b = 123.4567$, $c = 0.4567891$

    $d = 123580.2 + 0.4567891 = 123580.7$

    $e = 123456.7 + 123.9135 = 123580.6$
- This concludes the rest of last week's lecture

The goal of numerics

- To solve «unsolvable» equations

$$\log(\cos(x^2)) = \frac{e^{x^3}}{1+\sqrt{x}}$$

- Saves **a lot of time** and lets us *do more with maths*!
- Based on **algorithms**
  - Recipes expressed mathematically
  - **Implementation** done by programming
- Three main questions we will discuss for each topic:
  - ***What*** do the algorithms look like?
  - ***When*** do the algorithms work?
  - ***How well*** do they work?

What do the algorithms look like?

- Example: Bisection method
  - Simple algorithm (*root finder*) for finding zeroes of functions: $f(x) = 0$
  - Root finders can also be used to solve equations:

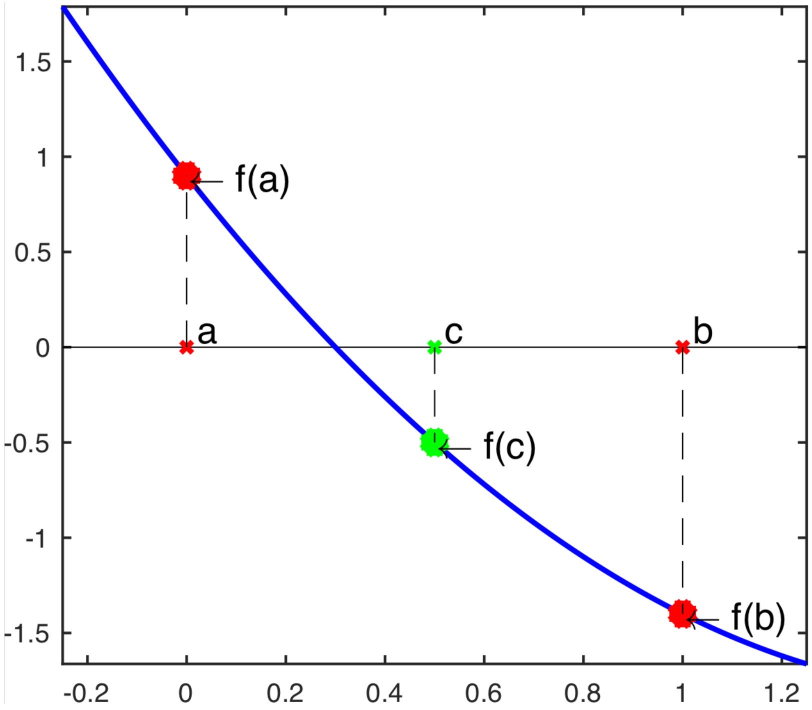
$$f(x) = g(x) \quad \Leftrightarrow \quad f(x) - g(x) = h(x) = 0$$

- Start with two points a and b such that $f(a) < 0$, $f(b) > 0$
  - Then, there is a point z between a and b where $f(z) = 0$
  - This point is also called a *root* of f
- Let $c = (a+b)/2$, check the value of $f(c)$
  - If $f(c) < 0$, swap a for c and repeat
  - If $f(c) > 0$, swap b for c and repeat
  - If $f(c) = 0$, we have a solution!
- Start again with new a and b.
  - A single step like this is called an *iteration.*
  - Repeated iterations makes a smaller and smaller interval around z.

Iteration 1

f(a) > 0, f(b) < 0, f(c) < 0

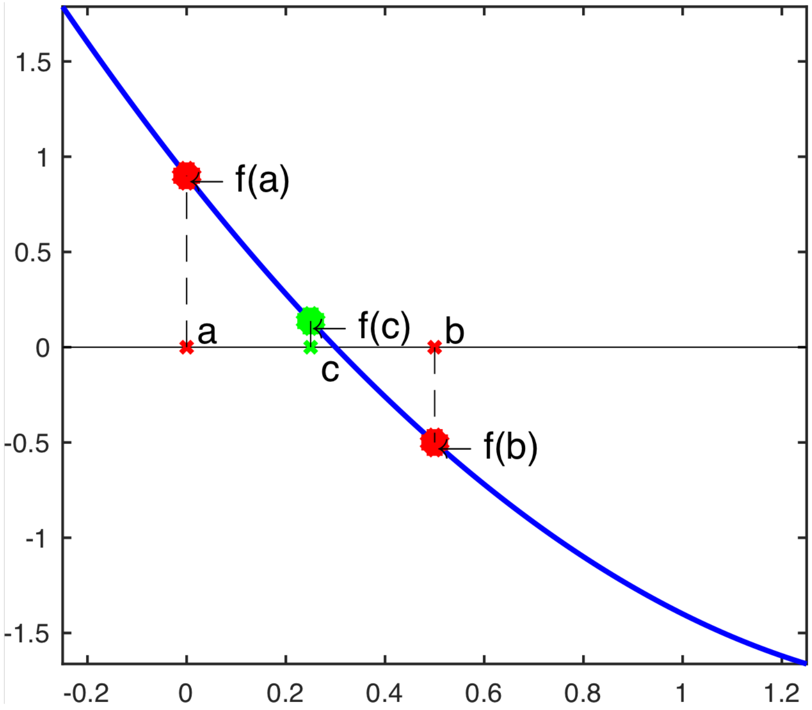
-> Swap b for c



Iteration 2

f(a) > 0, f(b) < 0, f(c) > 0

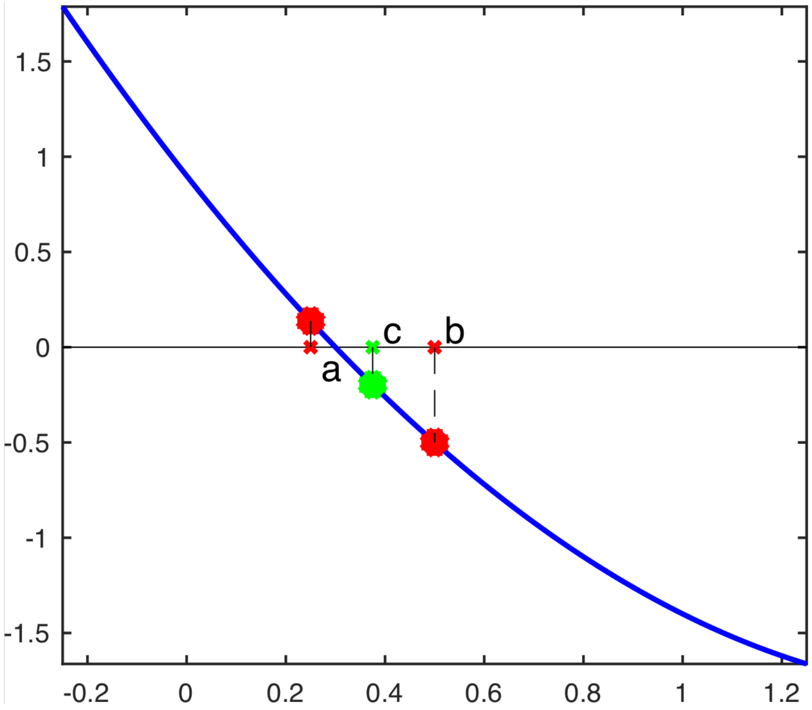
-> Swap a for c



Iteration 3

f(a) > 0, f(b) < 0, f(c) < 0

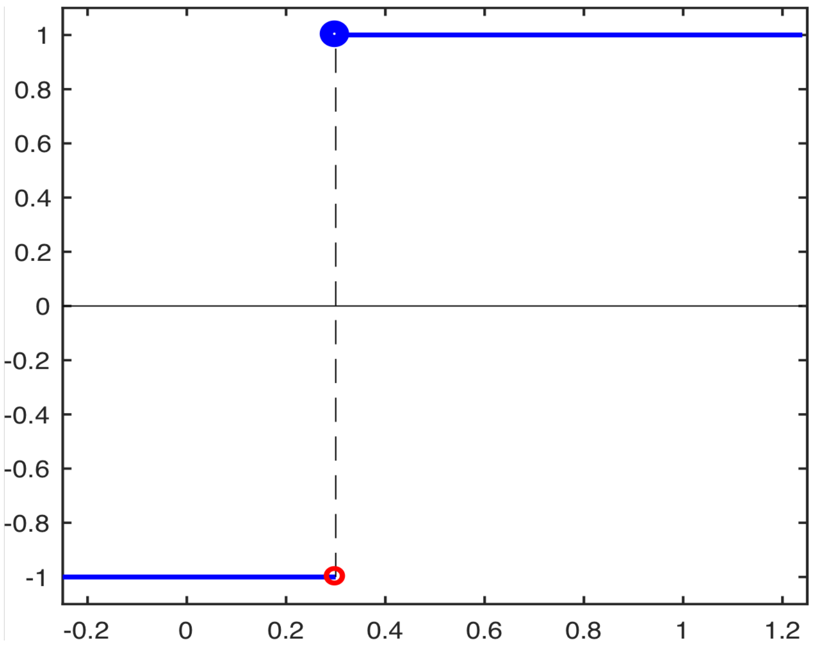
-> Swap b for c



When do the algorithms work?

- Algorithms work based on *requirements*, and it is important to meet them
  - Otherwise, absurd results can occur
- Example: Bisection method
  - Correct initialization: Need to start with two point a and b such that $f(a)$ and $f(b)$ have different signs.
    - Otherwise, we don't know if there is a zero in the interval
  - Properties of the function f
    - We require that f is *continuous*
    - A continuous function does not make jumps
    - Otherwise, our intuition that there is a point z between a and b where $f(z) = 0$ does not hold!

A discontinuous function: $f(x) = \begin{cases} -1, & x < 0.3 \\ 1, & x \geq 0.3 \end{cases}$



- No zeroes, but the starting interval *a* = 0, *b* = 1 is still OK!
- If we run the algorithm, it will try to find a non-existant root. Absurd!

How well do the algorithms work?

- How fast is the *convergence*?
- Example: Bisection vs Newton's method to find root of $f(x) = (x-0.3)(x-3)$
  - Newton's method is taught in week 39

Iteration no.	(a,b), Bisection	x, Newton
0	(0,1)	1
1	(0,0.5)	-0.0769230
2	(0.25,0.5)	0.25886586
3	(0.25,0.375)	0.29939185
4	(0.25,0.3125)	0.29999986
5	(0.28125,0.3125)	0.30000000

- Newton's method is extremely fast! 5 iterations to get 8 digits of accuracy.

The «user manual» for algorithms

- Several issues to keep in mind, all of which we will go through for every algorithm:
  - Convergence speed
  - Accuracy of solution
  - Error estimates
  - Conditions for use

Timeline

Week	Numerikk	Algorithms
34	Introduction	
35	Programming, floating points	
36	Refresh maths, floating points	Bisection method
37	Numerical integration	Trapezoidal rule, Simpson's rule
38	Numerical equation solvers in 1D	Newton's method
39	Numerical equation solvers in 1D	
40	Solving linear systems	Gaussian elimination
41	Solving linear systems	
42	Numerical equation solvers in nD	Newton's method for systems
43	Numerical solution of differential equations	Euler's method, Heun's method, Runge-Kutta methods
44	Numerical solution of differential equations	
45	Numerical integration with adaptive Simpson's rule	Adaptive Simpson's rule
46	Repetition	
47	Repetition	

Summary

- Floating point numbers are used for real (decimal) numbers and are **inexact**
- Addition of small and large numbers can cause problems
- Do not make code that relies on checking whether two floats are equal
  - Integers, on the other hand, are okay!
- Numerics solve practical mathematical problems
  - Algorithms behave differently
    - Some are faster than others
    - Some put stricter requirements on the problem
  - We will learn algorithms for numerical integration, equation solving and differential equation solving.

Questions?