A Railway Track Deflection Model Based on Nonlinear Discrete Support

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ABSTRACT: A nonlinear deflection model for railway tracks is developed for static or semi-static loads. The model is motivated by the fact that the load-deflection curve for a railway track is often, if not always, nonlinear with a hardening response. Hence a Winkler foundation is not appropriate as this represents a linear model - the beam on elastic foundation model. Euler-Bernoulli beam elements are used to model the rails, and the sleeper support is modelled by discrete nonlinear springs. The spring response is modelled by a simple power function with two parameters, thus providing a nonlinear behaviour. The analysis is based on measuring the rail deflection for two different load levels. A back-analysis using the nonlinear load-deflection relationship measurements is then carried out, which will give the numerical values of the two parameters in the nonlinear spring model. The rail deflection, rail bending moment and the rail seat loads can now be calculated. Assuming a hardening behaviour, the new model predicts a somewhat more extended deflection basin as compared with the beam on elastic foundation model with the same maximum deflection. Also, the maximum rail bending moment is somewhat smaller, while the maximum rail seat load attains a somewhat larger value. Thus, the new model can be said to give a stiff track response near the load application point and a softer response farther away from the load as compared with the beam on elastic foundation model.

KEY WORDS: Railway track deflection, nonlinear discrete support, beam elements, back-calculation

1 INTRODUCTION

When measuring the rail deflection for different levels of axle loads, there is often, if not always, a nonlinear relationship between the load and the deflection. This nonlinear behaviour is usually of the hardening type with increasing track stiffness as the load increases. Obviously, a Winkler foundation is not capable of modelling such behaviour as this type of foundation represents a linear model – the beam on elastic foundation model (abbr. BOEF model). A good reference for the BOEF model is Hetényi (1946). The BOEF model assigns a track modulus, often denoted k, to the foundation. Habitually the track modulus is tacitly assumed to be valid for a larger range of loads, if not all load levels. The task of measuring the track modulus has been described by Selig and Li (1994) and by Cai et al. (1994).

On the other hand, confronted with the true nonlinear relationship between load and deflection it is therefore difficult to assign a track modulus that you can relay on. If you pick a track modulus suitable for predicting the maximum deflection you have no guarantee that this will be the proper one when, say, estimating the maximum rail seat forces. The present study is an attempt to explore in a quite simple way the true nonlinear nature of the load-deflection relationship. The model that is developed has been investigated with a hardening behaviour in mind, see Figure 1 below. Nevertheless, the model also allows a softening behaviour, but the results hereof are to date unknown.

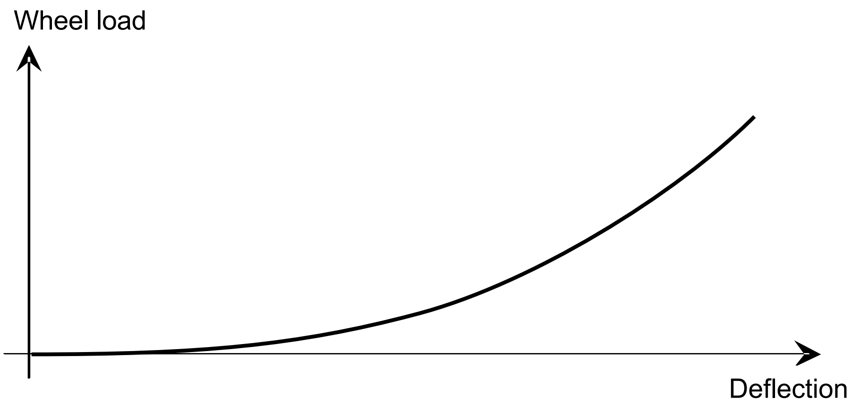

Figure 1: Sketch of a typical relationship between rail deflection and wheel load.

It is emphasised that this study is a mere possibility study as a quite limited number of problems has been looked at. Also, no concrete measurements form the basis for the modelling although the model can be adapted to any load-deflection relationship that can be approximated to a curve like the one in Figure 1.

An early version of the nonlinear model has been published by Skoglund (2002).

2 THE NEW NONLINEAR MODEL

2.1 The basic idea

The idea of the new model goes like this: Measure the loads and the corresponding deflections at one (or more) point in the track. Use a simple nonlinear relationship to establish a load-deflection curve that approximates the measurements. Calculate form the nonlinear relationship the seat loads, rail moments etc. assuming that the discrete supports also act nonlinear. In this way a nonlinear model is obtained with possibly only one measuring point in the track. Also, there is no need a priori to know the nonlinear properties of the individual components of the track.

Apart from the origin, at least two sets of corresponding loads and deflections are needed as input to the model. A simple power function with two parameters is then used to model the overall nonlinear load-deflection relationship, see Eqn. (1):

$$Q(y) = A \cdot \left(\frac{y}{y_{ref}} \right)^B \tag{1}$$

where

Q = wheel load [kN]

y = deflection of the rail directly below the wheel load [mm]
y_{ref} = a reference deflection, set equal to 1.0 mm
A = regression constant with same unit as Q_m
B = regression constant

In case more than two measurements are made, Eqn. (1) may be regarded a regression curve, and the constants A and B should then be chosen in a way that makes the best fit with the measured data. A and B form the needed parameters for the new model, much in the same way as the track modulus defines the linear BOEF model. The reason why an expression like the one in Eqn. (1) was chosen was that this expression would give a sufficient nonlinear behaviour at a cost of only one more parameter compared with the linear model. As will be evident later, this limited number of parameters will also be beneficial for the beam element model.

As can be seen by inspection of Eqn. (1), the track behaves linearly if $B = 1$, it hardens if $B > 1$, and if $B < 1$ the track shows a softening behaviour.

The nonlinear track stiffness, K_{nl}, is the derivative of the expression in Eqn. (1) and is given by Eqn. (2):

$$K_{nl}(y) = \frac{dQ}{dy} = AB\left(\frac{y}{y_{ref}}\right)^{B-1} \tag{2}$$

provided that K_{nl} has the unit of kN/mm.

2.2 Aspects of measuring loads and deflections for the new model

A minimum of three pairs of loads and deflections are needed, of which the origin constitutes one such pair. Compared with the beam-on-elastic-foundation model, only one more measurement is required.

Since the new model does not explicitly incorporate any dynamic effects, it is the static wheel loads that are of interest. For empty cars running at low speed these loads are close to those given in the car specifications by the rolling stock manufacturers. For freight trains the load of the cargo is often known, consequently a different load level is obtained. The two needed load levels should differ substantially in magnitude in order to get reliable data for the model. It is also possible to get load data from a weigh-in-motion site installed in the track (e.g. a wheel flat detector).

Having the loads, the corresponding deflections must be measured. An easy way of getting the deflections is to use LVDTs measuring the relative displacement between the rail and an anchor in the track bed. Of course, the deflections must be measured at low train speeds in order to avoid any dynamic load contributions.

A simple and appealing alternative in load-deflection measurements is to use a jack together with appropriate load and deflection measurement equipment. However, some precautions should be taken. First, both rails should be loaded with the same load. Loading only one rail at a time will cause false twist in the superstructure, and thereby creating track reactions that will not occur for a real axle load. Second, the device providing the counterbalance for the jack (e.g. a big digger) should be at a sufficient distance in order to avoid the effects from any nearby track onloading.

The expression in Eqn. (1) also applies to cases of two axles in a bogie. Then Q is still the wheel load, while y is the maximum deflection measured. There may be a risk that the maximum deflection is not directly below a wheel load when you have multiple axle loads, but the effect of this phenomenon is small and may be neglected. Another important property with a nonlinear model is that a summation of the rail deflections or moments is not possible when these reactions are produced by multiple axles. Instead, a separate analysis has to be made for each axle configuration.

3 CREATING A DISCRETE MODEL AND SOLVING THE MODEL EQUATIONS

3.1 Creating a discretely supported beam element model

The superstructure is modelled with Euler-Bernoulli beam elements with only one element per sleeper spacing if there are no loads between the sleepers. The rail support, i.e. the reaction from the sleepers, is modelled as nonlinear springs using the same nonlinear concept as in Eqn. (1) using parameters a and b instead of *A* and *B*, see Eqn. (3):

$$S_n(y_n) = \mathrm{a} \cdot \left(\frac{y_n}{y_{ref}} \right)^{\mathrm{b}} \tag{3}$$

where

S_n	=	rail seat load for sleeper no. n [kN]
y_n	=	deflection for sleeper no. n [mm]
a, b	=	parameters

It is emphasised that Eqn. (3) represents an assumption on the behaviour of the rail support. No tests have been carried out to establish this relation. However, Eqn. (3) was an appealing choice since the overall track behaviour was modelled by the same type of expression, conf. Eqn. (1). Also, if the rail was completely stiff and of finite length, a would equal *A* divided by the number of rail supports and b would be identical with *B*.

The numerical values of a and b will be found as part of the analysis, hence no information regarding their numerical value is needed prior to the numerical solution process. This feature is different from most of the other solution schemes for nonlinear models and eases the data acquisition for the present model.

3.2 Establishing and solving the model equations

The next step is to establish equations for the model. This is done by applying ordinary stiffness relations from structural mechanics valid for Euler-Bernoulli beam elements, but with the model in Eqn. (3) for the rail support. In structural mechanics this process is known as the direct method. The stiffness relations provide as many equations as there are unspecified degrees of freedom (dofs), but two more concurrent relations have to be established in order to solve also for a and b. One additional equation is established by regarding the deflection at the position of the wheel load as a specified dof, i.e. the known overall deflection. It is then necessary that this relation contains the two unknowns a and b, and the only requirement for this to be satisfied is that the known deflection is measured at a sleeper position.

The last necessary equation is established by equilibrium between external energy applied by the load and internal energy stored in the track model. The external energy, E_{ext}, supplied by a single wheel load is given by Eqn. (4):

$$E_{ext} = \int_{0}^{y} Q(y)dy = \frac{A}{B+1} \cdot \left(\frac{y}{y_{ref}} \right)^{B+1} \tag{4}$$

The energy stored internally in the track, E_{int}, because of the wheel load is a sum of the energies stored in the nonlinear support springs, $\sum E_n$, and the bending energy of the rail, E_{rail}, conf. Eqn. (5):

$$E_{\text{int}} = \sum_{n} E_n + E_{rail} = \sum_{n} \frac{\mathsf{a}}{\mathsf{b}+1} \cdot \left(\frac{y_n}{y_{ref}} \right)^{\mathsf{b}+1} + \frac{1}{2EI} \int_{rail} [M(x)]^2 \, dx \tag{5}$$

where n is rail support no. n and $M(x)$ is the rail moment.

As the principle of conservation of energy forms one of the model equations the model is energy consistent. This also provides uniqueness to the model, i.e. there is a one-to-one correspondence between load and deflection. As such the model could be said to be nonlinear elastic.

The model is currently expanded to cover 14 rail spans resting on 15 equidistant sleepers, thus dealing with 30 unknowns (14 deflections, 14 rotations, and a and b). A general purpose calculation programme (Mathcad Professional 2000) run on a PC was used to solve the resulting set of nonlinear equations. The software utilises a kind of Newton-Raphson technique to solve the equation set.

4 SOME RESULTS FOR THE NEW BEAM ELEMENT MODEL

From various analyses of a single wheel load the a and b are found to be reasonably constant for different load levels within the same overall model (same A and B). By way of example, a calculation with A = 20000 N, B = 1.5 and with 54E3 (formerly designated S54) rail elements spanning seven sleepers on each side of the load shows that the variation is at most around 1 % for both parameters, but they deviate in opposite directions. The deviation in the support load (seat load) by using a and b from different load levels is thus only in the order of 0.2 %. If Eqn. (5) should be valid, it is important that a and b do not vary with the load level.

4.1 Some results for a single axle load

In all cases there were a single wheel load Q = 100 000 N with A = 25 000 N, B = 2.0, and with maximum deflection y = 2.0 mm. The track consisted of 60E1 (formerly designated UIC60) rails and the sleeper spacing was 600 mm. Analysed track length was 8400 mm to each side of the wheel load (spanning 15 sleepers), but only the half to the right of the wheel load was analysed, as the two parts are symmetric. a and b were calculated to be 7670 N and 2.33, respectively. Also for the BOEF model the maximum deflection was set to 2 mm. The deflection consistent track modulus was calculated to be 24.8 N/mm^2. Rotation, but no vertical deformation is allowed at the rightmost sleeper. In Figure 2 the deflection distribution of the new model is compared with the BOEF model.

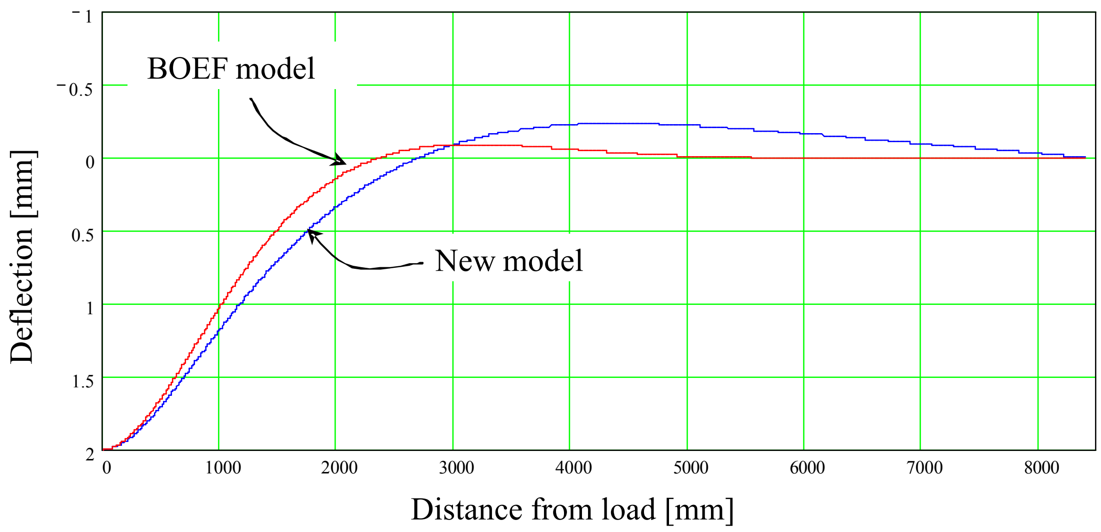

Figure 2: Rail deflection for the new model as compared to the BOEF model, single axle load.

The corresponding rail moment is depicted in Figure 3.

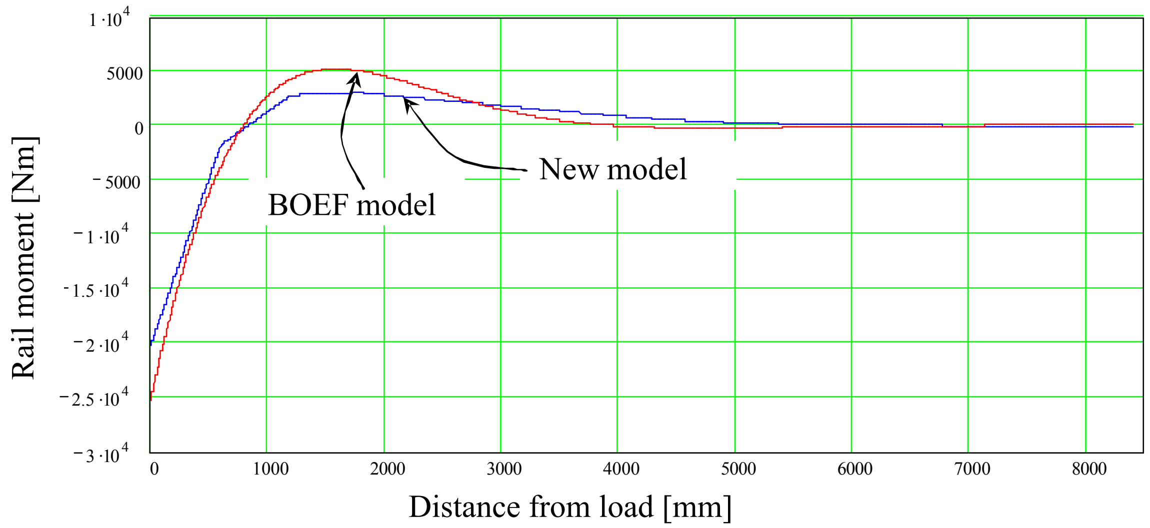

Figure 3: Rail moment for the new model as compared to the BOEF model, single axle load.

As can be seen from the diagrams in Figure 2 and Figure 3 there are no big differences either in rail deflection or in rail moment between the BOEF and the current new model, especially near the point of load application. It seems however that he new model has an impact on the track farther away from the load than that of the BOEF model, confer Figure 2. In 'linear response terms' this should imply a softer behaviour for the new model. It also seems as if the new model predicts a bigger uplift than the BOEF model, but here we have to bear in mind that the models do not incorporate any weight of the rails and sleepers.

With regard to the seat loads, the new model concentrates these loads nearer to the axle load, as can be seen from Figure 4. The seat loads for the new model thus indicates a stiffer behaviour near the point of load application. All in all it can be said that the new model gives a track that is stiff where the load is applied, but soft farther away from the load.

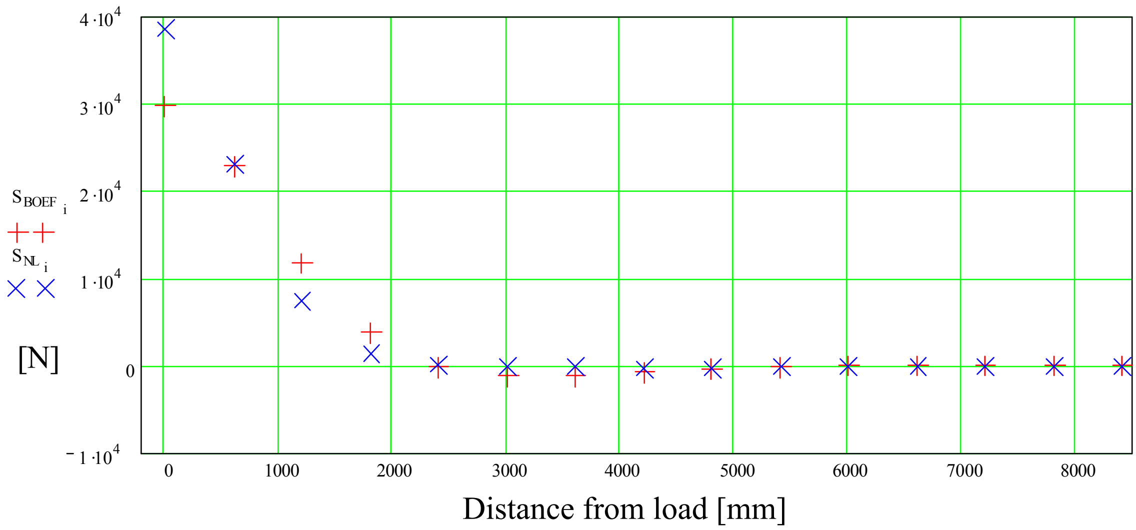

Figure 4: Seat load for the new model as compared to the BOEF model, single axle load.

4.2 Some results for a double axle load

In all cases there were a double wheel load Q = 100 000 N on each wheel with A = 25 000 N, B = 2.0, and with maximum deflection y = 2.0 mm. The wheel loads were separated at a distance of 2400 mm, which approximates a typical axle spacing in a bogie. The track and the analysed length were the same as for the single axle load. a and b were in this case calculated to be 7470 N and 2.25, respectively. Also for the BOEF model the maximum deflection was set to 2 mm. The deflection consistent track modulus was calculated to be 24.8 N/mm^2. In Figure 5 the deflection distribution of the new model is compared with the BOEF model.

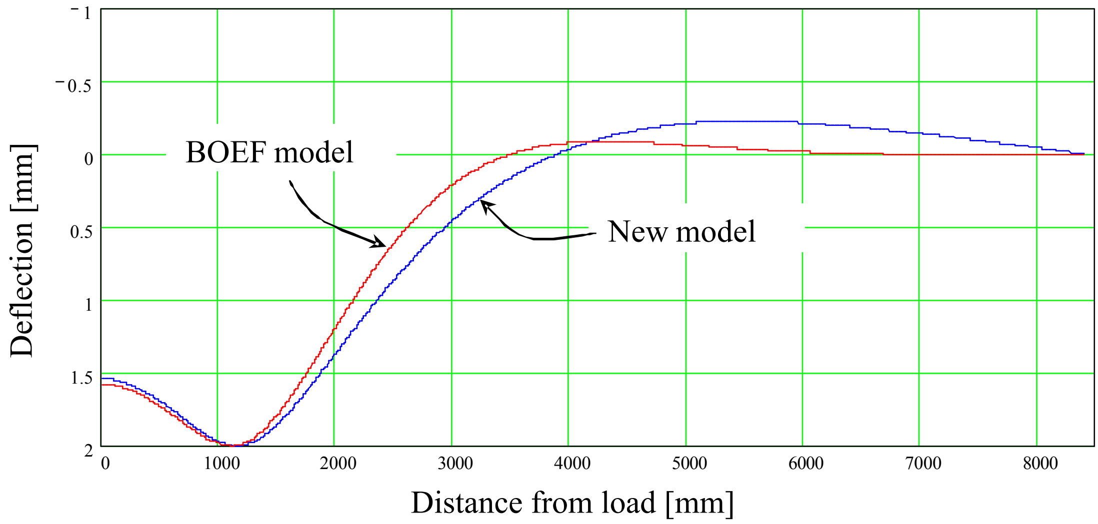

Figure 5: Rail deflection for the new model as compared to the BOEF model, double axle load.

The corresponding rail moment is depicted in Figure 6.

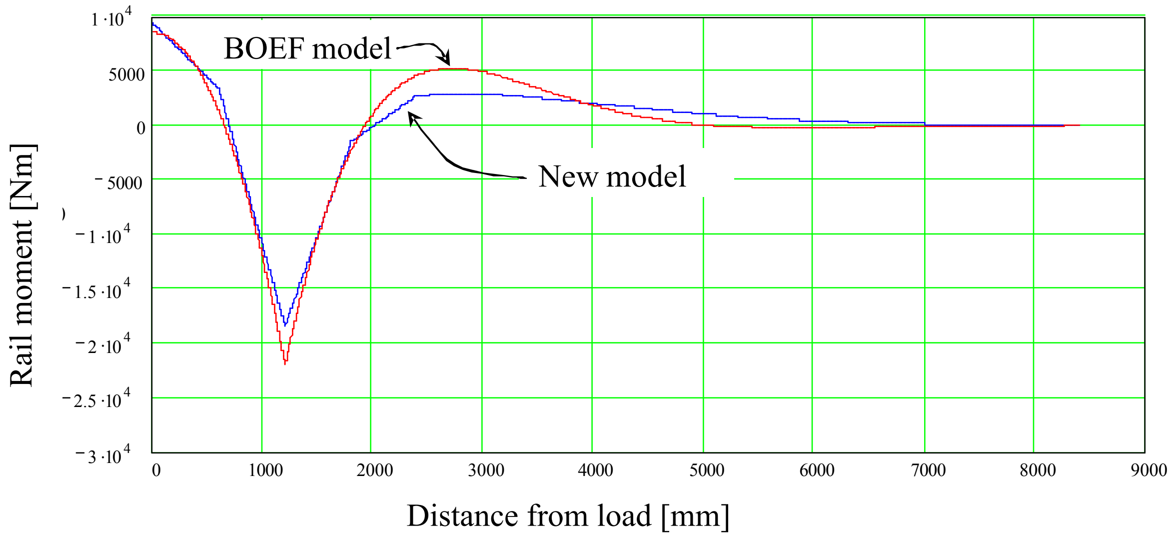

Figure 6: Rail moment for the new model as compared to the BOEF model, double axle load.

The rail seat loads are given in Figure 7.

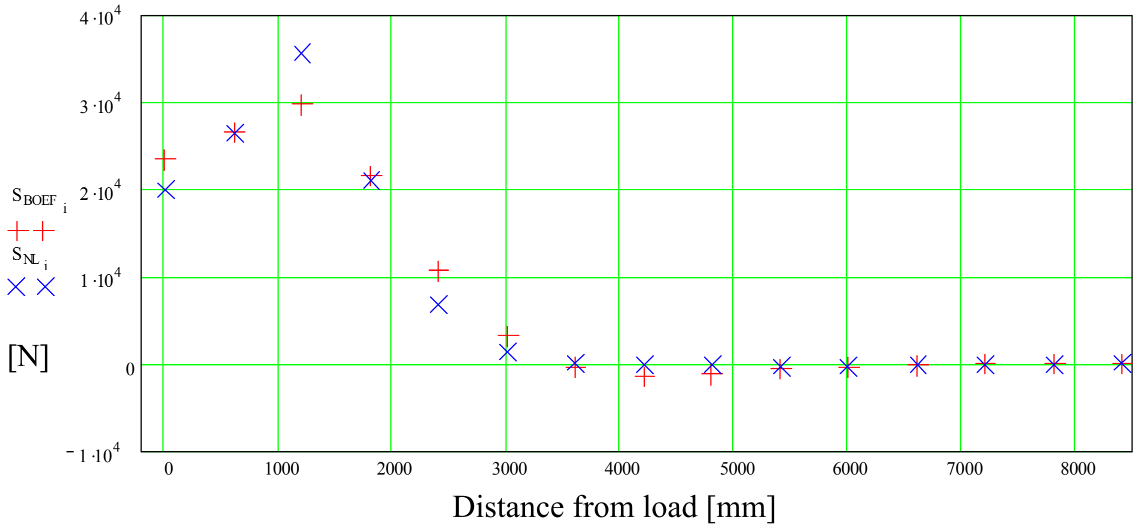

Figure 7: Seat load for the new model as compared to the BOEF model, double axle load.

It is seen from the preceding figures that the conclusions for a single axle load are also valid for a double axle load.

5 CONCLUDING REMARKS

Normally a finite element analysis is a 'bottom up' approach as the element properties (geometric properties and material properties) are used to build a total model. The new model represents a 'top down' approach in the sense that you first establish the overall track model (Eqn. (1)) then you proceed to the element reactions by calculating the various responses and the needed nonlinear spring data in the same operation. In principle, this is much the same approach as the ordinary BOEF method. In the BOEF method you establish the overall model by assuming a Winkler foundation and solving the resulting differential equation. Then you measure the track modulus, which is a parameter created by the BOEF model, and you may then calculate the various reactions. The main difference in the general solution procedures is that for the BOEF model it is only needed to solve the differential equation once and for all, while for the new nonlinear model you have to solve the equation set each time you change the input data.

Some of the findings in the new model can be stated as follows:

1. A somewhat smaller maximum rail moment is estimated compared with the BOEF model.
2. The maximum rail seat force is estimated to be larger compared with the BOEF model.
3. The uplift seems to be larger than in the BOEF model, but the effect of the weight of the track ladder has not been accounted for.

The capabilities of the new model have not been fully explored, hence the notion of a 'possibility study' as stated in the Introduction may be appropriate. Some of the most important facets of the model that need further exploration may be summarised as follows:

1. Data from track measurements, especially deflections, but also rail stresses and seat loads, are needed to explore the usefulness and reliability of the model. To get reliable data for rail stresses and seat loads is however more challenging than for the deflections.
2. The effect of the track ladder weight may be advantageous to include in the model.
3. It may be possible to separate the effects from the rail pads and the sleepers and then deduce from the overall behaviour what behaviour should be left with the ballast and substructure.

In the future, this method will hopefully provide a conceptually simple but still a more accurate tool for the railway track engineer. Especially when the track shows a clear nonlinear load-deflection relationship the present model will be advantageous, and in particular when it comes to seat load calculations. The fact that the method may be based on measuring the rail deflection only makes the data capturing as easy as for the BOEF method.

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