Pavement Response due to Torsional Surface Loading

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ABSTRACT: When heavy vehicles such as truck and trailer make a sharp turn, it is known that torsional as well as vertical loads are exerted on the pavement surface. However, analytical solution of elastic multi-layered systems is not known. This paper presents the solution for multilayered elastic systems subjected to uniformly distributed torsional loading over a circular area. By combining this solution with the solutions of uniform vertical loading on the same circular area, one can estimate pavement response due to both types of loadings.

KEY WORDS: Multilayered elastic system, aggregate dislodgement, torsional load, vertical load, Hankel transform.

1 INTRODUCTION

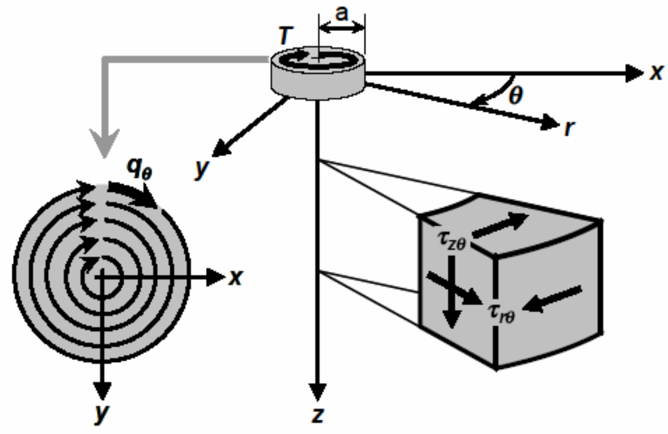
Currently, method of pavement design is shifting from empirical to mechanistic-empirical procedure. In order to achieve this goal, it is imperative that reliable and efficient tools of analysis are available to highway as well as airport agencies. Authors of this paper have already developed GAMES software, where pavement is modeled as multilayered elastic system on the surface of which vertical and/or horizontal surface loads act. Current version of GAMES software has same functions as BISAR, which is highly rated by pavement researchers around the world.

In this paper, extension of theoretical formulation to torsional surface load is introduced. The background to this development is based on the fact that when large-sized vehicles like trucks and trailers turn left or right, vertical and turning (torsional) loads are exerted on the pavement surface (Working Group, 2001). But since there is very little information in the literature on the analytical solution of multilayered elastic structure subjected to torsional surface loading, most methods for pavement analysis tend to ignore effects of torsional load.

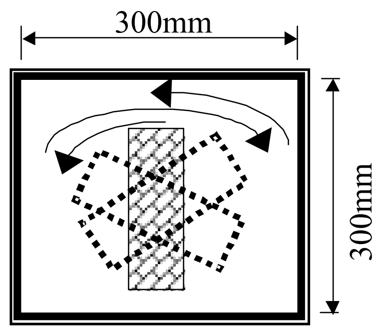
Burmister developed the well known solutions for axi-symmetric analysis of vertical load based on stress function. For the case of horizontal loading, solutions for half-space were developed by Muki (Muki, 1956) and for a two layered system Kimura's works are well known (Kimura, 1966). All these methods employ Hankel transformation. Compilations of the research works on half-space and two layered system can be found in references and (Kimura, 1978) and (Poulos and Davis, 1974). Miyamoto (Miyamoto, 1967) presented detailed classical mechanics of elasticity with application of Hankel transforms.

In Europe, Shell developed BISAR program for analysis of vertical and horizontal loads that takes into consideration slip in the layer interface (De Jong et al., 1979). This software is widely used among pavement researchers. Authors of this paper applied Hankel transforms to develop GAMES software (Maina and Matsui 2004, Matsui et al., 2002, Matsui et al., 2002), with functions similar to or better than BISAR, for axi-symmetric and asymmetric analyses of multilayered elastic systems subjected to vertical and/or horizontal surface loads. All the above mentioned research works considered only surface vertical load due to wheel load and surface horizontal load due to starting/stopping.

There has been very few research works on torsional surface loading. Solutions for the case of half-space due torsional surface load were presented by Reissner and Sagoci (1944). They considered problems for oscillating torsional load in elliptic coordinate systems and it was difficult to extend these solutions to multilayered systems. Bekheet et al. (2003) performed FEM analysis of field dynamic test to determine influence of dynamic shear stiffness on the rutting of asphalt mixes. These kinds of research works indicate that theoretical analysis of the action of torsional load on a multilayered elastic surface may not have been published before. This paper seeks to present analytical solutions for problems related to circular torsional shear stress acting on the surface of a multilayered elastic system by applying Hankel transforms. A combination with solutions for axi-symmetric vertical load will enable accurate analysis of internal pavement stresses and strains that will develop due to turning load developed from turning wheels of, for example, trucks.



a) Torsional load



b) Alternating torsional load

Figure 1: Surface torsional load on multilayered elastic system

a) Alternating torsional load test by passenger car

b) Sample surface condition before and after the test

Figure 2: Aggregate dislodgement test due to alternating torsional load

1.1 Background

It generally very common for asphalt concrete layers at parking areas as well as their entrances and road intersections to experience extreme damages at the surface and this particularly serious for porous pavements. When trailers turn left or right, inner rear tires act as fixed point of rotation. In addition to vertical loading, the surface will also be subjected to torsional load as shown in Figure 1a. Moreover, when the wheels are turned quickly to the opposite direction, pavement surface will be subjected to alternating torsional loads as shown in Figure 1b. Figure 2a shows alternating torsional load on the pavement surface exerted by a passenger car, while Figure 2b shows the state of aggregate dislodgement before and after the test. Dislodgement of aggregate particles occurs at relatively few number of wheel turns and this phenomenon is analogous to low cycle fatigue problems mostly seen in cement concrete and steel structures.

2 GOVERNING EQUATIONS

Figure 2a shows radially distributed uniform load, which is assumed to act on the surface of an elastic multilayered structure. Figure 2b represents the relationship between points of interest and the load.

The equilibrium equation for an infinitesimal element in a pavement structure due to turning (torsional) load is given as follows:

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{r^2} = 0 \tag{1}$$

where u_θ is a function of r and z. Furthermore, shearing stresses $\tau_{z\theta}$ and $\tau_{r\theta}$ may be determined in terms of u_θ and shear modulus of elasticity, μ, as follows:

$$\tau_{z\theta} = \mu \frac{\partial u_\theta}{\partial z} \tag{2}$$

$$\tau_{r\theta} = \mu \left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) \tag{3}$$

Since external shear stress, q_θ, on the pavement surface is uniformly distributed, boundary conditions may be given as:

$$\begin{aligned} \tau_{rz} &= 0 \\ \tau_{z\theta} &= -q_\theta \end{aligned} \tag{4a}$$

for $r \le a$, while the boundary condition for $r > a$ may be given as:

$$\tau_{r\theta} = \tau_{z\theta} = 0 \tag{4b}$$

2.1 Interface condition

If layers i and i+1 are completely bonded, the following relation is obtained:

$$\begin{pmatrix} u_\theta^{(i)} \\ \tau_{z\theta}^{(i)} \end{pmatrix} = \begin{pmatrix} u_\theta^{(i+1)} \\ \tau_{z\theta}^{(i+1)} \end{pmatrix} \tag{5}$$

However, torsion moment (torque) T may be determined as:

$$T = -\int_0^{2\pi}\int_0^{a} q_\theta \, r dr d\theta = -\pi a^2 q_\theta \tag{6}$$

3 DERIVATION OF SOLUTION

Performing Hankel transform on Equation (3) yields:

$$\int_0^\infty r\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{r^2}\right) J_1(\xi r)dr = \frac{d^2 \tilde{u}_\theta}{dz^2} - \xi^2 \tilde{u}_\theta = 0 \tag{7}$$

where ξ is Hankel transform parameter. $\tilde{u}_\theta$ may be represented by the following equation:

$$\tilde{u}_\theta = \int_0^\infty r u_\theta(r,z) J_1(\xi r)dr \tag{8}$$

Solving Equation (4) for $\tilde{u}_\theta$ yields:

$$\tilde{u}_\theta(\xi, z) = Ae^{\xi z} + Be^{-\xi z} \tag{9}$$

Performing Hankel transform on Equation (2) yields:

$$\tilde{\tau}_{z\theta}(\xi, z) = \int_0^\infty r \tau_{z\theta}(r,z) J_1(\xi r)dr = \mu\left(A\xi e^{\xi z} - B\xi e^{-\xi z}\right) \tag{10}$$

Equation Hankel transform of Equation (3) to $\tilde{\tau}_{r\theta}(\xi, z)$ yields:

$$\tilde{\tau}_{r\theta}(\xi, z) = \int_0^\infty r\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) J_2(\xi r)dr = -\mu\xi\tilde{u}_\theta = -\mu\xi\left(Ae^{\xi z} + Be^{-\xi z}\right) \tag{11}$$

Rearranging Equations (7) and (8) yields:

$$\begin{Bmatrix} \tilde{u}_\theta(\xi, z) \\ \tilde{\tau}_{z\theta}(\xi, z) \end{Bmatrix} = \begin{bmatrix} e^{\xi z} & e^{-\xi z} \\ \mu\xi e^{\xi z} & -\mu\xi e^{-\xi z} \end{bmatrix} \begin{Bmatrix} A(\xi) \\ B(\xi) \end{Bmatrix} \tag{12}$$

Considering boundary conditions at the surface gives:

$$\tilde{\tau}_{z\theta}(\xi, z) = -\int_0^a r q_\theta J_1(r\xi)dr = -\frac{q_\theta}{\xi^2}\left[J_1(\xi a)\left(2\xi a - \pi \mathbf{H}_0(\xi a)\right) + \pi \mathbf{H}_1(\xi a)\right] \tag{13}$$

where $\mathbf{H}_0(\xi a)$ and $\mathbf{H}_1(\xi a)$ are Struve functions explained in the appendix.

3.1 Elastic multi-layered systems

Considering an n-layered elastic body as shown in Figure 2a, response at i^{th} layer may be represented using Equation (12) with superscript (i) as:

$$\begin{Bmatrix} \tilde{u}_\theta^{(i)}(\xi, z) \\ \tilde{\tau}_{z\theta}^{(i)}(\xi, z) \end{Bmatrix} = \begin{bmatrix} e^{\xi z} & e^{-\xi z} \\ \mu^{(i)}\xi e^{\xi z} & -\mu^{(i)}\xi e^{-\xi z} \end{bmatrix} \begin{Bmatrix} A^{(i)}(\xi) \\ B^{(i)}(\xi) \end{Bmatrix} \tag{14}$$

The following equation may be derived for responses at the surface of layer i $(z = 0)$:

$$\begin{Bmatrix} \tilde{u}_\theta^{(i)}(\xi, 0) \\ \tilde{\tau}_{z\theta}^{(i)}(\xi, 0) \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ \mu^{(i)}\xi & -\mu^{(i)}\xi \end{bmatrix} \begin{Bmatrix} A^{(i)}(\xi) \\ B^{(i)}(\xi) \end{Bmatrix} \tag{15}$$

While responses at the bottom of layer i $(z = h_i)$ may be written as:

$$\begin{Bmatrix} \tilde{u}_\theta^{(i)}(\xi, h_i) \\ \tilde{\tau}_{z\theta}^{(i)}(\xi, h_i) \end{Bmatrix} = \begin{bmatrix} e^{\xi h_i} & e^{-\xi h_i} \\ \mu^{(i)}\xi e^{\xi h_i} & -\mu^{(i)}\xi e^{-\xi h_i} \end{bmatrix} \begin{Bmatrix} A^{(i)}(\xi) \\ B^{(i)}(\xi) \end{Bmatrix} \tag{16}$$

The relation between responses at the top and bottom of layer i may be written as:

$$\begin{Bmatrix} \tilde{u}_\theta^{(i)}(\xi, 0) \\ \tilde{\tau}_{z\theta}^{(i)}(\xi, 0) \end{Bmatrix} = \mathbf{T}^{(i)}(\xi, h_i) \begin{Bmatrix} \tilde{u}_\theta^{(i)}(\xi, h_i) \\ \tilde{\tau}_{z\theta}^{(i)}(\xi, h_i) \end{Bmatrix} \tag{17a}$$

where;

$$\mathbf{T}^{(i)}(\xi, h_i) = \begin{bmatrix} \cosh \xi h_i & -\dfrac{1}{\mu^{(i)}\xi} \sinh \xi h_i \\ -\mu^{(i)} \sinh \xi h_i & \cosh \xi h_i \end{bmatrix} \tag{17b}$$

$\mathbf{T}(h_i)$ is a transfer matrix representing the relationship between responses at the top and bottom of layer i. Furthermore, responses for the n^{th} layer where $z \to \infty$ would be; $u_\theta^{(n)} \to 0, \tau_{z\theta}^{(n)} \to 0$, which means $A^{(n)}(\xi) = 0$.

Hence, response at the top of n^{th} layer may be written as:

$$\begin{Bmatrix} \tilde{u}_\theta^{(n)}(\xi, 0) \\ \tilde{\tau}_{z\theta}^{(n)}(\xi, 0) \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ \mu^{(n)}\xi & -\mu^{(n)}\xi \end{bmatrix} \begin{Bmatrix} 0 \\ B^{(n)}(\xi) \end{Bmatrix} = \mathbf{P}^{(n)}(\xi, 0) \begin{Bmatrix} 0 \\ B^{(n)}(\xi) \end{Bmatrix} \tag{18a}$$

where;

$$\mathbf{P}^{(n)}(\xi, 0) = \begin{bmatrix} 1 & 1 \\ \mu^{(n)}\xi & -\mu^{(n)}\xi \end{bmatrix} \tag{18b}$$

Making use of Equations (17) and (18), the relationship between responses for the 1^{st} and n^{th} layers may be obtained as:

$$\begin{Bmatrix} \tilde{u}_\theta^{(1)}(\xi, 0) \\ \tilde{\tau}_{z\theta}^{(1)}(\xi, 0) \end{Bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{Bmatrix} 0 \\ B^{(n)}(\xi) \end{Bmatrix} \tag{19a}$$

where;

$$\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} = \mathbf{T}^{(1)}(\xi, h_1)\mathbf{T}^{(2)}(\xi, h_2)\cdots\mathbf{T}^{(i)}(\xi, h_i)\cdots\mathbf{T}^{(n-1)}(\xi, h_{n-1})\mathbf{P}^{(n)}(\xi, 0) \tag{19b}$$

Hence, introducing this boundary condition under the Hankel transform domain, the relation between responses at the surface of the 1^{st} layer and coefficients of integration of the n^{th} layer may be written as follows:

$$\begin{Bmatrix} \tilde{u}_\theta^{(0)}(\xi, 0) \\ \tilde{\tau}_{z\theta}^{(0)}(\xi, 0) \end{Bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{Bmatrix} 0 \\ B^{(n)}(\xi) \end{Bmatrix} \tag{20}$$

Solving for $B^{(n)}$ from Equation (20) yields:

$$B^{(n)}(\xi) = \frac{\tilde{\tau}_{z\theta}^{(0)}}{t_{22}} \tag{21}$$

Responses for the i^{th} layer may be determined by performing inverse Hankel transform on Equations (8) and (10):

$$u_\theta^{(i)}(r,z) = \int_0^\infty \xi \, \tilde{u}_\theta^{(i)}(\xi, z) J_1(\xi r) d\xi \tag{22}$$

$$\tau_{z\theta}^{(i)}(r,z) = \int_0^\infty \xi \, \tilde{\tau}_{z\theta}^{(i)}(\xi, z) J_1(\xi r) d\xi \tag{23}$$

Performing Hankel transform on Equation (13) yields:

$$\tau_{r\theta}^{(i)}(r,z) = \int_0^\infty \xi \, \tilde{\tau}_{r\theta}^{(i)}(\xi, z) J_2(\xi r) d\xi \tag{24}$$

Solutions of Equations (22)-(24) may be obtained through numerical integration. However, ensuring accuracy of the numerical integration for sections closer to the surface is very difficult. In this research double exponential formula together with Richardson's extrapolation were used to obtain solutions with excellent accuracy.

4 TWO LAYER SYSTEMS

4.1 Torsional loading

Two types of 2-layered pavement structure with 5cm and 10cm thick surface layer were compared. Young's moduli for the first and second layers were $E_1 = 4000\,\mathrm{MPa}$ and $E_2 = 100\,\mathrm{MPa}$, respectively. Poisson's ratio for both layers was 0.35. Assuming coefficient of friction to be 0.5 when a 49kN vertical load acts on the surface of the 2-layered structure, shearing stress due to torsional load would be $q_\theta = 0.347\,\mathrm{MPa}$. Figure 3 shows u_θ for the two structures with $h = 5\,\mathrm{cm}$ and $h = 10\,\mathrm{cm}$ as surface layer thicknesses. This figure shows the displacement, u_θ, for the structure with 5cm thick surface layer is about 40% higher than displacement for the structure with 10cm thick surface layer. The circumferential displacements in both structures decrease with depth.

Figure 4 shows $\tau_{z\theta}$ for the two structures with $h = 5\,\mathrm{cm}$ and $h = 10\,\mathrm{cm}$ as surface layer thicknesses. $\tau_{z\theta}$ values at the surface of these structures were similar. These values were relatively very small at the bottom of the surface layers; i.e. 1/50 decrease for the case of 5cm thick surface layer and 1/100 decrease for the case of the 10cm thick surface layer. Figure 5 shows $\tau_{r\theta}$ results for the two structures. Comparisons at $r = 15\,\mathrm{cm}$ show that $\tau_{r\theta}$ values for the 5cm thick surface layer was 18% higher than for the 10cm thick surface layer. Furthermore, comparisons of maximum shear stress at the bottom of the surface layer indicated the results for the 5cm thick layer was three times higher than that of the structure with 10cm thick surface layer.

Results show that $\tau_{z\theta}$ at the pavement surface is not affected by the thickness of the surface layer, while $\tau_{r\theta}$ is affected by the thickness of the surface layer. $\tau_{r\theta}$ at the surface of the 10cm thick surface layer was 1.37 times $q_\theta\,(= 0.347\,\mathrm{MPa})$ while at the surface of the 5cm thick surface layer, $\tau_{r\theta}$ was 1.67 time q_θ. This means in thinner surface layers there will be higher shearing stress outside the loaded area (in the periphery of the loaded area). Results of shearing stress on the second layer were relatively smaller than on the first layer. The inference drawn from this is there is very low level of shearing stress on inner pavement layers that have lower Young's modulus. When there is sharp turn of wheels, alternating torsional load will be exerted on the pavement surface resulting in alternating stress. When the direction of shearing stress changes at boundary the responses will also change in sign. Thus, pavement responses due to alternating torsional load will also alternated between negative and positive values as shown in Figures 3-5.

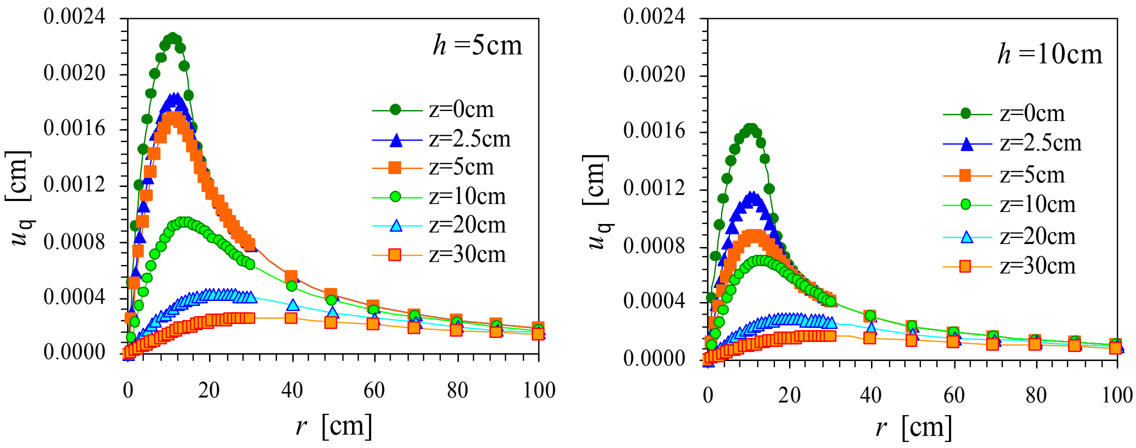


Figure 3. Displacement, u_θ, for $h = 5\,\mathrm{cm}$ and $h = 10\,\mathrm{cm}$.

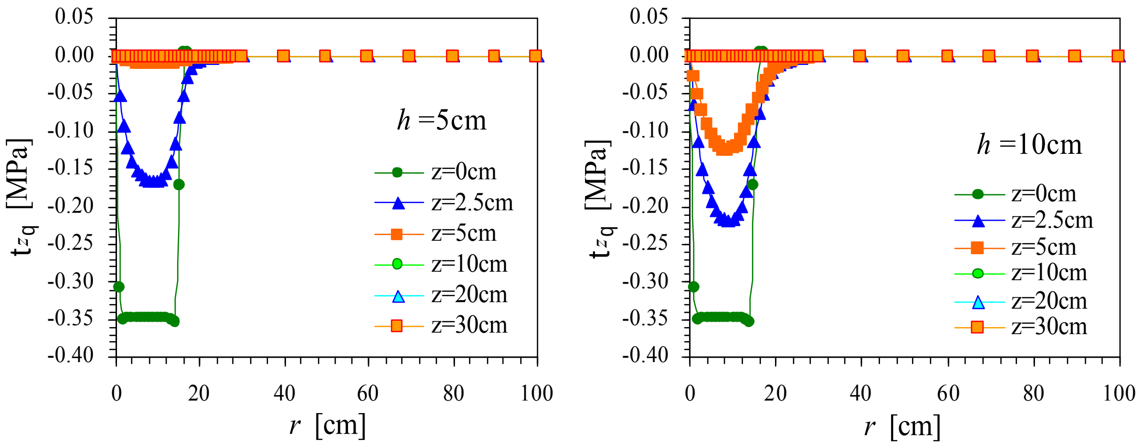


Figure 4. Shear stress, $\tau_{z\theta}$, for $h = 5\,\mathrm{cm}$ and $h = 10\,\mathrm{cm}$.

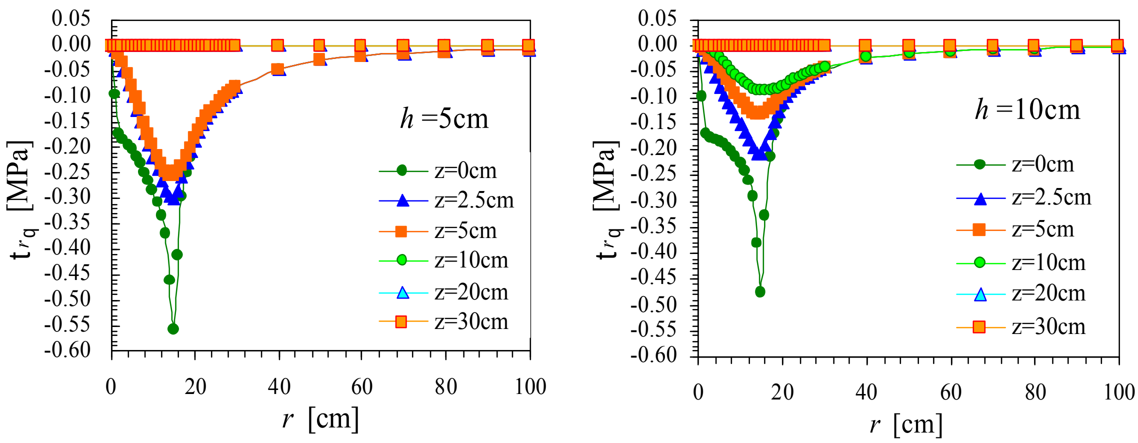


Figure 5. Shear stress, $\tau_{r\theta}$, for $h = 5\,\mathrm{cm}$ and $h = 10\,\mathrm{cm}$.

4.2 Vertical and torsional loading

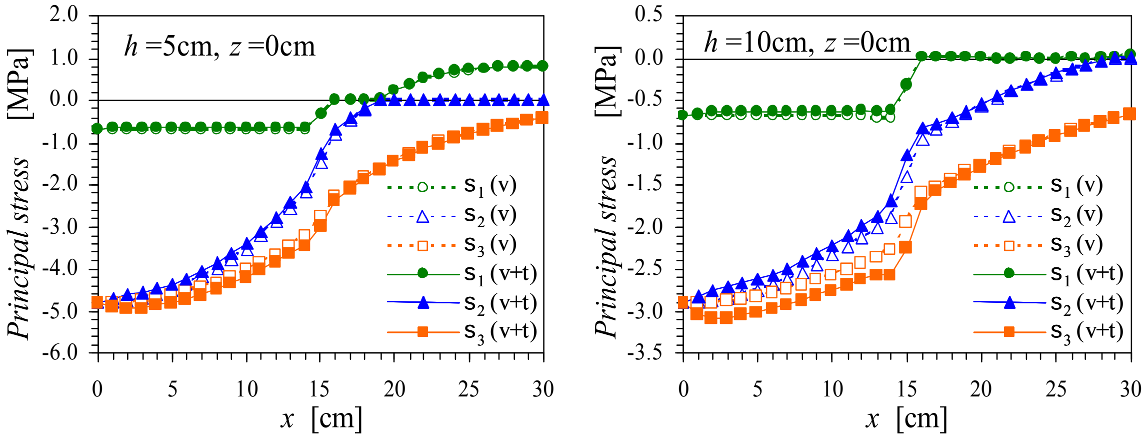


Figure 6. Variation of principal stresses at the surface ($z = 0$) [v: vertical load, t: torsional load].

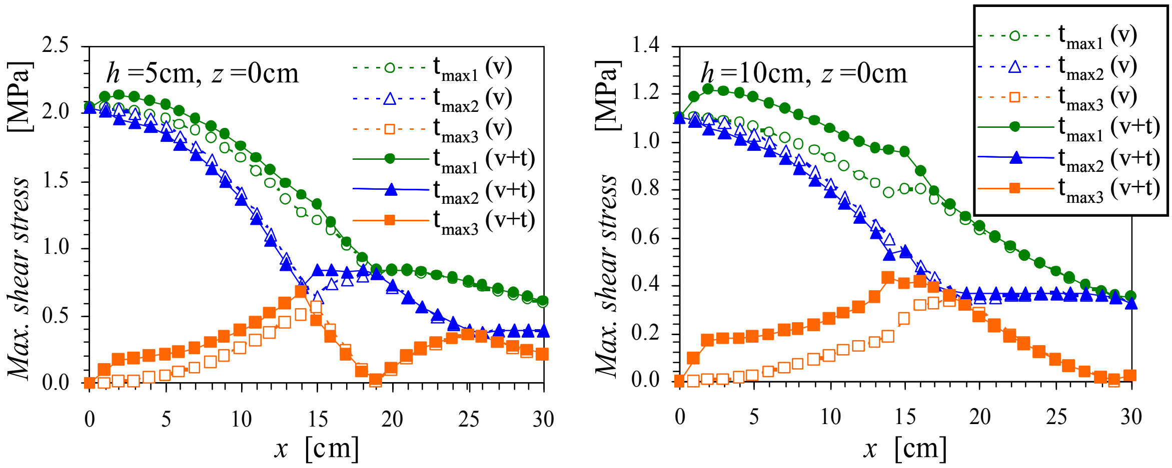


Figure 7. Comparison of maximum shear stress [v: vertical load, t: torsional load].

In real pavement structures, vertical and torsional loads exist. Considering the two pavement structures with different cross sections (h = 5cm, h = 10cm) discussed above, on the surface of which 49kN vertical load and the resulting torsional load act. The moment load resulted from multiplying coefficient of friction (=0.5) by the vertical load. Responses due to vertical load were the displacements u_r and u_z while stresses s_r, s_q, s_z and t_{rz} . Displacement u_q and stresses t_{zq} and t_{rq} were all zeros. Meanwhile, responses due to torsional load were u_q, t_{zq} and t_{rq} while the other components of displacement and stress were all zeros. This implies that when torsional load is applied in addition to the vertical load, u_q, t_{zq} and t_{rq} components appear in pavement structural responses.

The best way to evaluate the influence of torsional load is to examine the differences in principal stresses and maximum shear stresses when there was action of vertical load only and simultaneous actions of vertical load and torsional load on the surface of the pavement structure. Figure 6 shows variation of principal stresses at the surface ($z = 0$). The difference in principal stresses between the h = 5cm thick layer and h = 10cm thick was at most 10%. Figure 7 shows comparison of maximum shear stress. The influence of torsional load is relatively significant, where there is a 20% increase in maximum shear stress. When the direction of torsional load was changed, there was only a change in the direction of principal stresses and maximum shear stress without any increase in their values. The maximum principal stress was compressive and increased by 10% due to the action of a torsional load, whereas maximum shear stress increased by 20% due to the action of torsional load. At the bottom of the surface, since responses due to vertical loading are dominant, no distinct differences were observed.

When uniformly distributed torsional load was considered to act between the tire and pavement surface interface, the effect of torsional load was found to be small. However considering the fact that problems related to dislodgement of aggregated particles have been reported in various places, there appears to be stress concentration between the aggregates and binder that may be higher than the values obtained in this research.

5 CONCLUSIONS

Solutions for torsional load acting on the surface of a layered system were derived using cylindrical coordinate system and Hankel transforms. Solutions to this problem consist of displacement in the circumferential direction, u_q and stresses t_{zq} and t_{rq}. The following conclusions were derived from the worked examples presented in this paper:

1) t_{zq} at the surface of the top layer was similar to the boundary conditions provided, which confirms the accuracy of the algorithm developed.
2) There was a sharp increase of t_{rq} starting from the edge of the load, and overall t_{rq} values were larger for thinner layers. This value is much greater than q_q
3) u_q for the top layer was larger for thinner layers.
4) The maximum principal stress at the surface was compressive and increased by 10% due to the action of a torsional load, whereas maximum shear stress increased by 20% due to the action of torsional load.

Analyses of the simultaneous actions of vertical load and torsional load have shown the influence of torsional load on the principal stress and maximum shear stress was small. This may be due to the assumption that uniformly distributed torsional load was acting between the tire and pavement surface interface. Dislodgement of aggregate particles is considered to be the result of concentrated stress. Analysis based on concentrated stress will be the topic of future research.

APPENDIX

Struve function $\mathbf{H}_n(z)$ is defined as follows:

$$\mathbf{H}_n(z) = \left(\frac{z}{2}\right)^{n+1} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k}}{\Gamma(k+\frac{3}{2})\Gamma(k+n+\frac{3}{2})}$$

When $n = 0$,: $\mathbf{H}_0(z) = \frac{2}{p}\left[z - \frac{z^3}{1^2 \cdot 3^2} + \frac{z^5}{1^2 \cdot 3^2 \cdot 5^2} - \Lambda\right]$

When $n = 1$: $\mathbf{H}_1(z) = \frac{2}{p}\left[\frac{z^2}{1^2 \cdot 3} - \frac{z^4}{1^2 \cdot 3^2 \cdot 5} + \frac{z^5}{1^2 \cdot 3^2 \cdot 5^2 \cdot 7} - \Lambda\right]$

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